McGraw Hill Education WE Series

2nd EDITION

Functions & Graphs

for JEE Main & Advanced

Basics
Graphs
Sets and Relations
Functions

Vinay Kumar

McGRAW HILL EDUCATION WINNING EDGE SERIES

Functions & Graphs

for JEE Main & Advanced

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Second Edition

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B.Tech., IIT Delhi

Mc Graw Hill Education

McGraw Hill Education (India) Private Limited

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McGraw Hill Education (India) Private Limited

Published by McGraw Hill Education (India) Private Limited,
P-24, Green Park Extension, New Delhi – 110 016

Functions and Graphs for JEE Main & Advanced, *2/e*

ISBN (13) : 978-1-25-906421-0
ISBN (10) : 1-25-906421-2

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Typeset at Kaushik Laser Point & Printers, Tis Hazari Court, Delhi-53 and printed at

Cover printed at:

Cover Design: K. Anoop

PREFACE

This book is meant for students who aspire to join the Indian Institute of Technologies (IITs) and various other engineering institutes through the JEE Main and Advanced examinations. The content has been devised to cover the syllabi of JEE and other engineering entrance examinations on the topic *Functions and Graphs*. The book will serve as a text book as well as practice problem book for these competitive examinations.

As a tutor with more than thirteen years of teaching this topic in the coaching institutes of Kota, I have realised the need for a comprehensive textbook in this subject.

I am grateful to McGraw-Hill Education for providing me an opportunity to translate my years of teaching experience into a comprehensive textbook on this subject.

This book will help to develop a deep understanding of the topics through concise theory and problem solving. The detailed table of contents will enable teachers and students to easily access their topics of interest.

Each chapter is divided into several segments. Each segment contains theory with illustrative examples. It is followed by Concept Problems and Practice Problems, which will help students assess the basic concepts. At the end of the theory portion, a collection of Target Problems have been given to develop mastery over the chapter.

The problems for JEE Advanced have been clearly indicated.

The collection of objective type questions will help in a thorough revision of the chapter. The Review Exercises contain problems of a moderate level while the Target Exercises will assess the students' ability to solve tougher problems. For teachers, this book could be quite helpful as it provides numerous problems graded by difficulty level which can be given to students as assignments.

I am thankful to all teachers who have motivated me and have given their valuable recommendations. I thank my family for their whole-hearted support in writing this book. I specially thank Mr. Devendra Kumar and Mr. S. Suman for their co-operation in bringing this book.

Suggestions for improvement are always welcomed and shall be gratefully acknowledged.

Vinay Kumar

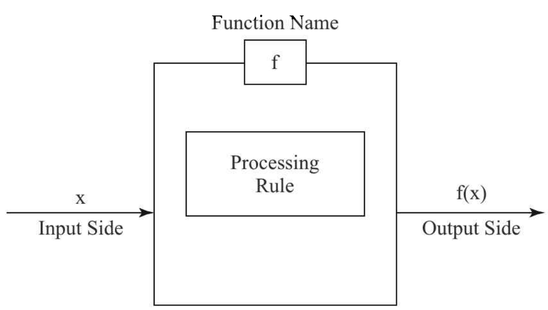
CONTENT

CHAPTER 1

BASICS

1.1 | INTRODUCTION TO FUNCTIONS

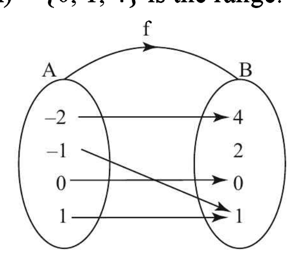
A function is any process where numerical input is transformed into numerical output with the operating restriction that each unique input must lead to one and only one output.



A function f from a set A to a set B is a rule of correspondence that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain which is the set of inputs of the function f, and the set B is the co-domain which contains the outputs. The actual set of outputs is called the range of the function.

Usually, after having agreed on a domain and a co-domain, f is given by a rule, e.g. $f(x) = x^2$, $f(t) = \sin t$, $f(n) = n + 1$.

We sometimes use a diagram to describe a function; for example, the diagram below describes the function $f : A \rightarrow B$, where $A = \{-2, -1, 0, 1\}$ is the domain of f, $B = \{0, 1, 2, 4\}$ is its co-domain, and $f = \{(1, 1), (-1, 1), (0, 0), (-2, 4)\}$. The set $f(A) = \{0, 1, 4\}$ is the range.



Observe that the range of f is always a subset of the co-domain. Observe carefully the distinction between f(x) and f : whereas f(x) is an element of the co-domain, f is the rule of assignment, conveniently expressed as a subset of $A \times B$.

Here, we will only study real valued functions of a real variable. These are functions where both the input and output variable must be a real number.

We often drop the formal notation, which involves its name, specifications of domain and co-domain, direction of relation etc. Rather, we work with the rule alone. For example,

$$f(x) = x^2 + 2x + 3$$

This simplification is based on the fact that domain, co-domain and range are subsets of real numbers. In case these sets have some specific intervals other than "R" itself, then we mention the same with a semicolon (;) or a comma(,) or with a combination of them:

$$f(x) = \sqrt{(x+1)^2 - 1}\ ;\ x < -2,\ x \geq 0$$

Note that the interval "$x < -2, x \geq 0$" specifies a subset of real numbers and defines the domain of function. In general, co-domain of real function is "R".

In some cases, we specify domain, which involves exclusion of certain value(s), like:

$$f(x) = \frac{1}{1-x},\ x \neq 1$$

This means that domain of the function is $R - \{1\}$.

The domain of a function may be explicitly described along with the function, or it may be implied by the expression used to define the function. The implied domain is the set of all real numbers for which the expression is defined.

For instance, the function $f(x) = \dfrac{1}{x^2-4}$ has an implied domain that consists of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is used to avoid even roots of negative numbers. For example, the function $f(x) = \sqrt{x}$ is defined only for $x \geq 0$. Hence, its implied domain is the interval $[0, \infty)$.

The range of a function is more difficult to find, and can best be obtained from the graph of the function.

EXAMPLE 1.1 Find the domain of each of the following functions.

(i) $f : \{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

(ii) Volume of a sphere : $V = \dfrac{4}{3}\pi r^3$

(iii) g(x) = (iv) $h(x) = \sqrt{4-x}$

SOLUTION

(i) The domain of f consists of all first coordinates in the set of ordered pairs, and is therefore the set $\{-3, -1, 0, 2, 4\}$.

(ii) For the volume of a sphere we must choose positive values for the radius r. Thus, the domain is the set of all real numbers r such that $r > 0$.

(iii) Excluding x-values that yield zero in the denominator, the domain of g is the set of all real numbers with $x \neq -5$.

(iv) Choose x-values for which $4 - x \geq 0$. The domain is all real numbers that are less than or equal to 4.

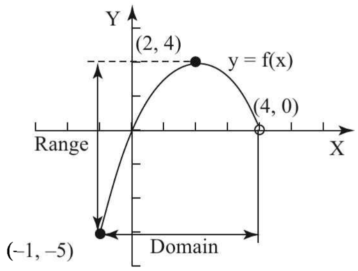
The Graph of a Function

Earlier we studied functions from an algebraic point of view. Here we will study functions from a geometric perspective. The graph of a function f is the collection of ordered pairs (x, f(x)) such that *x* is in the domain of f. The following is the geometrical interpretation of x and f(x).

x = the directed distance from the y-axis.

f(x) = the directed distance from the x-axis.

EXAMPLE 1.2 Using the graph of the function f, shown in figure (i) find the domain of f (ii) find the function values f(–1) and f(2) (iii) find the range of f.



SOLUTION

(i) The closed dot (on the left) indicates that $x = -1$ is in the domain of f, whereas the open dot (on the right) indicates $x = 4$ is not in the domain. Thus, the domain of f is all x in the interval $[-1, 4)$.

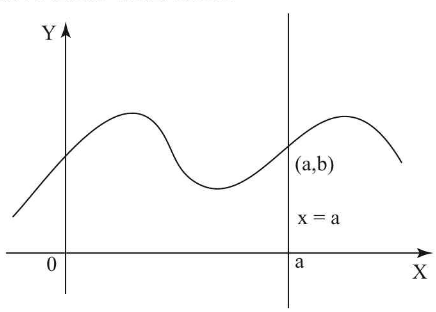
(ii) Because $(-1, -5)$ is a point on the graph of f, it follows that $f(-1) = -5$. Similarly, because $(2, 4)$ is a point on the graph of f, it follows that $f(2) = 4$.

(iii) Since the graph lies between $f(-1) = -5$ and $f(2) = 4$, the range of f is the interval $[-5, 4]$.

The Vertical Line Test

Which curves in the xy-plane are graphs of functions? This is answered by the following test.

A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.



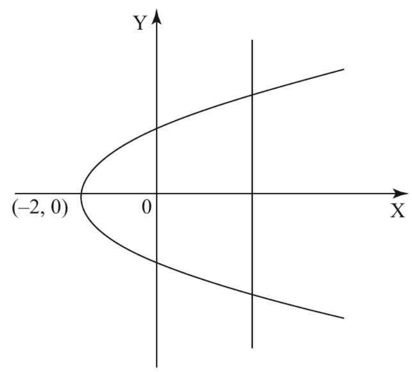
The reason for the truth of the vertical line test can be seen in the figure. If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersect the curve twice, at (a, b) and (a, c), then the curve cannot represent a function because a function cannot assign two different values to a.

For example, the parabola $x = y^2 - 2$ shown in Figure (a), is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice.

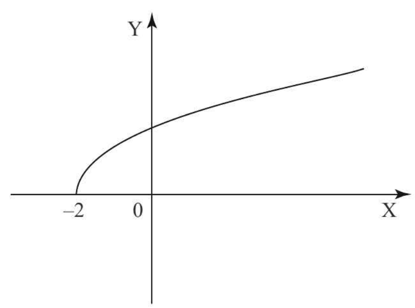
The parabola, however, does contain the graphs of two functions of x. Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x+2}$. Thus, the upper and lower halves of the parabola are the graphs of the functions

$f(x) = \sqrt{x+2}$ [Figure (b)] and

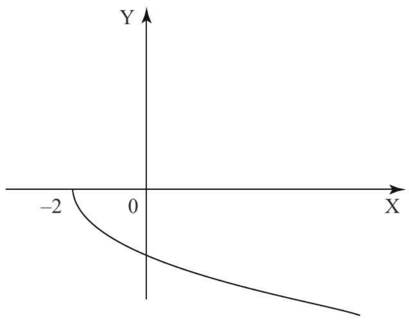
$g(x) = -\sqrt{x+2}$ [Figure (c)].



(a) $y^2 = x + 2$



(b) $y = \sqrt{x+2}$



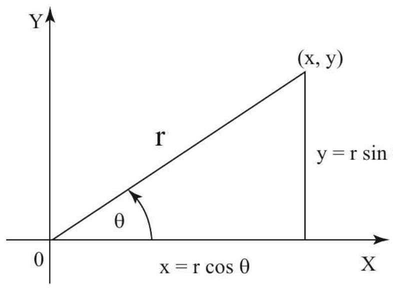
(c) $y = -\sqrt{x+2}$

Polar Coordinates

Up to now we have located points in the plane with rectangular coordinates. We can also locate them with polar coordinates. This is done as follows. Let P be a point distinct from the origin. Suppose the line segment joining P to the origin has length $r > 0$ and makes an angle of θ radians with the positive x-axis.

The two numbers r and θ are called polar coordinates of P. They are related to the rectangular coordinates (x, y) by the equations

$x = r \cos \theta, \qquad y = r \sin \theta$ ...(1)



The positive number r is called the radial distance of *P*, and *θ* is called a polar angle.

The radial distance r is uniquely determined, $r = \sqrt{x^2 + y^2}$, but the polar angle θ is determined only up to integer multiples of 2π.

The set of all points with polar coordinates (r, θ) satisfying $r = f(\theta)$ is called the graph of f in polar coordinates. The equation $r = f(\theta)$ is called the graph of f in polar coordinates. The equation $r = f(\theta)$ is called a polar equation of this graph.

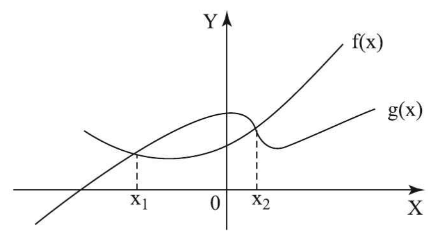
For some curves, polar equations may be simpler and more convenient to use than Cartesian equations. For example, the circle with Cartesian equation $x^2 + y^2 = 4$ has the simpler polar equation $r = 2$. The equations (1) show how to convert from rectangular to polar coordinates.

Graphical Solution of Equations by Intersection of Graphs

An equation is a statement that two algebraic expressions are equal. To solve an equation in x means to find all values of x for which the equation is true. Such values are solutions. For instance, $x = 4$ is a solution of the equation $3x - 5 = 7$, because $3(4) - 5 = 7$ is a true statement.

To solve the equation $f(x) = g(x)$ graphically, we draw the graphs of $y_1 = f(x)$ and $y_2 = g(x)$. The x-coordinate of any point of intersection of the two graphs is a solution of the equation.

Consider the graph shown below:



From the graph, we observe that the equation $f(x) = g(x)$ has two solutions. The solutions of the equation are x_1 and x_2, which are the x-coordinates of the points of intersection.

An equation that is true for every real number in the domain of the variable is an **identity**. For instance, $x^2 - 9 = (x + 3)(x - 3)$ is an identity because it is a true statement for any real value of x. Similarly, $x/(3 \times 2) = 1/(3x)$, where $x \neq 0$, is an identity because it is true for any non-zero real value of x.

Inequality

Inequality is an important concept in understanding function and its properties — particularly domain and range. Many function forms are valid in certain interval(s) of real numbers. This means definition of function is subjected to certain restriction of values with respect to dependent and independent variables. The restriction is generally evaluated in terms of algebraic inequalities,

which may involve linear, quadratic, higher degree polynomials or rational polynomials.

Standard deductions on inequalities:

(i) Equal numbers can be added or subtracted to both sides of an inequality.

(ii) Both sides of an inequality can be multiplied or divided by a positive number without any change in the inequality relation.

(iii) Both sides of an inequality can be multiplied or divided by a negative number with reversal of inequality relation.

(iv) Both sides of an inequality can be squared, provided expressions are nonnegative. As a matter of fact, this conclusion results from rule that we can multiply both sides with a positive number.

(v) When both sides (with same sign) are replaced by their reciprocal, the inequality is reversed.

Equivalently, we may state the above deductions as:

If $x > y$, then:

$x + a > y + a$

$ax > ay; \quad a > 0$

$ax < ay; \quad a < 0$

$x^2 > y^2; \quad x, y > 0$

$\frac{1}{x} < \frac{1}{y}$; when x and y have same sign.

It is evident that we can deduce similar conclusions with the remaining three inequality signs.

Note

(i) Positive number means $x > 0$ (excludes "0").

(ii) Negative number means $x < 0$ (excludes "0").

(iii) Non-negative number means $x \geq 0$ (includes "0").

(iv) Non-positive number means $x \leq 0$ (includes "0").

Graphical Solution of Inequality by Intersection of Graphs

To solve the inequality $f(x) < g(x)$ graphically, we draw the graphs of $y_1 = f(x)$ and $y_2 = g(x)$. The x-coordinates of the points of intersection of the two graphs are solutions of the equation $f(x) = g(x)$.

The solution of the inequality $f(x) < g(x)$ consists of x-coordinates of all points lying on the graph of $y_1 = f(x)$ which lie below the graph of $y_2 = g(x)$.

In the figure shown above, the solution of the inequality $f(x) < g(x)$ is $x_1 < x < x_2$.

1.2 | IMPORTANT TYPES OF FUNCTIONS

In mathematics, we deal with specific real functions, which are characterised by specific domain, range and rules. Some of the familiar functions are polynomial, rational, irrational, trigonometric, exponential, logarithmic functions and piecewise defined functions, etc. These functions are further combined to form more complex functions following certain definition or rule so that the function is meaningful for real values.

There are varieties of functions. These functions are broadly classified under three headings:

(i) Algebraic Functions

An algebraic function is one which is obtained by performing with the variable and known constants any finite number of operations of addition, subtraction, multiplication, division and extraction of integral roots. Polynomials, rational and irrational functions are algebraic.

The function $y = f(x)$ is an algebraic function of x, if y is the root of an equation of the nth degree in y whose coefficients are polynomial functions of x,

$P_0(x)y^n + P_1(x)y^{n-1} + \dots\dots + P_{n-1}(x)y + P_n(x) = 0$

where n is a positive integer and $P_0(x)$, $P_1(x)$.......... are polynomials in x.

For example, $y = x^2 - 3x + 2$, $y = \frac{x^2 - 2x^{3/2}}{x - 1}$, etc. are algebraic functions.

We shall assume that the above equation is irreducible, that is to say it is incapable of resolution into factors whose coefficients are also rational functions of x.

For example the equation $y^4 - x^2 = 0$ is

reducible, since it implies that either $y^2 + x = 0$ or else $y^2 - x = 0$, and each of these equations is irreducible.

$y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note that $y = 1 + x + x^2 + x^3 + ... + \infty$ is not an algebraic function.

(ii) Transcendental Functions

All functions which are not algebraic like trigonometric, inverse trigonometric, exponential and logarithmic functions are called transcendental. This is a wide class of functions, and includes such well known functions as sin x, cos x and log x, as well as many which are less familiar.

(iii) Piecewise Defined Functions

These are functions which are defined by different formulae in different parts of their domains, like modulus, greatest integer, least integer, fraction part functions, etc.

EXAMPLE 1.1 A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

Evaluate f(0), f(1) and f(2) and sketch the graph.

SOLUTION For this function the rule is as follows:

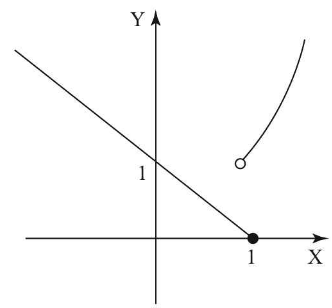
First look at the value of the input x. If it happens that $x \le 1$, then the value of f(x) is $1 - x$. On the other hand, if $x > 1$, then the value of f(x) is x^2.

Since $0 \le 1$, we have $f(0) = 1 - 0 = 1$

Since $1 \le 1$, we have $f(1) = 1 - 1 = 0$

Since $2 > 1$, we have $f(2) = 2^2 = 4$.

How do we draw the graph of f ? We observe that if $x \le 1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = 1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1. If $x > 1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = 1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph. The solid dot indicates that the point (1, 0) is included on the graph; the hollow ring indicates that the point (1, 1) is excluded from the graph.



1.3 | LINEAR FUNCTION

A real polynomial, simply referred as polynomial in our study, is an algebraic expression having terms of "x" raised to nonnegative numbers, separated by "+" or "–" sign.

A real polynomial function in one variable is an algebraic expression having terms of real variable "x" raised to nonnegative numbers. The general form of representation is:

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$

or $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

Here, a_0, a_1, ..., a_n are real numbers. For real function, "x" is real variable and "n" is a nonnegative number. An expression like $2x^2 + 2$ is a valid polynomial in x. But, $x + 1/x$ is not as $1/x = x^{-1}$ has negative integer power. Also, $3x^{1.2} + 2x$ is not a polynomial as it contains a term with fractional power.

Sum and difference of two real polynomials is also a polynomial. Its domain is the real number set R.

A polynomial function is further classified on the basis of

(i) numbers of terms, eg., monomial, binomial, trinomial, etc.

(ii) numbers of variables involved, eg., function in one or two variables and

(iii) degree of the polynomial, eg., linear, quadratic, cubic, biquadratic, etc.

Degree of Polynomial Function

The highest power in the expression is the degree of the polynomial. The degree of the polynomial $x^3 + x^2 + 3$ is 3. The degree "1" corresponds to linear, degree "2" to quadratic, "3" to cubic and "4" to biquadratic polynomial. The general form of quadratic function is:

$f(x) = ax^2 + bx + c;\ a, b, c \in R; a \neq 0$

Note that "a" can not be zero because degree of function reduces to 1. Extending this requirement for maintaining degree of polynomial, we define a polynomial of degree "n" as:

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n;\ a_0 \neq 0$

Zero Polynomial

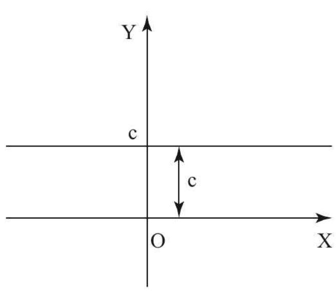
The function is defined as: $y = f(x) = 0$

The polynomial "0", which has no term at all, is called zero polynomial. The graph of zero polynomial is the x-axis itself. Clearly, domain is the real number set R, whereas range is a singleton set {0}.

Constant Function

It is a polynomial of degree 0. The value of constant function is constant irrespective of values of x. The image of the constant function (y) is constant for all values of pre-images (x).

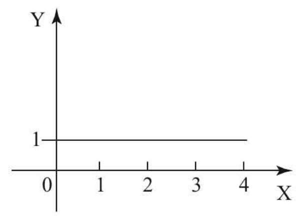
$y = f(x) = c$



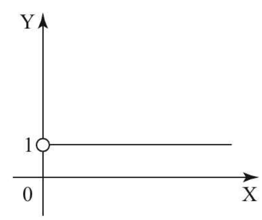
The graph of a constant function is a straight line parallel to x-axis. As "y = (f(x) = c" holds for real values of x, the domain of constant function is R. On the other hand, the value of y is a single valued constant, hence range of the constant function is a singleton set {c}.

For example, $f : R \to R$

$f(x) = 1$



Also, $f(x) = \dfrac{e^{\ln x}}{x} = \dfrac{x}{x} = 1$, is a constant function for $x > 0$.



EXAMPLE 1.1 If $f(x) = \cos 4x - 8\cos^4 x + 8\cos^2 x$, find $f(\pi/9)$.

SOLUTION

$f(x) = 2\cos^2 2x - 1 - 8\cos^4 x + 8\cos^2 x$

$= 2(2\cos^2 x - 1)^2 - 1 - 8\cos^4 x + 8\cos^2 x$

$= 2 - 1 = 1$

Thus, $f(x) = 1$, is a constant function for $x \in R$.

$\therefore\ f(\pi/9) = 1$.

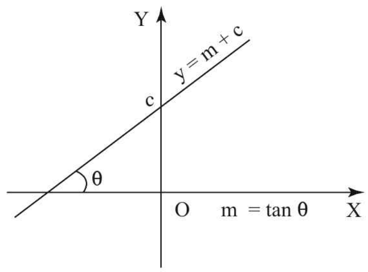
Linear Function

Linear function is a polynomial of degree 1.

$f(x) = a_0x + a_1$

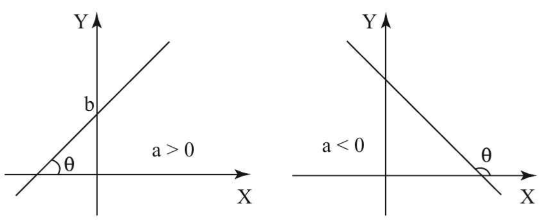
It is also expressed as:

$f(x) = mx + c$

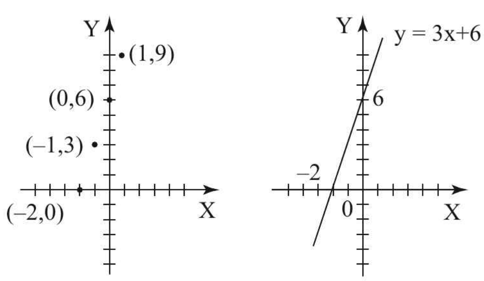


The graph of a linear function is a straight line. The coefficient of x i.e. m is slope of the line and c is y-intercept, which is obtained for x = 0 such that f(0) = c. It is clear from the graph that its domain and range both are real number set R.

The graph of y = ax + b when b > 0 and "a" changes sign is shown below.



Draw the graph of the function y = 3x + 6.



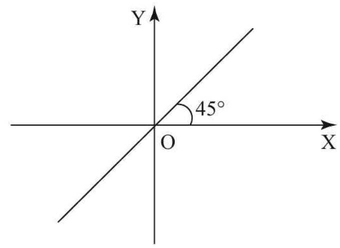
(a) (b)

Identity Function

The dependent (y) and independent (x) variables have same value. It is a linear function in which m=1 and c = 0. An identity function is represented as:

$f : A \to A$

$y = f(x) = x.$



The graph of identity function is a straight line bisecting first and third quadrants of the coordinate system.

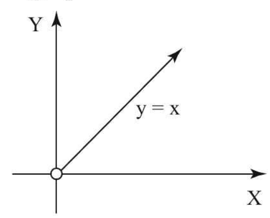
Note

That slope of straight line is 45º. It is clear from the graph that if its domain is R then the range is also R.

For example, $f : R \to R$

$f(x) = \ell n\, e^x = x$, is an identity function for $x \in R$

However, $y = e^{\ell n x} = x$, is an identity function for $x \in R^+$. Its graph is shown below.



EXAMPLE 1.2 Let f be a linear function for which f(6) – f(2) = 12. Find the value of f(12) – f(2).

SOLUTION Since the slope of linear function is constant

$$\therefore \frac{f(6)-f(2)}{6-2} = \frac{f(12)-f(2)}{12-2}$$

$\Rightarrow f(12) - f(2) = 30.$

Linear Inequalities

Each of the inequalities

$2x + 3 > 4$, $3x - 4 \le 2x + 5$, and $3 \ge -5x + 4$ is a linear inequality in the variable x.

EXAMPLE 1.3 Solve the linear inequality $5x - 7 > 3x + 16$.

SOLUTION $5x - 7 > 3x + 16$

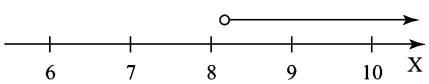
$5x > 3x + 16$

$5x - 3x > 16$

$2x > 16$

$x > 8$

Thus, the solution set consists of all real numbers that are greater than 8. The interval notation for this solution set is $(8, \infty)$. The number line graph of this solution set is shown in the figure.



Solution interval: $(8, \infty)$

CAUTION Remember that when you multiply or divide an inequality by a negative number, you must reverse the inequality symbol.

EXAMPLE 1.4 Solve the linear inequality $1 - \frac{3x}{2} \geq x - 4$ algebraically as well as graphically.

SOLUTION $1 - \frac{3x}{2} \geq x - 4$

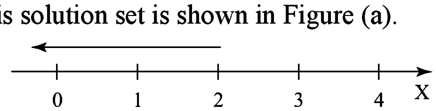
$2 - 3x \geq 2x - 8$

$-3x \geq 2x - 10$

$-5x \geq -10$

$x \leq 2$.

The solution set consists of all real numbers that are less than or equal to 2. The interval notation for this solution set is $(-\infty, 2]$. The number line graph of this solution set is shown in Figure (a).

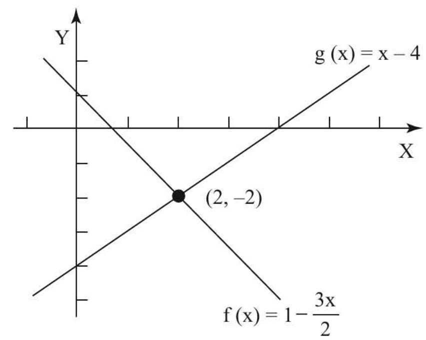


Solution interval: $(-\infty, 2)$

(a)

Graphically, the given inequality can be solved by sketching the graphs of the left and right sides.

$$f(x) = 1 - \frac{3x}{2} \quad \text{and} \quad g(x) = x - 4.$$



(b)

From Figure (b) the graphs appear to intersect at the point $(2, -2)$. This is confirmed by the fact that $f(2) = -2 = g(2)$. Moreover, the graph of f lies above the graph of g to the left of their point of intersection, which implies that $f(x) \geq g(x)$ for $x \leq 2$.

EXAMPLE 1.5 Find the values of "a" for which the function $f(x) = \begin{cases} 2x - 3 & , x < 1 \\ ax + 2a - 3 & , x \geq 1 \end{cases}$ is negative for all real x.

SOLUTION Since f is negative for all real x,

$f(1) = 3a - 3 < 0 \Rightarrow a < 1$.

Further the slope should be nonpositive when

$f(x) = ax + 2a - 3$

$\Rightarrow a \leq 0$.

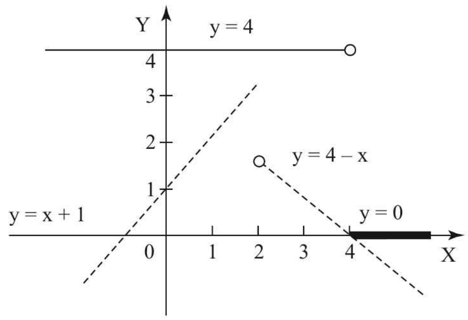
Hence, $a \in (-\infty, 0]$.

EXAMPLE 1.6 Find the values of "a" for which $f(x) \leq g(x)$, for $x \in R$,

where $f(x) = \begin{cases} x + 1 & , \quad x \leq 2 \\ a & , \quad x = 2 \\ 4 - x & , \quad x > 2 \end{cases}$ and

$$g(x) = \begin{cases} 4, & x < 4 \\ 0, & x \geq 4 \end{cases}.$$

SOLUTION We solve this problem graphically. The idea is that the graph of f should lie below the graph of g for all x.



We can see that $f(x) \le g(x)$, for $x \in R$ if $a \le 4$.

Region Represented by a Linear Inequality

To find the region represented by linear inequalities $ax + by \le c$ and $ax + by \ge c$, we proceed as follows:

(i) Convert the inequality into equality to obtain a linear equation in x, y.

(ii) Draw the straight line represented by it.

(iii) The straight line obtained in (ii) divides the xy-plane in two parts.

To determine the region represented by the inequality choose some reference point, for example, origin or some other point on the coordinate axes.

If the coordinates of the reference point satisfy the inequality, the region (half-plane) containing the reference point is the required region, otherwise the region not containing the point is the required region.

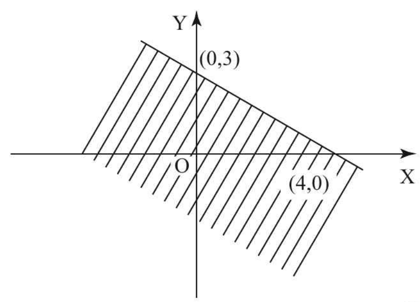
EXAMPLE 1.7 Mark the region represented by

$3x + 4y \le 12.$

SOLUTION Converting the inequality into equation we get $3x + 4y = 12$.

This line meets the coordinate axes at (4, 0) and (0, 3) respectively. We join these points to obtain a straight line represented by $3x + 4y = 12$.

This straight line divides the plane in two parts. One part contains the origin and the other does not contain the origin. Clearly (0, 0) satisfies the inequality $3x + 4y < 12$. So, the region containing the origin is the region represented by $3x + 4y < 12$. This is shown in the figure here.



PRACTICE PROBLEMS [A]

1. Find the domain of each of the following functions:

 (i) $y = f(x) = \sqrt{2x-3}$ (ii) $y = f(x) = 1/(x+1)$

 (iii) $y = f(x) = 1/(x^2 - 1)$ (iv) $y = f(x) = \sqrt{-1/x}$

2. Solve the following equation , $(a^2 - 9)\, x = a^3 + 27$.

3. r and s are integer such that $3r \ge 2s - 3$ and $4s \ge r + 12$. Find the smallest value of $\frac{r}{s}$

4. Solve the inequality, $ax \le 1$

5. Solve the inequality, $a^2 + ax < 1 - x$.

6. Find the number of positive integral roots of the equation $m x = m + 15 x$, $m \in N$

7. Find the complete set of values of ‘a’ for which the system $ax - 1 \le 0$, $x - 4a > 0$ possesses at least one solution.

8. Determine all non-negative integral pairs (x, y) for which $(xy - 7)^2 = x^2 + y^2$.

1.4 | QUADRATIC FUNCTION

The second degree polynomial function

$f(x) = ax^2 + bx + c,\ a \neq 0,$

is called a quadratic function. The domain of the function is $x \in R$. The graph of the function is a parabola with vertical axis.

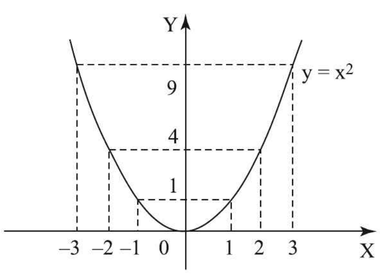
In general, $y = ax^2 + bx + c$

$$\Rightarrow \quad y = a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right)$$

$$\Rightarrow \quad y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a},$$

where $D = b^2 - 4ac$.

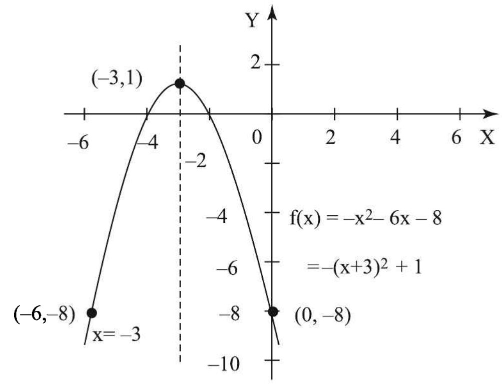
The graph of $y = ax^2 + bx + c$ is a parabola with axis $x = -\dfrac{b}{2a}$ and vertex $\left(-\dfrac{b}{2a}, -\dfrac{D}{4a}\right)$, which opens upward if $a > 0$ and downward if $a < 0$. We first draw the graph of $y = x^2$.



The graph of $y = x^2$ is symmetric about the y-axis. The point of intersection of the parabola with its axis of symmetry is called the vertex of the parabola. The vertex of the parabola $y = x^2$ is origin (0, 0).

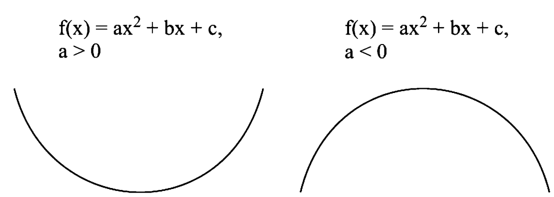
Also note that the function $y = x^2$ increases on the interval $(0, \infty)$ and decreases on the interval $(-\infty, 0)$.

The graph of $f(x) = -x^2 - 6x - 8 = -(x + 3)^2 + 1$ is shown in the figure.



The parabola is the graph of $y = x^2$, translated 3 units to the left and 1 unit upward. It opens downward because of the negative sign before $(x + 3)^2$. The line $x = -3$ is its axis of symmetry, since if it were folded along this line, the two halves would coincide. The vertex, (–3, 1), is the highest point on the graph. The domain is $(-\infty, \infty)$ and the range is $(-\infty, 1]$. The function increases on the interval $(-\infty, -3]$ and decreases on $[-3, \infty)$. Since $f(0) = -8$, the y-intercept is –8 , and since $f(-4) = f(-2) = 0$, the x-intercepts are –4 and –2.

Concavity



This graph is concave up.

This graph is concave down.

Using the quadratic function graph as an illustration, we see that if $a > 0$, the graph is at all times opening upward. If water were to be poured from above, the graph would, in a sense, "hold water." We say that this graph is concave up for all values in its domain. On the other hand, if $a < 0$, the graph opens downward at all times, and it would similarly " dispel water" if it were poured from above. In this case, the graph is concave down for all values in its domain.

A formal discussion of concavity will be done later.

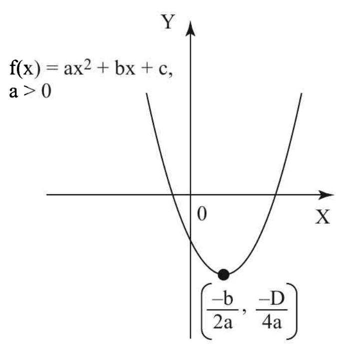
Maximum/ Minimum Values of a Quadratic Function

For the quadratic function $f(x) = ax^2 + bx + c$,

(a) if $a > 0$, the vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is called the minimum point of the graph. The minimum value of the function is $f\left(\frac{-b}{2a}\right) = \frac{-D}{4a}$.

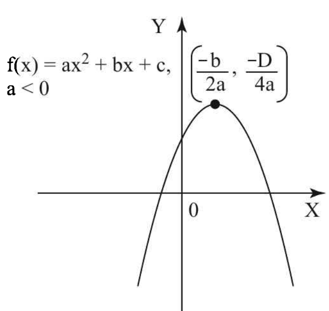
(b) if $a < 0$, the vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is called the maximum point of the graph. The maximum value of the function if $f\left(\frac{-b}{2a}\right) = \frac{-D}{4a}$.

The figures illustrate these points.



$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is the minimum point.

(a)



$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is the maximum point.

(b)

EXAMPLE 1.1 Give the coordinates of the extreme point of the graph of each function, and the corresponding maximum or minimum values of the function.

(a) $f(x) = 2x^2 + 4x - 16$

(b) $f(x) = -x^2 - 6x - 8$

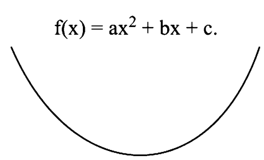
SOLUTION

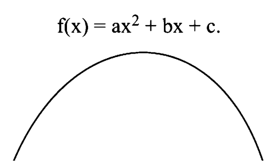
(a) The vertex of the graph of this function is $(-1, 18)$. It opens upward since $a > 0$, so the vertex $(-1, -18)$ is the minimum point and -18 is the minimum value of the function.

(b) The vertex $(-3, 1)$ is the maximum point and $f(-3) = 1$ is the maximum value of the function.

End Behaviour of the Graph of a Quadratic Function

We know that if the value of a is positive for the quadratic function $f(x) = ax^2 + bx + c$, the graph opens upward, and if a is negative, the graph opens downward. The sign of a determines the end behaviour of the graph also. If $a > 0$, as x approaches $-\infty$ or ∞ (writing $x \to -\infty$ or $x \to \infty$), the value of f(x) approaches ∞ (written $f(x) \to \infty$). The other situations similar to this are summarized below.





If $a > 0$,	If $a < 0$,
as $x \to -\infty$, $f(x) \to \infty$;	as $x \to -\infty$, $f(x) \to -\infty$;
as $x \to \infty$, $f(x) \to \infty$.	as $x \to \infty$, $f(x) \to -\infty$.

The graph of a quadratic function can be constructed with the following technique :

(i) Plot the axis $x = -\frac{b}{2a}$ and the vertex $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

(ii) Find the real roots of the quadratic function, if any, and plot the corresponding points of the parabola on the x-axis.

(iii) Mark (0, c), the point of intersection of the parabola with the y-axis.

(iv) The parabola is drawn, with mouth opening upwards if $a > 0$ and downward if $a < 0$.

Study Tip

It is easy to check that the x coordinate of the vertex is $\frac{\alpha+\beta}{2}$ where α, β are roots of the quadratic function. So, once we mark the roots (real) on the x-axis, we draw the axis of the parabola as a vertical line midway between the roots. If the roots are equal, the axis is drawn there itself.

EXAMPLE 1.2 Construct the graph of

(a) $y = -2x^2 - 4x + 6$

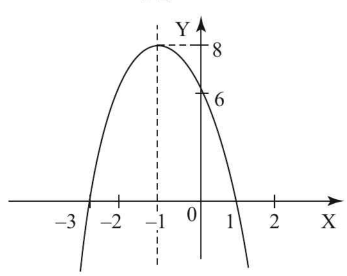
(b) $y = 2x^2 - 4x - 1$

SOLUTION

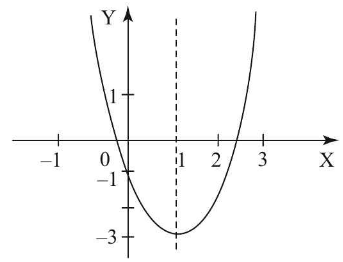
(a) The vertex is (–1, 8) using the formula $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$. The axis of symmetry is $x = -1$. $y = -2x^2 - 4x + 6 = -2(x - 1)(x + 3)$. Hence the roots are x = 1, –3. Observe that $\frac{1+(-3)}{2} = -1$ is the abscissa of the vertex. The graph has an y intercept of 6 units. Since $a = -2 < 0$, the parabola is drawn opening downward.

(b) We can transform $y = 2x^2 - 4x - 1$ as $y = 2(x - 1)^2 - 3$. Hence the vertex is (1, –3).

The roots are $1 \pm \sqrt{\frac{3}{2}}$.



(a)



(b)

EXAMPLE 1.3 Construct the graphs of

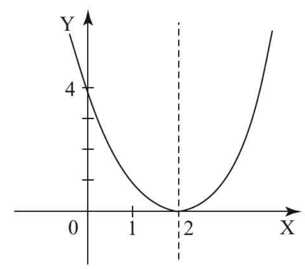
(a) $y = x^2 - 4x + 4$

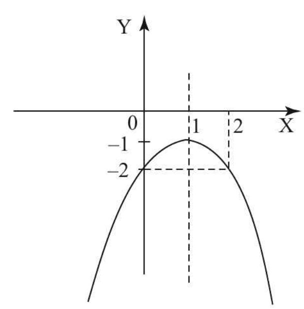
(b) $y = -x^2 + 2x - 2$

SOLUTION

(a) The vertex is (2, 0). Notice that the graph will touch the x-axis at (2, 0) where x = 2 appears as a repeated root. The graph cuts the y-axis at (0, 4).

(b) The vertex is (1, –1). Since there are no real roots, the graph does not intersect the x-axis. The y intercept is –2.





EXAMPLE 1.4 The quadratic polynomial p(x) has the following properties: $p(x) \geq 0$ for all real numbers, $p(1) = 0$ and $p(2) = 2$. Find the value of $p(0) + p(3)$.

SOLUTION As $p(1) = 0$ and $p(x) \geq 0$, hence let $p(x) = k(x-1)^2, k > 0$

$p(2) = k = 2 \quad \Rightarrow \quad k = 2$

$\therefore \quad p(x) = 2(x-1)^2 \quad \Rightarrow \quad p(0) + p(3) = 10.$

EXAMPLE 1.5 If a, b, c are distinct positive real number such that $b(a + c) = 2ac$, then show that the roots of $ax^2 + 2bx + c = 0$ are imaginary.

SOLUTION $b = \dfrac{2ac}{a+c}$ means that b is the H.M. of 'a' and 'c'

Now, the discriminant of $ax^2 + 2bx + c = 0$ is

$D = 4(b^2 - ac)$

Since the G.M. of 'a' and 'c' is $\sqrt{ac}$ and b is their H.M. and G.M. > H.M., we have $\sqrt{ac} > b$

$\therefore \quad ac > b^2 \quad \Rightarrow \quad D < 0.$

Hence, the roots of $ax^2 + 2bx + c = 0$ are imaginary.

EXAMPLE 1.6 Consider the quadratic polynomial $f(x) = x^2 - px + q$ where $f(x) = 0$ has prime roots. If $p + q = 11$ and $a = p^2 + q^2$ then find the value of f(a) where a is an odd positive integer.

SOLUTION $x^2 - px + q = 0$, where roots α, β are prime. $\alpha + \beta = p, \ \alpha\beta = q$

Given $p + q = 11$

$\therefore \quad \alpha + \beta + \alpha\beta = 11$

$\alpha(1 + \beta) = 11 - \beta$

$\Rightarrow \alpha = \dfrac{11-\beta}{1+\beta} = 1 - \dfrac{12}{1+\beta}$

With $\beta = 2, \ \alpha = \dfrac{9}{3} = 3$; check that $\beta \neq 5$.

Hence $\alpha = 3$ and $\beta = 2$

$\therefore \quad p = 5$ and $q = 6$

$p^2 + q^2 = 25 + 36 = 61 = a$

$f(a) = f(61) = (61)^2 - 6 \cdot 61 + 6$

$= 61 \cdot 56 + 6 = 3416 + 6 = 3422.$

Range of Quadratic Function

Consider the quadratic function $f(x) = ax^2 + bx + c$

(i) Range when $x \in R$

If $a > 0 \quad \Rightarrow \quad f(x) \in \left[-\dfrac{D}{4a}, \infty\right)$

$a < 0 \quad \Rightarrow \quad f(x) \in \left(-\infty, \dfrac{D}{4a}\right]$

Maximum or minimum value of

$y = ax^2 + bx + c$ occurs at $x = -(b/2a)$

according as $a < 0$ or $a > 0$.

EXAMPLE 1.7 Find maximum or minimum values of the following quadratic functions over $x \in R$.

(i) $f(x) = 4x^2 - 12x + 15$

(ii) $f(x) = -3x^2 + 5x - 4$

SOLUTION

(i) $f(x) = 4x^2 - 12x + 15$

As $a = 4 > 0$

f(x) has the minimum value at vertex

$f_{min} = \dfrac{-D}{4a} \quad$ at $\quad x = \dfrac{-b}{2a}$

$D = (-12)^2 - 4 \times 4 \times 15 = 144 - 240 = -96$

$\Rightarrow \quad f_{min} = \dfrac{-(-96)}{4 \times 4} = \dfrac{96}{16} = 6$ at $x = -\dfrac{-12}{2 \times 4} = \dfrac{3}{2}$

$\therefore \quad f_{min} = 6$ at $x = \dfrac{3}{2}$

$f_{max} = \infty$

Alternative

Method of perfect square

Let us isolate a perfect square

$$f(x) = 4\left(x^2 - 3x + \frac{9}{4}\right) + 15 - 9$$

$$= 4\left(x - \frac{3}{2}\right)^2 - 6$$

$$f_{min} = -6 \text{ at } x = \frac{3}{2}$$

(ii) $f(x) = -3x^2 + 5x - 4$

As $a = -3 < 0$

f(x) has the maximum value at vertex

$$f_{max} = \frac{-D}{4a} \text{ at } x = -\frac{b}{2a}$$

$D = (5)^2 - 4(-3)(-4) = 25 - 48 = -23$

$$f_{max} = -\frac{(-23)}{4(-3)} = -\frac{23}{12} \text{ at } x = \frac{-(5)}{2(-3)} = \frac{5}{6}$$

$$\therefore \quad f_{max} = \frac{-23}{12} \text{ at } x = \frac{5}{6}$$

$f_{min} = -\infty$

Alternative

Method of perfect square

$$f(x) = -3\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right) - 4 + \frac{25}{12}$$

$$= -3\left(x - \frac{5}{6}\right)^2 - \frac{23}{12}$$

$$f_{max} = \frac{-23}{12} \text{ at } x = \frac{5}{6}.$$

(ii) Range in restricted domain

Consider $f(x) = ax^2 + bx + c$, where $x \in [d, e]$.

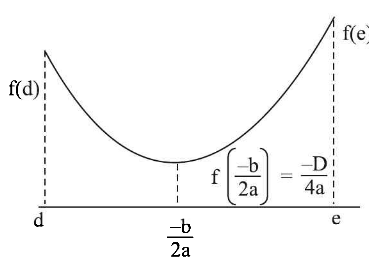
We evaluate $f(x)$ at the following points and choose the least and greatest values of f:

(i) f(d)

(ii) f(e)

(iii) If $x = \frac{-b}{2a} \in (d, e)$ then find $f\left(-\frac{b}{2a}\right)$ or $-\frac{D}{4a}$

otherwise leave the vertex point since it does not lie in the restricted domain.



If the least value is m and the greatest value is M then the range is [m, M].

Study Tip

If x belongs to a union of several intervals then the range is equal to the union of ranges found on each of these intervals.

EXAMPLE 1.8 Find the range of
$f(x) = 2x^2 - 3x + 2$ in [0, 2].

SOLUTION f(0) = 2

f(2) = 4

$f\left(\frac{3}{4}\right) = \frac{7}{8}$ where $x = \frac{-b}{2a} = \frac{3}{4} \in [0, 2]$

$f\left(\frac{3}{4}\right)$ is the least value and f(2) is the greatest value. Thus, the range is $\left[\frac{7}{8}, 4\right]$.

Alternative

Method of perfect square

$$f(x) = 2\left(x - \frac{3}{4}\right)^2 + \frac{7}{8}, x \in [0, 2].$$

To obtain the least value $x = \frac{3}{4}$ is required to make the square zero. The greatest value is obtained using x = 2 since $\left|2 - \frac{3}{4}\right| > \left|0 - \frac{3}{4}\right|$.

Hence the range is $\left[\frac{7}{8}, 4\right]$.

EXAMPLE 1.9 Find the range of
$y = 3 - 2\sin^2\theta - 6\sin\theta$.

SOLUTION Let $x = \sin\theta$, $x \in [-1, 1]$.

$f(x) = 3 - 2x^2 - 6x$, $x \in [-1, 1]$

$f(-1) = 3 - 2 + 6 = 7$

$f(1) = 3 - 2 - 6 = -5$

$$x = \frac{-b}{2a} = \frac{-6}{2 \times 2} = \frac{-3}{2} \notin [-1, 1]$$

Hence, we donot include the vertex.

The range is $y \in [-5, 7]$.

Alternative

$$f(x) = 3 - 2\left(x^2 + \frac{2.3}{2}x + \left(\frac{3}{2}\right)^2\right) + \frac{9}{2}$$

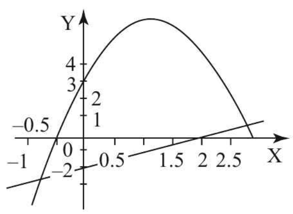
$$= \frac{15}{2} - 2\left(x + \frac{3}{2}\right)^2, x \in [-1, 1]$$

The least value is obtained when the square is made larger using $x = 1$.

The greatest value is obtained when the square is made smaller using $x = -1$.

Here we cannot make the square zero since $x = \frac{-3}{2}$ is not available.

EXAMPLE 1.10 Find the maximum vertical distance d between the parabola $y = -2x^2 + 4x + 3$ and the line $y = x - 2$ throughout the bounded region in the figure.



SOLUTION The vertical distance is given by

$d = -2x^2 + 4x + 3 - (x - 2) = -2x^2 + 3x + 5$, which is a quadratic expression with $a < 0$.

Its maximum value is the y-coordinate of the vertex which has x-coordinate equal to

$$\frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{3}{4}.$$

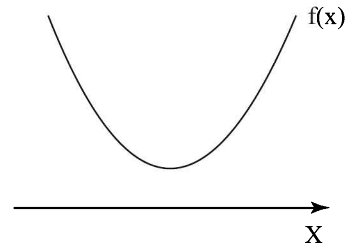
The y-coordinate, then, is $-2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) + 5$

$$= \frac{-9}{8} + \frac{18}{8} + \frac{40}{8} = \frac{49}{8}.$$

Quadratic inequalities

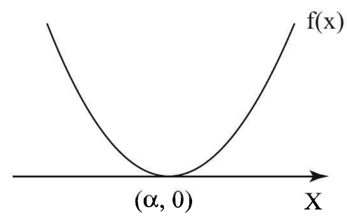
Different positions for graphs of quadratic functions with related inequalities are shown below:

I. $a > 0$ and $D < 0$



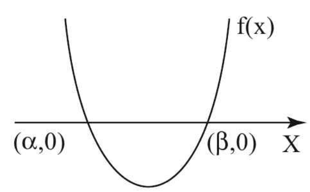
Question	Solution Set
$f(x) = 0$	ϕ
$f(x) < 0$	ϕ
$f(x) > 0$	$(-\infty, \infty)$

II. $a > 0$ and $D = 0$



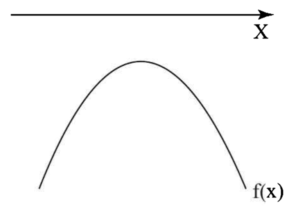
Question	Solution Set
$f(x) = 0$	$\{\alpha\}$
$f(x) < 0$	ϕ
$f(x) > 0$	$(-\infty, \alpha) \cup (\alpha, \infty)$

III. $a > 0$ and $D > 0$



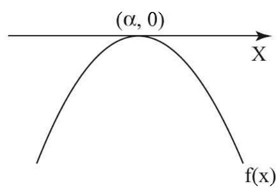
Question	Solution Set
$f(x) = 0$	$\{\alpha, \beta\}$
$f(x) < 0$	(α, β)
$f(x) > 0$	$(-\infty, \alpha) \cup (\beta, \infty)$

IV. $a < 0$ and $D < 0$



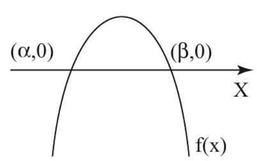
Question	Solution Set
$f(x) = 0$	ϕ
$f(x) < 0$	$(-\infty, \infty)$
$f(x) > 0$	ϕ

V. $a < 0$ and $D = 0$



Question	Solution Set
$f(x) = 0$	$\{\alpha\}$
$f(x) < 0$	$(-\infty, \alpha) \cup (\alpha, \infty)$
$f(x) > 0$	ϕ

VI. $a < 0$ and $D > 0$



Question	Solution Set
$f(x) = 0$	$\{\alpha, \beta\}$
$f(x) < 0$	$(-\infty, \alpha) \cup (\beta, \infty)$
$f(x) > 0$	(α, β)

Consider some examples of solving quadratic inequalities :

(i) The solution set of $x^2 + 2x + 4 > 0$ consists of the entire set of real numbers, $(-\infty, \infty)$ since $a > 0$ and $D < 0$. In other words, the quadratic $x^2 + 2x + 4$ is positive for every real value of x. (Note that this quadratic inequality has no critical numbers. In such a case, there is only one test interval – the entire real line.)

(ii) The solution set of $x^2 + 2x + 1 \le 0$ consists of the single real number, -1, because $a > 0$ and $D = 0$, in which case the quadratic is zero at $x = \dfrac{-b}{2a}$.

(iii) The solution set of $x^2 + 3x + 5 < 0$ is empty since $a > 0$ and $D < 0$. In other words, the quadratic $x^2 + 3x + 5$ is not less than zero for any value of x.

(iv) The solution set of $x^2 - 4x + 4 > 0$ consists of all real numbers except the number 2. In interval notation, this solution set can be written as $(-\infty, 2) \cup (2, \infty)$.

(v) The solution set of $x^2 - x - 6 < 0$ is the interval $(-2, 3)$. The roots of the quadratic $f(x) = x^2 - x - 6$ are $x_1 = 3$ and $x_2 = -2$. This means that the inequality is equivalent to $(x - 3)(x + 2) < 0$.
Applying the sign scheme as discussed above, we find that the set of all solutions of the inequality is the interval $(-2, 3)$

EXAMPLE 1.11 Determine interval of a for which graph of $y = x^2 + (a - 1)x + 16$ lies above the x-axis.

SOLUTION Here coefficient of x^2 is positive. Now, the graph of quadratic function lie above x-axis when $D < 0$ and $a > 0$.

$$D = (a-1)^2 - 4.1.16 < 0 \Rightarrow (a-1)^2 - 64 < 0$$

$$\Rightarrow (a-1)^2 - 8^2 < 0 \Rightarrow (a - 1 + 8)(a - 1 - 8) < 0$$

$$\Rightarrow (a + 7)(a - 9) < 0$$

$$\Rightarrow -7 < a < 9.$$

EXAMPLE 1.12 If the quadratic polynomials defined on real coefficients $P(x) = a_1x^2 + 2b_1x + c_1$ and $Q(x) = a_2x^2 + 2b_2x + c_2$ take positive values $\forall x \in R$, then prove that $g(x) = a_1a_2x^2 + b_1b_2x + c_1c_2$ attains only positive values.

SOLUTION $D_1 = 4b_1^2 - 4a_1c_1 < 0$

i.e. $a_1c_1 > b_1^2$ (1)

$$D_2 = 4b_2^2 - 4a_2c_2 < 0$$

Hence, $a_2c_2 > b_2^2$ (2)

Multiplying (1) and (2)

$$a_1a_2c_1c_2 > b_1^2\, b_2^2$$

Now consider for g(x)

$$D = b_1^2\, b_2^2 - 4a_1a_2\, c_1c_2 < b_1^2\, b_2^2 - 4\, b_1^2\, b_2^2$$

$$= -3\, b_1^2\, b_2^2$$

$\therefore$ $D < 0$ Also $a_1 > 0$ and $a_2 > 0 \Rightarrow a_1, a_2 > 0$

$g(x) > 0\ \forall\ x \in R$.

EXAMPLE 1.13 If $f(x) = 4x^2 + ax + (a - 3)$ is negative for atleast one negative x, find all possible values of a.

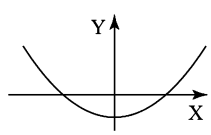
SOLUTION

Case I:

if $f(0) < 0$

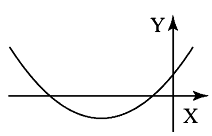
$a - 3 < 0 \Rightarrow a < 3$

$a \in (-\infty, 3)$



Case II:

If $f(0) \geq 0$ and $D > 0$ and $-\dfrac{b}{2a} < 0$



Now, $f(0) > 0$ gives $a - 3 \geq 0 \quad \Rightarrow a \geq 3$ ...(1)

$D > 0$ gives $a^2 - 16(a - 3) > 0$

$\Rightarrow a^2 - 16a - 48 > 0 \Rightarrow (a - 12)(a - 4) > 0$

$\Rightarrow a > 12$ or $a < 4$ ...(2)

$-\dfrac{b}{2a} < 0$ gives $-\dfrac{a}{8} < 0 \Rightarrow a > 0$ ...(3)

From (1), (2) and (3), $a \in [3, 4) \cup (12, \infty)$

Finally, $a \in (-\infty, 4) \cup (12, \infty)$

EXAMPLE 1.14 If set of values of 'a' for which $f(x) = ax^2 - (3 + 2a)x + 6$, $a \neq 0$ is positive for exactly three distinct negative integral values of x is (c, d] then find the value of $(c^2 + 16d^2)$.

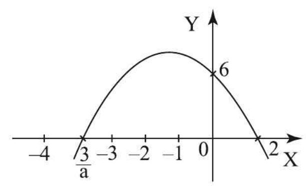
SOLUTION

$f(x) = ax^2 - (3 + 2a)x + 6 = (ax - 3)(x - 2)$

Here, roots of the equation $f(x) = 0$ are 2 and $\dfrac{3}{a}$, and $f(0) = 6$.

f(x) should be positive for exactly three negative integral values of x. Therefore, graph of f(x) must be a downward parabola passing through $x = 2$ and $x = \dfrac{3}{a}$

and $-4 \leq \dfrac{3}{a} < -3$.



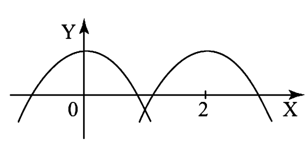
$\Rightarrow a \in \left(-1, -\dfrac{3}{4}\right]$

$\Rightarrow c = -1, d = -\dfrac{3}{4}$

$\Rightarrow c^2 + 16d^2 = 1 + 16 \times \dfrac{9}{16} = 1 + 9 = 10.$

EXAMPLE 1.15 Find all values of k such that the quadratic equation $-2x^2 + kx + k^2 + 5 = 0$ has two distinct roots and only one of the roots satisfies $0 < x < 2$.

SOLUTION



$f(0) \cdot f(2) < 0$

$(k^2 + 5)(k^2 + 2k - 3)$

$k^2 + 2k - 3 < 0$

$(k + 3)(k - 1) < 0$

$-3 < k < 1$

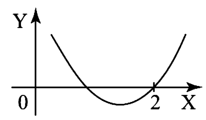
If one root is $x = 2$

then $f(2) = 0$

$k^2 + 2k - 3 = 0 \quad \Rightarrow k = 1$ or $k = -3$

If $k = 1$, $2x^2 - x - 6 = 0$

$\Rightarrow 2x^2 - 4x + 3x - 6 = 0$



$\Rightarrow 2x(x - 2) + 3(x - 2) = 0$

$\Rightarrow x = 2$ or $x = -3/2$

Other root does not lie in (0, 2).

$\Rightarrow k = 1$ is not possible

Similarly when $k = -3$, the roots are $2, -7$

$\Rightarrow k = -3$ is also not possible

Finally, we have $-3 < k < 1$.

PRACTICE PROBLEMS [B]

1. Draw the graph of the following functions:
   (a) $y = 3x - x^2 - 2$ (b) $y = 2x - x^2 - 1$
   (c) $y = x - x^2 - 1$
2. Solve the following inequalities :
   (i) $x^2 - 3x - 4 > 0$ (ii) $4x^2 + 4x + 1 \le 0$
   (iii) $2x^2 - x + 5 > 0$
3. Solve the following system of inequalities :
   (i) $\begin{cases} 2x^2 - 5x + 2 = 0 \\ x - 2 < 0 \end{cases}$ (ii) $\begin{cases} x^2 - 4 \ge 0 \\ x^2 - 2x - 8 \ge 0 \\ -x^2 + 5x - 4 \ge 0 \end{cases}$
4. Determine the intervals of the increase of the function:
   (a) $y = \frac{1}{3}x^2 + x - \sqrt{2}$ (b) $y = -2x^2 + 8x - 3$
5. At what values of a does the function $f(x) = -x^2 + (a - 1)x + 2$ increases on the interval (1, 2) ?
6. Find the maximum and minimum values of the functions $f(x) = 2x^2 - 6x + 3$ where
   (i) $x \in R$ (ii) $x \in [2, \infty)$
   (iii) $x \in [1, 5]$.
7. The point x is said to be a fixed point for the function f if $f(x) = x$. Find all the fixed points of $f(x) = (x - 2)^2$.
8. Solve $x^2 + ax + a > 0$ for different real values of a.
9. Find all values of a for which the inequality $(a - 1)x^2 - (a + 1)x + a + 1 > 0$ is satisfied for all real x.
10. Find all values of 'a' for which the inequality $(a + 4)x^2 - 2ax + 2a - 6 < 0$ is satisfied for all $x \in R$.
11. Find the least and the greatest value of the functions in the indicated intervals :
    (a) $y = 3x^2 - x + 5$ on the interval [1 , 2] (b) $y = -4x^2 + 5x - 8$ on the interval [2 , 3].
12. Find the difference between the least and greatest values of $y = -2x^2 + 3x - 2$ for $x \in [0, 2]$.
13. Find the range of $y = 3^{x+1} - 2 \cdot 3^{2x} - 2$

14. Find the values of k so that the least value of the expression $x^2 + 2kx + k^2 + 3k$ for $x \in [0, 2]$ is 4.
15. Find x when the least value of $2x^2 + 3y^2$ under the conditions $x + y = 2$, $x \geq 0$, $y \geq 0$ occurs.
16. Prove that the expression $3\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) - 8\left(\frac{x}{y} + \frac{y}{x}\right) + 10$ is non-negative for any real x and y different from zero.
17. Prove that the inequality $x^2 + 2xy + 3y^2 + 2x + 6y + 4 \geq 1$ holds for any real x and y.
18. Prove that if $2x + 4y = 1$, the inequality $x^2 + y^2 \geq \frac{1}{20}$ is fulfilled.

1.5 | POLYNOMIAL FUNCTION

A polynomial function is written in the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n.$$

If $a_0 \neq 0$, then we say that the polynomial is of degree n. If it is not known whether the coefficient a_0 is zero or not, then we say that the polynomial is of a degree not higher than n.

From this definition it follows, in particular, that polynomials of degree zero are numbers different from zero. The number zero is also considered to be a polynomial, being the only polynomial which is not assigned a degree. The following notations are used:

P(x), Q(x), , R(x), p(x), q(x), r(x).

If we want to emphasise that the polynomial P(x) is of degree n, we write $P_n(x)$.

Two polynomials, which are integral with respect to x, are **identical** if and only if their degrees are equal and the coefficients in like powers of x are equal.

Remainder Theorem

The remainder of the division of the polynomial P(x) by the binomial $(x - \alpha)$ is equal to the value of the polynomial P(x) for $x = \alpha$, i.e. $r = P(\alpha)$

Factor Theorem

The polynomial P(x) is exactly divisible by the binomial $(x - \alpha)$ if and only if the value of the polynomial is zero for $x = \alpha$,, i.e. $P(\alpha) = 0$

Roots of a Polynomial

The number α is a root of the polynomial P(x) if $P(\alpha) = 0$.

EXAMPLE 1.1 Without actual division prove that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

SOLUTION Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ and $g(x) = x^2 - 3x + 2$ be the given polynomials. Then

$$g(x) = x^2 - 3x + 2 = (x - 1)(x - 2)$$

In order to prove that f(x) is exactly divisible by g(x), it is sufficient to prove that $x - 1$ and $x - 2$ are factors of f(x). For this it is sufficient to prove that $f(1) = 0$ and $f(2) = 0$.

Now, $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

$\Rightarrow f(1) = 2 \times 1^4 - 6 \times 1^3 + 3 \times 1^2 + 3 \times 1 - 2$

and, $f(2) = 2 \times 2^2 - 6 \times 2^3 \times 2^2 + 3 \times 2 - 2$

$\Rightarrow f(1) = 2 - 6 + 3 + 3 - 2$

and $f(2) = 32 - 48 + 12 + 6 - 2$

$\Rightarrow f(1) = 8 - 8$ and $f(2) = 50 - 50$

$\Rightarrow f(1) = 0$ and $f(2) = 0$

$\Rightarrow (x - 1)$ and $(x - 2)$ are factors of f(x).

$\Rightarrow g(x) = (x - 1)(x - 2)$ is a factor of f(x).

Hence, f(x) is exactly divisible by g(x).

EXAMPLE 1.2 A polynomial in x of degree greater than 3 leaves the remainder 2, 1 and –1 when divided by (x – 1), (x + 2) and (x + 1) re-

spectively. Find the remainder, if the polynomial is divided by $(x^2 - 1)(x + 2)$.

SOLUTION Given that $f(1) = 2$, $f(-2) = 1$, $f(-1) = -1$

Let $f(x) = p(x).(x^2 - 1)(x + 2) + ax^2 + bx + c$

Hence, $a + b + c = 2$

$4a - 2b + c = 1$

and $a - b + c = -1$

Solving we get $f(x) = \frac{7}{6}x^2 + \frac{3x}{2} - \frac{2}{3}$.

EXAMPLE 1.3 If p(x) is a cubic polynomial with

$p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$, find $p(6)$.

SOLUTION Put $g(x) = p(x) - x$, which is a polynomial of degree 3 and vanishes at 1, 2, and 3.

Thus $g(x) = A(x - 1)(x - 2)(x - 3)$.

Since $g(4) = p(4) - 4 = 1$ we get

$1 = g(4) = A(4 - 1)(4 - 2)(4 - 3)$ so $A = \frac{1}{6}$.

Thus $p(x) = x + g(x) = x + \frac{(x-1)(x-2)(x-3)}{6}$

and $p(6) = 16$.

EXAMPLE 1.4 Find all cubic polynomials p(x) such that $(x - 1)^2$ is a factor of $p(x) + 2$ and $(x + 1)^2$ is a factor of $p(x) - 2$.

SOLUTION If $(x - \alpha)$ divides a polynomial $q(x)$ then $q(\alpha) = 0$. Let $p(x) = ax^3 + bx^2 + cx + d$. Since $(x - 1)$ divides $p(x) + 2$, we get $a + b + c + d + 2 = 0$.

Hence $d = -a - b - c - 2$ and

$$p(x) + 2 = a(x^3 - 1) + b(x^2 - 1) + c(x - 1)$$
$$= (x - 1)\{a(x^2 + x + 1) + b(x + 1) + c\}.$$

Since $(x - 1)^2$ divides $p(x) + 2$, we conclude that $(x - 1)$ divides $a(x^2 + x + 1) + b(x + 1)$ c. This implies that $3a + 2b + c = 0$. Similarly, using the information that $(x + 1)^2$ divides $p(x) - 2$, we get two more relations: $-a + b - c + d - 2 = 0$; $3a - 2b + c = 0$. Solving these for a, b, c, d, we obtain $b = d = 0$, and $a = 1$, $c = -3$. Thus there is only one polynomial satisfying the given condition:

$$p(x) = x^3 - 3x.$$

EXAMPLE 1.5 Let p be the product of the non-real roots of the equation $x^4 - 4x^3 + 6x^2 - 4x = 2008$, then find p.

SOLUTION $x^4 - 4x^3 + 6x^2 - 4x + 1 = 2009$

$(x - 1)^4 = 2009$

$(x - 1)^2 = \pm\sqrt{2009}$

As we want only the product of non-real roots,

$(x - 1)^2 = -\sqrt{2009} = -7\sqrt{41}$

$x^2 - 2x + 1 + 7\sqrt{41} = 0$

$\Rightarrow$ product of roots $= 1 + 7\sqrt{41}$

$\therefore$ $p = 1 + 7\sqrt{41}$

EXAMPLE 1.6 Compute

$$\sqrt{(1000000)(1000001)(1000002)(1000003)+1}$$

without a calculator.

SOLUTION Put $x = 1\,000\,000 = 10^6$. Then

$x(x + 1)(x + 2)(x + 3) = x(x + 3)(x + 1)(x + 2)$

$= (x^2 + 3x)(x^2 + 3x + 2)$

Again, put $y = x^2 + 3x$.

Then $x(x + 1)(x + 2)(x + 3) + 1$

$= (x^2 + 3x)(x^2 + 3x + 2) + 1$

$= y(y + 2) + 1 = (y + 1)^2$

All this gives, $\sqrt{x(x+1)(x+2)(x+3)+1} = y + 1$

$= x^2 + 3x + 1 = 10^{12} + 3.\ 10^6 + 1$

$= 1\,000\,003\,000\,001$

Roots using sign change of function

If the graph of a continuous function f(x) crosses the x-axis between β_1 and β_2 then f(x) changes sign between β_1 and β_2 i.e. $f(\beta_1)$ and $f(\beta_2)$ have opposite signs.

Hence, if $f(\beta_1)$ and $f(\beta_2)$ have opposite signs, then $f(x) = 0$ has atleast one real root between β_1 and β_2.

If f(x) is a polynomial function then it is continuous for all x and hence, if f(a) and f(b) are of opposite signs, then atleast one root of the equation f(x) = 0 must lie between a and b.

As x changes gradually from a to b, the function f(x) changes gradually from f(a) to f(b), and therefore must pass through all intermediate values; but since f(a) and f(b) have contrary signs the value zero must lie between them; that is, f(x) = 0 for some value of x between a and b.

It does not follow that f(x) = 0 has only one root between a and b; neither does it follow that if f(a) and f(b) have the same sign f(x) = 0 has no root between a and b.

If there are two values of polynomial f(a) and f(b) such that f(a)f(b) > 0, then there are either no real roots or there are even numbers of real roots between a and b. The condition f(a)f(b) > 0 means that function values f(a) and f(b) are either both negative or both positive, i.e., they lie on the same side of x-axis. Since graph is continuous, it may not cross at all or may cross x-axis several times. Clearly, there is either no real root or there are several real roots.

EXAMPLE 1.7 If a < b < c< d, then prove that the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are real and distinct.

SOLUTION Consider

$$f(x) = (x - a)(x - c) + 2(x - b)(x - d)$$
$$f(a) = 2(a - b)(a - d) > 0$$
$$f(b) = (b - a)(b - c) < 0$$
$$f(c) = 2(c - b)(c - d) < 0$$
$$f(d) = (d - a)(d - c) > 0$$

Note that f has degree 2 and hance can have at most two real roots

We see that f(x) changes sign between a and b and again between c and d. Hence f(x) = 0 has one real between a and b and other between c and d.

Note

1. Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term, provided the leading coefficient is positive.

   In the function f(x) if we substitute for x the values ∞, 0, –∞ successively, then we get
   $f(\infty) = \infty,\ f(0) = a_n,\ f(-\infty) = -\infty.$

   If a_n is positive, then f(x) = 0 has a root lying between 0 and –∞, and if a_n is negative f(x) = 0 has a root lying between 0 and ∞.

Study Tip

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} \ldots\ldots + a_n$, then for sufficiently large values of x, f(x) has the same sign as a_0.

2. Every equation which is of an even degree and has its last term negative has atleast two real roots, one positive and one negative, provided the leading coefficient is positive.

   In the function f(x) if we substitute for x the values ∞, 0, –∞ successively, then we get
   $f(\infty) = \infty,\ f(0) = a_n,\ f(-\infty) = \infty.$

   Since a_n is negative, f(x) = 0 has a root lying between 0 and ∞ and a root lying between –∞ and 0 .

3. If f(x) is a polynomial function and the expressions f(a) and f(b) have contrary signs, then an odd number of roots of f(x) = 0 will lie between a and b; and if f(a) and f(b) have the same sign, either no root or an even number of roots will lie between a and b.

EXAMPLE 1.8 Show that the equation

$$10x^3 - 17x^2 + x + 6 = 0$$

has atleast one root between –1 and 0.

SOLUTION Let $f(x) = 10x^3 - 17x^2 + x + 6 = 0$

$$f(-1) = -10 - 17 - 1 + 6 = -22 < 0$$
$$f(0) = 6 > 0$$

Since f(–1). f(0) < 0, f(x) = 0 has atleast one root between –1 and 0.

Study Tip

To determine the nature of roots of some equations the following statements are helpful:

(i) If the coefficients are all positive, the equation has no positive root; thus the equation $x^5 + 4x^3 + 2x + 1 = 0$ cannot have a positive root.

(ii) If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers of x are all of opposite sign, the equation has no negative root. Thus the equation $x^7 + x^5 - 3x^4 + x^3 - 3x^2 + 2x - 5 = 0$ cannot have negative roots.

(iii) If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root. Thus the equation $x^8 + 3x^4 + 2x^2 + 1 = 0$ cannot have a real root.

(iv) If the equation contains only odd powers of x, and the coefficients are all of the same sign, the equation has no real root except $x = 0$. Thus the equation $x^9 + 4x^5 + 5x^3 + 3x = 0$ has no real root except $x = 0$.

Graph of Polynomial Function

Graph of polynomial is continuous and non-periodic. If degree is greater than 1, then it is a nonlinear graph. Polynomial graphs are analysed with the help of function properties like intercepts, slopes, concavity, and end behaviours. They may or may not intersect x-axis. This means that it may or may not have real roots. As the maximum number of roots of a polynomial is atmost equal to the degree of the polynomial, we can deduce that graph can at the most intersect x-axis n times, as maximum numbers of real roots are n. For drawing graphs of polynomial functions refer to the chapter of graphs.

Polynomial Inequalities

EXAMPLE 1.9 Solve the inequality

$$x^3 - 3x^2 - x + 3 > 0$$

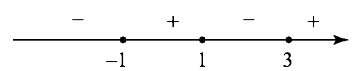
SOLUTION The roots of the polynomial can be easily found.

In fact, $x^3 - 3x^2 - x + 3 = x^2 (x - 3) - (x - 3)$
$= (x - 3) (x - 1) (x + 1)$.

The inequality can now be written as follows:
$(x - 3) (x - 1) (x + 1) > 0$.

The critical numbers of the polynomial are
$x = -1, x = 1, x = 3$

These points partition the number line into 4 subintervals. Let us mark the points on the number line. To determine the sign of the polynomial on each subinterval, we can do the following: note that for $x > 3$ all the linear factors in the polynomial are positive and, consequently, on the interval $(3, \infty)$ the polynomial assumes only positive values. We mark the interval $(3, \infty)$ with plus sign.



When passing from the interval $(3, \infty)$ to the interval $(1, 3)$, through the point $x = 3$, only one of the linear factors, $x - 3$, changes sign and, consequently, the polynomial becomes negative. So we mark the interval $(1, 3)$ with negative sign.

Proceeding to the left, to the interval $(-1, 1)$, we find that only the factor $x - 1$ changes sign. This means that on passing through the point $x = 1$ the left-hand side of the inequality changes sign. Finally, the passage to the last interval, $(-\infty, -1)$ is again followed by a change of sign of the function.

The inequality being strict, the critical numbers themselves are not the solutions.

Hence the solution set is $(-1, 1) \cup (3, \infty)$.

EXAMPLE 1.10 Solve the inequality

$$(x^2 - 3x + 2)(x^3 - 3x^2)(4 - x^2) \geq 0 \quad (1)$$

SOLUTION Since the identities

$x^2 - 3x + 2 = (x - 2)(x - 1)$,
$x^3 - 3x^2 = x^2 (x - 3)$
$4 - x^2 = -(x - 2)(x + 2)$

hold true, it follows that inequality (1) is equivalent to the inequality

$$[x -(-2)]x^2 (x - 1) x - 2)^2 (x - 3) \leq 0. \quad (2)$$

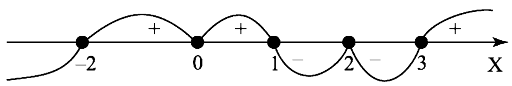
We first solve the equation

$$[x-(-2)]x^2(x-1)(x-2)^2(x-3)=0 \qquad (3)$$

It has only five roots: $x_1=-2$, $x_2=0$, $x_3=1$, $x_4=2$, and $x_5=3$. Then we solve the strict inequality

$$[x-(-2)]x^2(x-1)(x-2)^2(x-3)<0 \qquad (4)$$

using the method of intervals. The set of all its solutions is the set $(-\infty,-2)\cup(1,3)-\{2\}$.



Uniting the set of solutions of equation (3) and that of the strict inequality (4), we get the set of all solutions of inequality (1):

$(-\infty,-2)\cup\{0\}\cup[1,3]$

EXAMPLE 1.11 Prove that

$f(x)=x^{12}-x^9+x^4-x+1$ is positive for all x.

SOLUTION $f(x)=x^{12}-x^9+x^4-x+1$

$=x^9(x^3-1)+x(x^3-1)+1$

In $(-\infty,-1)$, $x^9<0$, $x^3-1<0$, $x<0$

$\Rightarrow f(x)>0$

In $(1,\infty)$, $x^9>0$, $x^3-1>0$, $x>0$

$\Rightarrow f(x)>0$

In $(0,1)$, $0<x^9<1$, $-1<x^3-1<0$, $0<x<1$

$\Rightarrow (x^9+x)<1$, $x^9(x^3-1)+x(x^3-1)>-1$

$\Rightarrow 1+x^9(x^3-1)+x(x^3-1)>0$

Similarly we can prove for $(-1,0)$.

PRACTICE PROBLEMS [C]

1. Find a polynomial in x of the third degree which will vanish when $x=1$ and $x=-2$ and will have the values 4 and 28 when $x=-1$ and $x=2$ respectively
2. The polynomial p(x) satisfies $p(-x)=-p(x)$. When p(x) is divided by $x-3$ the remainder is 6. Find the remainder when p(x) is divided by x^2-9.
3. Solve $(x+4)^5(x+3)^6(x+2)^7(x-1)^8\le 0$
4. Solve $x^4-3x^2+x^2+3x-2\ge 0$
5. Solve $x^4+6x^3+6x^2+6x+5<0$.
6. Solve $(x+3)^4+(x-1)^4\ge 82$.
7. Find the minimum value of x for which $2x^3-5x^2+x+2\ge 0$.
8. In each case, find all polynomials p of degree ≤ 2 which satisfy the given conditions for all real x.
   (a) $p(x)=p(1-x)$ (b) $p(x)=p(1+x)$
   (c) $p(2x)=2p(x)$ (d) $p(3x)=p(x+3)$
9. Solve
   (a) $(x^2-x)^2+3(x^2-x)+2\ge 0$
   (b) $x(x+1)(x+2)(x+3)<48$
10. Find the maximum value of
    $f(x)=x^2-3x$ if $x^4+36\le 13x^2$.
11. Find the number of real roots of the equation
    $(1+x)^8+(1+x^2)^4=2x^4$.

12. Prove that the equation $x^3 + 2x^2 + x + 4 = 0$ has exactly one root in the interval $(-3, -2)$.
13. The polynomial p(x) has integral coefficients and p(x) = 7 for four different integral values of x. Show that p(x) never equals 14.
14. Let $f(x) = (2x-1)^5 (x^2 + x - 1)^{20} = a_{45}x^{45} + a_{44}x^{44} + ... + a_1x + a_0$ be a polynomial of degree 45. Find
    (i) a_0
    (ii) $a_0 + a_1 + a_2 + ...+ a_{44} + a_{45}$
    (iii) $a_0 - a_1 + a_2 + ... - a_{43} + a_{44} - a_{45}$
    (iv) $a_0 + a_2 + a_4 + ... + a_{42} + a_{44}$
    (v) $a_1 + a_3 + a_5 + ... + a_{43} + a_{45}$

1.6 | RATIONAL FUNCTION

Rational function is defined in similar fashion as rational number is defined in terms of numerator and denominator. Implicitly, we refer to "real" rational function here. It is defined as the ratio of two real polynomials with the condition that polynomial in the denominator is not a zero polynomial.

$$f(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0$$

Rational function is not defined for values of x for which denominator of the polynomial evaluates to zero as ratio "p(x)/0" is not defined. Some examples of rational function are:

$$f(x) = \frac{2x^2 - x + 1}{2x^2 - 5x - 3},\ x \neq -\frac{1}{2},\ x \neq 3$$

$$g(x) = \frac{x+1}{2x^2 - x + 1}$$

$$h(x) = \frac{2x^4 - x^2 + 1}{x - 1},\ x \neq -1$$

Note

Observe the second function, g(x) above. There is a no exclusion point for this rational polynomial. The denominator polynomial is $2x^2 - x + 1$, whose discriminant is negative and coefficient of x^2 term is positive. It means denominator of g(x) is positive for all values of x.

We should also note that values of x being excluded are points – not a continuous interval.

Domain of rational function is domain of numerator polynomial minus exclusion points as determined by zeroes of denominator polynomial. Since domain of polynomial is R, domain of rational polynomial is R minus exclusion points determined by denominator. The domains for three rational functions given above are:

Domain of $f(x) = R - \left\{-\frac{1}{2}, 3\right\}$

Domain of g(x) = R

Domain of $h(x) = R - \{-1\}$

EXAMPLE 1.1 Solve the equation

$$\frac{3x}{x-1} - \frac{2x}{x+2} = \frac{3x-6}{(x-1)(x+2)}$$

SOLUTION Any solution of this equation is also a solution of the equation

$3x(x+2) - 2x(x-1) = 3x - 6,$

which results from multiplying the given equation by $(x-1)(x+2)$. This equation is, evidently, equivalent to the following quadratic equation:

$x^2 + 5x + 6 = 0.$

We can find its roots $x = -3$ and $x = -2$. Consequently, only the numbers -3 and -2 can be solution of this equation. Verification confirms that the number -3 is its root while the number -2 is not since for $x = -2$ the sides of the given equation are not defined. The solution is $\{-3\}$

EXAMPLE 1.2 If for any real x, we have

$$-1 \leq \frac{x^2 + nx - 2}{x^2 - 3x + 4} \leq 2,$$ then find the values of n.

SOLUTION The right hand inequality can be written as

$$\frac{x^2+nx-2}{x^2-3x+4}-2\le 0$$

$$\Rightarrow \frac{x^2-(n+6)x+10}{x^2-3x+4}\ge 0$$

$\Rightarrow x^2-(n+6)x+10\ge 0$

[As $x^2-3x+4>0\ \forall\ x\in R$, since $D<0$ and $a>0$]

The above inequality will be true $\forall\ x \in R$ if its $D \le 0$

This gives $-\sqrt{40}-6\le n\le \sqrt{40}-6$ (1)

The left hand inequality can be written as

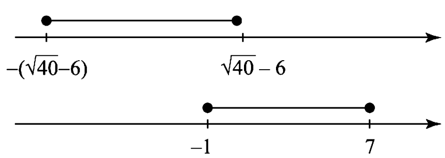
$$\frac{x^2+nx-2}{x^2-3x+4}+1\ge 0$$

$$\Rightarrow \frac{2x^2+(n-3)x+2}{x^2-3x+4}\ge 0$$

$\Rightarrow 2x^2+(n-3)x+2\ge 0$

The above inequality will be true $\forall\ x\in R$ if $D \le 0$ which gives $-1\le n\le 7$

Drawing the number lines for inequalities (1), (2) and taking their intersection gives $x \in [-1, \sqrt{40}-6]$.

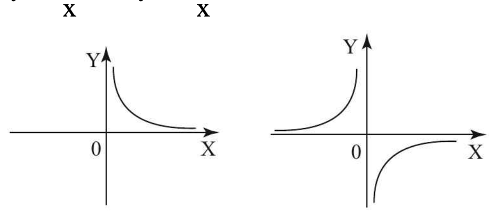


Linear Fractional Functions

The function $f(x)=\dfrac{ax+b}{cx+d}$, $x\in R-\{-d/c\}$, is a called a linear fractional function whose graph is a rectangular hyperbola.

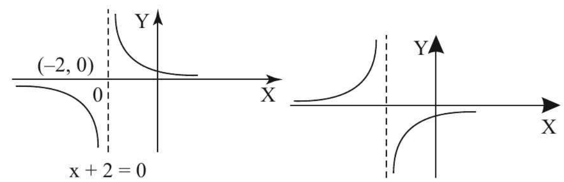
The graphs of rectangular hyperbolas

$y=\dfrac{1}{x}$ and $y=\dfrac{-1}{x}$ are shown below :



The graphs of $y=\dfrac{1}{x+2}$ and $y=\dfrac{-1}{x+1}$ can be drawn using shift of origin.

$$y=\frac{1}{x+2}\qquad y=\frac{-1}{x+1}$$



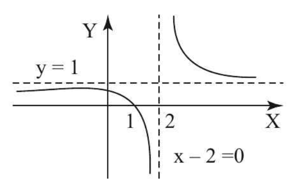
EXAMPLE 1.3 Draw the graph of $y=\dfrac{x-1}{x-2}$ and find its range.

SOLUTION

$$y=\frac{x-1}{x-2}=\frac{x-2+1}{x-2}=1+\frac{1}{x-2}$$

$$\Rightarrow \quad y-1=\frac{1}{x-2}$$

We now draw the graph using shift of origin.



From the graph we observe that, the domain is $x\in R-\{2\}$ and the range is $y\in R-\{1\}$.

In general, the range of the function $y=\dfrac{ax+b}{cx+d}$ is $y\in R-\left\{\dfrac{a}{c}\right\}$.

For example, the range of $y=\dfrac{1-2x}{3x-4}$ is $R-\left\{-\dfrac{2}{3}\right\}$

Range of Rational Function

Let us consider a rational polynomial of degree 2 (we mean that the highest degree of polynomials involved in the ratio is 2). In order to find real values of rational function for real x, we rearrange the given rational function to form a quadratic equation in x.

Let $y = f(x) = \dfrac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$

Rearranging to form a quadratic equation in x, we have:

$y(a_2x^2 + b_2x + c_2) = a_1x^2 + b_1x + c_1$

$\Rightarrow (ya_2 - a_1)x^2 + (yb_2 - b_1)x + c_2y - c_1 = 0$

For x to be real, $D \geq 0$. Hence,

$(yb_2 - b_1)^2 - 4x(ya_2 - a_1)(c_2y - c_1) \geq 0$

We see that discriminant itself is a quadratic inequality. Depending on the nature of coefficient of "y^2" in the quadratic equation and discriminant of the corresponding quadratic equation, the inequality is solved for "y". This, in turn, allows us to determine the real interval(s) of y corresponding to real x.

EXAMPLE 1.4 If x be a real, show that the expression $\dfrac{x^2 + 2x - 11}{x - 3}$ can take all values which do not lie in the open interval (4, 12).

SOLUTION Let $y = \dfrac{x^2 + 2x - 11}{x - 3}$

Writing this as a quadratic equation in x, we have

$x^2 + x(2 - y) + (3y - 11) = 0$ ....(1)

The values of x and y are related by this equation and for each value of y, there is a value of x, which is a root of this quadratic equation. In order that this x (or root) is real, discriminant ≥ 0.

$(2 - y)^2 - 4(3y - 11) \geq 0$

$y^2 - 16y + 48 \geq 48$

$(y - 4)(y - 12) \geq 0$

$\therefore\ y \leq 4$ or $y \geq 12$

Hence y (or the given expression) does not take any value between 4 and 12.

Note

If any value between 4 and 12, say 5, is given for y the equation (1) becomes $x^2 - 3x + 4 = 0$ whose roots are imaginary.

EXAMPLE 1.5 Show that, by giving real values to x, $\dfrac{4x^2 + 36x + 9}{12x^2 + 8x + 1}$ can be made to assume any real value.

SOLUTION Put $\lambda = \dfrac{4x^2 + 36x + 9}{12x^2 + 8x + 1}$

then $x^2(4 - 12\lambda) + (36 - 8\lambda)x + 9 - \lambda = 0$.

Now in order that x may be real it is necessary and sufficient that

$(36 - 8\lambda)^2 - 4(4 - 12\lambda)(9 - \lambda) > 0$,

or that $\lambda^2 - 8\lambda + 72 > 0$,

or $(l - 4)^2 + 56 > 0$,

which is clearly true for all real values of λ. Thus we can find real values of x corresponding to any real value of λ.

EXAMPLE 1.6 Find the value of x for which the function $f(x) = \dfrac{x^2 - 6x + 5}{x^2 + 2x + 1}$ has the least value.

SOLUTION $y = \dfrac{x^2 - 6x + 5}{x^2 + 2x + 1}$

Rearranging to form a quadratic equation in x, we have:

$yx^2 + 2yx + y = x^2 - 6x + 5$

$\Rightarrow (y - 1)x^2 + 2(y + 3)x + y - 5 = 0$

For x real, $D \geq 0$.

$\Rightarrow 4(y + 3)^2 - 4(y - 1)(y - 5) \geq 0$

$\Rightarrow (y^2 + 6y + 9) - (y^2 - 6y + 5) \geq 0$

$\Rightarrow 12y + 4 \geq 0$

$\Rightarrow y \geq -\dfrac{1}{3}$

The values of y, therefore, lies in the interval $[-1/3, \infty)$. The least value of y is $-1/3$. Now, we calculate value of x corresponding to y as:

$-\dfrac{1}{3} = \dfrac{x^2 - 6x + 5}{x^2 + 2x + 1}$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x - 2)^2 = 0 \ \Rightarrow\ x = 2$

This point belongs to the domain of the function. Hence, the least value of y is attained at x = 2.

EXAMPLE 1.7 Find the range of the function

$$y = f(x) = \frac{x^2 - 5x + 4}{x^2 - 3x + 2}$$

SOLUTION We see that discriminants of numerator and denominator polynomials are positive. On factorising,

$$y = \frac{x^2 - 5x + 4}{x^2 - 3x + 2} = \frac{(x-1)(x-4)}{(x-1)(x-2)}$$

Clearly, the rational function is not defined for x = 1 and x = 2. Domain of the function is R – {1, 2}. For the sake of determining range, the limiting values of function for these values of x are obtained by canceling (x – 1) from numerator and denominator:

$$\Rightarrow y = \frac{(x-4)}{(x-2)}$$

For x = 1, y = 3. For x = 2, however, the function value is infinity. In totality, we need to exclude y = 3 from the interval of real values of y. Now, in order to determine real values of y, we rearrange the given function to form a quadratic equation in x:

$yx^2 - 3yx + 2y = x^2 + 4$

$\Rightarrow (y - 1)x^2 + (5 - 3y)x + 2y - 4 = 0$

We should analyse for coefficient of "x^2" in the quadratic equation. For quadratic equation, coefficient of "x^2" can not be zero i.e. $y - 1 \neq 0$. For real x, we have $D \geq 0$.

For y = 1. Putting this value in the quadratic equation,

$0 + (5 - 3)x + 2 - 4 = 0$

$\Rightarrow x = 1$

We see that x = 1 is not part of domain. This is actually the value which reduces denominator to zero. Hence, we should exclude y = 1 from the real values of y. Now for $D \geq 0$,

$D = (5 - 3y)^2 - 4(y - 1)(2y - 4) \geq 0$

$\Rightarrow 25 + 9y^2 - 30y - 4\{2y^2 - 6y + 4\} \geq 0$

$\Rightarrow 25 + 9y^2 - 30y - 4(2y^2 - 6y + 4\} \geq 0$

The coefficient of y^2 is positive. The discriminant is 0. Clearly, following sign rule, $f(x) \geq 0$ for all real values of y. Hence, the values of y are real number set R. However, we need to exclude y = {1, 3) as discussed above. Therefore, range of given function is R – {1, 3}.

Alternative

Once, exception points are noted, we can find the range from the reduced form:

$$y = \frac{(x-4)}{(x-2)}$$

Solving,

$$x = \frac{2y-4}{(y-1)}$$

clearly, $y \neq 1$. But we have seen that $y \neq 3$ as well. Hence, range of the function is R –{1, 3}.

EXAMPLE 1.8 Find the values of m for which the expression $\frac{2x^2 - 5x + 3}{4x - m}$ can take all real values for

$$x \in R - \left\{\frac{m}{4}\right\}.$$

SOLUTION Let $y = \frac{2x^2 - 5x + 3}{4x - m}$

$\Rightarrow 2x^2 - (4y + 5)x + 3 + my = 0$

$\Rightarrow$ As x is real , discriminant ≥ 0

$\Rightarrow (4y + 5)^2 - 8(3 + my) \geq 0$

$\Rightarrow 16y^2 + (40 - 8m)y + 1 \geq 0$

A quadratic in y is nonnegative for all values of y if coefficient of y^2 is positive and discriminant ≤ 0.

$\Rightarrow (40 - 8m)^2 - 4(16)(1) \leq 0$

$\Rightarrow (5 - m)^2 - 1 \leq 0$

$\Rightarrow (m - 5 - 1)(m - 5 + 1) \leq 0$

$\Rightarrow (m - 6)(m - 4) \leq 0$

$\Rightarrow m \in [4, 6]$

Consider m = 4

$$y = \frac{(2x-3)(x-1)}{4(x-1)}$$

$$y = \frac{2x-3}{4}, \; x \neq 1$$

Since $x \neq 1, y \neq -\frac{1}{4}$

Hence y cannot take all real values for m = 4. So m = 4 is not acceptable.
Similarly m = 6 is to be rejected.
So for the given expression to take all real values, m should take values: $m \in (4, 6)$.

CAUTION In such problems the endpoints of the result are rejected, in general due to cancellation of common factor from numerator and denominator and the resulting expression being incapable of assuming all real values.

Procedure for solving rational inequalities

1. Write the inequality so that one side is 0.
2. If necessary, combine terms on the other side into a single quotient.
3. Locate points on the number line for which the quotient is zero (numbers making the numerator zero) or undefined (numbers making the denominator zero). These critical numbers will divide the number line into test intervals.
4. Choose one representative number for each test interval. If substituting that value into the original inequality produces a true statement, then all real numbers in the test interval belong to the solution set. The solution set is the union of all such test intervals.

EXAMPLE 1.9 Solve $\dfrac{x}{x-2} > \dfrac{1}{x+3}$

SOLUTION $\dfrac{x}{x-2} > \dfrac{1}{x+3}$

$$\Rightarrow \frac{x}{x-2} > \frac{1}{x+3} > 0$$

$$\Rightarrow \frac{x^2+3x-x+2}{(x-2)(x+3)} > 0$$

$$\Rightarrow \frac{x^2+2x+2}{(x-2)(x+3)} > 0$$

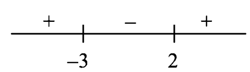
The discriminant of the numerator is

$b^2 - 4ac = 4 - 8 = -4 < 0$

which indicates that the numerator has no zeros. Therefore, x = 2 and x = –3 are the only critical numbers

Critical numbers: $x = -3, x = 2$
Test intervals: $(-\infty\ -3), (-3, 2)\ (2, \infty)$

Test : Is $\dfrac{x^2+2x+2}{(x-2)(x+3)} > 0$?



By testing these intervals, we can determine that the rational expression $(x^2 + 2x + 2)/[(x - 2)(x + 3)]$ is positive in the open intervals $(-\infty, -3)$ and $(2, \infty)$. Therefore, the solution set of the inequality is $(-\infty, -3) \cup (2, \infty)$.

EXAMPLE 1.10 Find solution of the rational inequality:

$$\frac{3x^2+6x-15}{(2x-1)(x+3)} \geq 1$$

SOLUTION We first convert the given inequality to the form $f(x) \geq 0$.

$$\Rightarrow \frac{3x^2+6x-15}{(2x-1)(x+3)} - 1 \geq 0$$

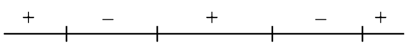
$$\Rightarrow \frac{3x^2+6x-15-(2x-1)(x+3)}{(2x-1)(x+3)} \geq 0$$

$$\Rightarrow \frac{x^2+x-12}{(2x-1)(x+3)} \geq 0$$

$$\Rightarrow \frac{x^2+4x-3x-12}{(3x-1)(x+3)} \geq 0$$

$$\Rightarrow \frac{(x-3)(x+4)}{(2x-1)(x+3)} \geq 0$$

Critical numbers are –4, –3, 1/2, 3. The corresponding sign scheme is:



The solution of inequality is:

$x \in (-\infty, -4] \cup (-3, 1/2) \cup [3, \infty)$

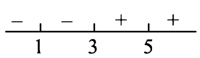
We do not include "–3" and "1" as they reduce denominator to zero.

EXAMPLE 1.11 Solve the inequality

$$\frac{(x-1)^4(x-3)^3}{(x+5)^2} \geq 0 .$$

SOLUTION We first find the sign scheme of the function

$$\frac{(x-1)^4(x-3)^3}{(x+5)^2}$$



There is repetition of sign at such points which appear even number of times. The sign changes alternately at points which appear odd number of times. Hence, the solution of the inequality is

$x \in \{1\} \cup [3, 5) \cup (5, \infty)$.

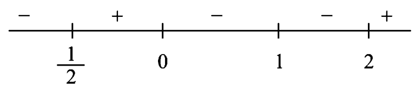
EXAMPLE 1.12 Find interval of x satisfying the inequality given by :

$$\frac{(2x+1)(x-1)}{(x^3-3x^2+2x)} \geq 0$$

SOLUTION We factorise each of the polynomials in the numerator and denominator:

$$\frac{(2x+1)(x-1)}{(x^3-3x^2+2x)} = \frac{(2x+1)(x-1)}{x(x-1)(x-2)}$$

It is important that we do not cancel common factors or terms. Here, critical numbers are –1/2,1,0,1 and 2. The critical number "1" is repeated even times. Hence, we do not change sign about "1" while drawing sign scheme.



While writing interval, we drop equality sign for critical number, which corresponds to denominator.

We do not include "–1" and "1" as they reduce denominator to zero.

Thus, $-1/2 \leq x < 0$, $2 < x < \infty$

$\Rightarrow x \in [-1/2, 0) \cup (2, \infty)$.

EXAMPLE 1.13 Solve the inequality

$$\frac{(x^2-3x+2)(x^2+2x+2)}{(x-3)(x-1-2x^2)} \geq 0$$

SOLUTION Note that the discriminants of the quadratic trinomials $x^2 + 2x + 2$ and $-2x^2 + x - 1$ are negative. Therefore, the trinomials attain values of the same sign coinciding with the sign of the coefficient of x^2:

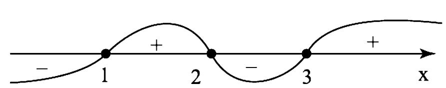
$x^2 + 2x + 2 > 0$ and $-2x^2 + x - 1 < 0$ for any x.

We then rewrite the given inequality as follows:

$$\frac{(x^2-3x+2)}{(x-3)} \leq 0 \text{ or } \frac{(x-1)(x-2)}{x-3} \leq 0$$

Mark the points x = 1, x = 2 and x = 3 on the number line. In the interval x < 1 we take an arbitrary point, say x = 0. At this point $\frac{(x-1)(x-2)}{x-3} = -\frac{3}{2} < 0.$

The wavy curve in the figure indicates alternation in sign



At the points x = 1 and x = 2 the fraction vanishes ; x = 1 and x = 2 are solutions of the given inequality. For x = 3 the fraction loses meaning.

Thus, the solution of the inequality will be:

$x \leq 1, 2 \leq x < 3.$

PRACTICE PROBLEMS [D]

1. If x is real, show that $\dfrac{2x^2-3x+2}{2x^2+3x+2}$ lies between 1/7 and 7.

2. For real values of x, prove that $\dfrac{11x^2+12x+6}{x^2+4x+2}$ cannot lie between –5 and 3.

3. Solve the following inequations:

   (i) $\dfrac{x^2-5x+4}{x-4} \geq 0$ (ii) $\dfrac{x+3}{x^2-6x+8} < 0$

   (iii) $\dfrac{1}{2x-3} \leq \dfrac{1}{2x+5}$ (iv) $\dfrac{1}{x^2+x} \leq \dfrac{1}{2x^2+2x+3}$

   (v) $\dfrac{1}{x-1} - \dfrac{4}{x-2} + \dfrac{4}{x-3} - \dfrac{1}{x-4} < \dfrac{1}{30}$ (vi) $\dfrac{5-2x}{3x^2-2x-16} < 0$

   (vii) $\dfrac{(x+2)(x^2-2x+1)}{4+3x+x^2} \geq 0$ (viii) $\dfrac{1-2x-3x^2}{3x-x^2-5} > 0$

4. Write the set $\{x \in R : (x-1)(x+2) \geq 0\} \cap \left\{x \in R : \dfrac{x^3-2x^2}{x^2-x-6} \leq 0\right\}$ as an interval or as a union of intervals.

5. Find the complete set of values of 'a' for which the equation $\dfrac{1}{x^2} = \dfrac{2a-3}{4-a}$ has solution.

6. Find all values of $n \in N - \{10\}$ for which the equation $\dfrac{x-8}{n-10} = \dfrac{n}{x}$ has no solution.

7. For what values of a is the inequality $\dfrac{x-2a-1}{x-a} < 0$ satisfied for all x lying in the interval [1, 2] ?

8. If $\dfrac{mx^2+3x+4}{x^2+3x+4} < 5$ for all $x \in R$, find all possible values of m.

9. Solve the following inequalities:

   (i) $\dfrac{x+4}{x-2} < \dfrac{2}{x+1}$ (ii) $\dfrac{1}{x+1} - \dfrac{1}{x} \leq \dfrac{1}{x-1} - \dfrac{1}{x-2}$

   (iii) $\dfrac{x^3-2x^2+5x+2}{x^2+3x+2} \geq 1$

10. Solve the following inequalities:

    (i) $x^2 + \dfrac{x^2}{(x+1)^2} < \dfrac{5}{4}$ (ii) $x^2 + \dfrac{4x^2}{(x-2)^2} \leq 5$

11. Solve the following inequalities:

    (i) $\dfrac{(x+1)^4}{x(x^2+1)} > \dfrac{128}{15}$ (ii) $x^3 - \dfrac{1}{x^3} \geq 4\left(x - \dfrac{1}{x}\right)$

1.7 | IRRATIONAL FUNCTION

The term radical is the name given to nth root sign $\sqrt[n]{\ }$. A radical number is n^{th} root of a real number. If y is n^{th} root of x, then $x = y^n$. Interchangeably, we write

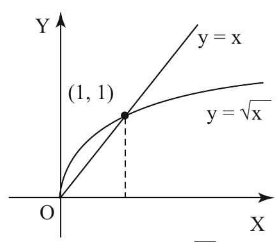
$y = x^{1/n} = \sqrt[n]{(x)}$ when $n \in N, n \geq 2$.

If n is even integer, then x cannot be negative. For n = 2, we drop "n" from the notation and we write,

$y = \sqrt{x}$.

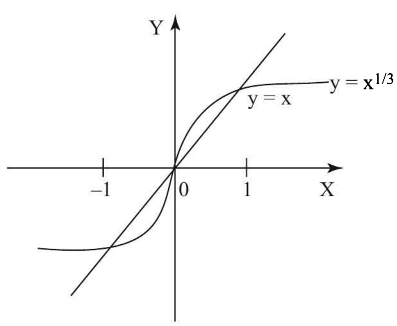
Irrational functions are those which contain atleast one fractional power of x.

For example $f(x) = \sqrt{x}$ is the square root function which is an example of an irrational function.



Graph of $y = \sqrt{x}$

$f(x) = \sqrt[3]{x}$ is the cube root function



Graph of $y = x^{1/3}$

Graph of $y = x^{1/3}$

Note

1. If roots are all even (i.e. $\sqrt{x}, \sqrt[4]{x}, \sqrt[6]{x}$, ... etc.), then it is defined for non-negative real values of the radicand. In other words, if the radicand is negative (i.e. $x < 0$), then the root is imaginary, if the radicand is zero, then the root is also zero, and if the radicand is positive, then the value of the root is also positive.
2. If roots are all odd (i.e. $\sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x}$, etc.) of an equation then it is defined for all real values of the radicand. If the radicand is negative, then the root is negative, if the radicand is zero then the root is zero and if the ra m m,m,m m,m ,m,m,m,n dicand is positive then the root is positive.

EXAMPLE 1.1 Find the domain of the function

$$y = \sqrt{\frac{x^2 - 7x + 12}{x^2 - 2x - 3}}.$$

SOLUTION For the square root function to be defined,

$$\frac{x^2 - 7x + 12}{x^2 - 2x - 3} \geq 0$$

$$\frac{x^2 - 4x - 3x + 12}{x^2 + x - 3x - 3} \geq 0$$

$$\frac{x(x-4) - 3(x-4)}{x(x+1) + 3(x+1)} \geq 0$$

$$\Rightarrow \frac{(x-4)\ (x-3)}{(x-3)(x+1)} \geq 0$$

$$\Rightarrow x \neq 3, \frac{x-4}{x+1} \geq 0$$

$$\begin{array}{ccccc} + & & - & & + \\ & -1 & & 4 & \end{array}$$

Hence, the domain is $x \in (-\infty, -1) \cup [4, \infty)$.

EXAMPLE 1.2 Find the domain of the function

$$y = \sqrt{\frac{\sqrt[4]{17 - 15x - 2x^2}}{x + 3}}.$$

SOLUTION For the square root function to be defined, $\dfrac{\sqrt[4]{17 - 15x - 2x^2}}{x + 3} \geq 0$ (1)

and for the fourth root function to be defined,

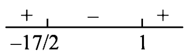
$$17 - 15x - 2x^2 \geq 0 \quad (2)$$

$\Rightarrow 2x^2 + 15x - 17 \leq 0$

$\Rightarrow 2x^2 - 2x + 17x - 17 \leq 0$

$\Rightarrow 2x(x-1) + 17(x-1) \leq 0$

$\Rightarrow (x - 1)(2x + 17) \leq 0$



$x \in [-17/2, 1]$.

We solve (1) in two cases:

Case I : $\begin{cases} \sqrt[4]{17 - 15x - 2x^2} \geq 0 \\ x + 3 > 0 \end{cases}$

$\Rightarrow x \in [-17/2, 1]$ and $x > -3$

$\Rightarrow x \in (-3, 1]$.

Case II : $\begin{cases} \sqrt[4]{17 - 15x - 2x^2} \leq 0 \\ x + 3 < 0 \end{cases}$

$\Rightarrow x = -17/2, 1$ and $x + 3 < 0$

$\Rightarrow x = -17/2$.

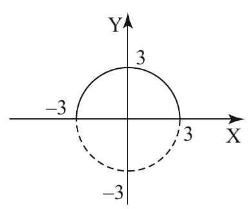
Finally, the domain is $x \in (-3, 1] \cup \{-17/2\}$.

Graphs of Simple Irrational Functions

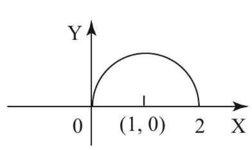
Some irrational functions can be sketched using some familiar curves of coordinate geometry. For sketching complicated irrational functions, the concepts of differential calculus are applied.

The function $y = \sqrt{9 - x^2}$ can be found as the upper semicircular part ($y \geq 0$) of the circle

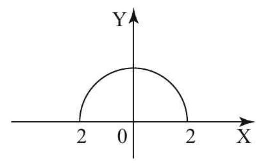
$y^2 = 9 - x^2$ i.e. $x^2 + y^2 = 9$.



Similarly, $y = \sqrt{2x - x^2}$ is drawn using the circle $x^2 + y^2 - 2x = 0$.

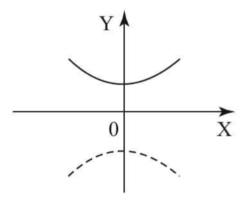


The semi-ellipse $y = \dfrac{\sqrt{4 - x^2}}{2}$ can be drawn with the help of $4y^2 + x^2 = 4$ i.e. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$



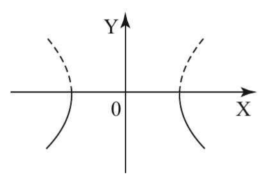
The function $y = \sqrt{x^2 + 1}$ represents a branch of a hyperbola.

$$y = \sqrt{x^2 + 1} \Rightarrow y^2 = x^2 + 1 \Rightarrow x^2 - y^2 = -1$$



The function $y = -\sqrt{x^2 - 1}$, where $y \leq 0$, represents the portion of the hyperbola lying below the x-axis.

$y^2 = x^2 - 1 \quad \Rightarrow \quad x^2 - y^2 = 1$



Irrational Equations

Irrational equation is an equation in which the unknown quantities are under the radical sign. The domain of permissible values of an irrational equation consists of the values of the unknown for which all the expressions under the radical signs of an even degree are non-negative.

One of the ways of solving an irrational equation is to raise both sides of the equation successively to a power which is the least common multiple of the exponents of all the radicals entering into the given equation. If the power to which the

equation is raised is even, the resulting equation can have extraneous roots. In that case the roots must be verified.

Study Tip

When solving irrational equations, we frequently use the formula $\left(\sqrt[n]{f(x)}\right)^n = f(x)$. In the case of an even n, its application may lead to extending the domain of the given equation (for $\left(\sqrt[n]{f(x)}\right)^n$, the constraint $f(x) \ge 0$ is naturally used for an even n, whereas with $\left(\sqrt[n]{f(x)}\right)^n$ replaced by f(x) this constraint is removed). For this reason, when solving irrational equations, we must check the found solutions in the original equation.

The Method of Raising both sides of an Equation to the Same Power.

EXAMPLE 1.3 Solve the equation

$$\sqrt{(6-4x-x^2)} = x+4$$

SOLUTION The equation $\sqrt{(6-4x-x^2)} = x+4$ is equivalent to the system

$$\begin{cases} x+4 \ge 0 \\ 6-4x-x^2 = (x+4)^2 \end{cases}$$

$$\Rightarrow \begin{cases} x \ge -4 \\ x^2+6x+5 = 0 \end{cases}$$

Solving the equation $x^2 + 6x + 5 = 0$

We find that $x_1 = -1$, and $x_2 = -5$ only $x_1 = -1$ satisfies the condition $x \ge -4$.

Consequently, the number –1 is the only solution of the given equation.

EXAMPLE 1.4 Solve the equation

$$\sqrt{x-1} + \sqrt{2x+6} = 6$$

SOLUTION $\sqrt{x-1} + \sqrt{2x+6} = 6$ (1)

Squaring both sides of the equation, we get:

$$x-1+2\sqrt{(x-1)(2x+6)}+2x+6 = 36,$$

and further $2\sqrt{2x^2+4x-6} = -3x+31$.

Squaring the last equation, we get

$$8x^2 + 16x - 24 = 9x^2 - 186x + 961,$$

$\Rightarrow x^2 - 202x + 985 = 0 \quad \Rightarrow x_1 = 5, x_2 = 197$

The found roots are readily checked directly by substituting them into equation (1).

(i) $\sqrt{x_1-1} + \sqrt{2x_1+6} = \sqrt{5-1} + \sqrt{2\times5+6}$
$= 6$

Thus $x_1 = 5$ is a root of the given equation.

(ii) $\sqrt{x_2-1} + \sqrt{2x_2+6} = \sqrt{197-1} + \sqrt{2\times197+6} \ne 6$

that is, $x^2 = 197$ is an extraneous root. Thus, x = 5 is the only root of the given equation.

EXAMPLE 1.5 Solve the equation

$$\sqrt{3x+4} + \sqrt{x-4} = 2\sqrt{x}.$$

SOLUTION

$$\sqrt{3x+4} + \sqrt{x-4} = 2\sqrt{x} \quad (1)$$

We square both sides of the equation

$$3x+4+2\sqrt{(3x+4)(x-4)}+x-4 = 4x \quad (2)$$

$$\Rightarrow 2\sqrt{(3x+4)(x-4)} = 0$$

whose roots are x = –4/3 and x = 4. One of the roots obtained, namely, x = –4/3, does not satisfy the initial equation since it does not belong to the set of its permissible values. Verification shows that for x = 4 satisfies the equation.

EXAMPLE 1.6 Solve the equation

$$\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)} = 1$$

SOLUTION We have

$$\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)} = 1 \quad ...(1)$$

Cubing both sides of (1), we obtain

$$2x-1+x-1 + 3\sqrt[3]{(2x-1)(x-1)}\left[\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)}\right] = 1$$

$\Rightarrow 3x - 2 + 3 \,.\, \sqrt[3]{(2x^2 - 3x + 1)}\ (1) = 1$ [using (1)]

$\Rightarrow 3 \,.\, \sqrt[3]{(2x^2 - 3x + 1)} = 3 - 3x$

$\Rightarrow \sqrt[3]{(2x^2 - 3x + 1)} = (1 - x)$

Again cubing both sides, we obtain

$2x^2 - 3x + 1 = (1 - x)^3$

$\Rightarrow (2x - 1)(x - 1) = (1 - x)^3$

$\Rightarrow (2x - 1)(x - 1) = -(x - 1)^3$

$\Rightarrow (x - 1)\{2x - 1 + (x - 1)^2\} = 0$

$\Rightarrow (x - 1)(x^2) = 0$

$\therefore x_1 = 0$ and $x^2 = 1$

$\because x_1 = 0$ doesnot satisfy the equation (1).

$x_2 = 1$ is the only root of the original equation.

The Method of Introducing New Variables

EXAMPLE 1.7 Solve the equation

$$x^2 + 3 - \sqrt{2x^2 - 3x + 2} = 1.5\,(x + 4)$$

SOLUTION $x^2 + 3 - \sqrt{2x^2 - 3x + 2} = 1.5\,(x + 4)$ (1)

Isolating the radical and squaring both sides of equation (1) would lead to a complicated equation.

Multiplying both sides by 2, we get:

$$2x^2 + 6 - 2\sqrt{2x^2 - 3x + 2} = 3x + 12,$$

and further $2x^2 - 3x + 2 - 2\sqrt{2x^2 - 3x + 2} - 8 = 0$

Setting $y = \sqrt{2x^2 - 3x + 2}$, we get:

$$y^2 - 2y - 8 = 0 \Rightarrow y_1 = 4,\ y_2 = -2.$$

Hence, $\sqrt{2x^2 - 3x + 2} = 4$ or $\sqrt{2x^2 - 3x + 2} = -2$

From the first equation we find: $x_1 = \dfrac{7}{2}$, $x_2 = -2$.

the second equation has no real roots.

Since equation (1) is equivalent to the equation $\sqrt{2x^2 - 3x + 2} = 4$ (because the second equation has no solution), the found values can be checked by substituting them into the equation $\sqrt{2x^2 - 3x + 2} = 4$. This substitution shows that both values of x are roots of the indicated equation, and hence of equation (1).

EXAMPLE 1.8 Assume that p is a real number. Find the values of p so that $\sqrt[3]{x + 3p + 1} - \sqrt[3]{x} = 1$ has real solutions.

SOLUTION $\sqrt[3]{x + 3p + 1} = \sqrt[3]{x} + 1.$

Put $\sqrt[3]{x} = h \quad \sqrt[3]{x + 3p + 1} = h + 1$

$x + 3p + 1 = h^3 + 3h^2 + 3h + 1$

$h^3 + 3p + 1 = h^3 + 3h^2 + 3h + 1$

$3h^2 + 3h - 3p = 0$

$h^2 + h - p = 0$

For real solution $D \ge 0$

$D = 1 + 4p \ge 0$ or, $p \ge -1/4$.

Alternative

$\sqrt[3]{x + 3p + 1} + (-\sqrt[3]{x}) + (-1) = 0$

If $a + b + c = 0$ we have $a^3 + b^3 + c^3 = 3abc$

$x + 3p + 1 - x - 1 = 3\,[(x + 3p + 1)(x)]^{1/3}$

$3p = 3[(x + 3p + 1)(x)]^{1/3}$

$\Rightarrow p^3 = x(x + 3p + 1)$

$\Rightarrow x^2 + (3p + 1)x - p^3 = 0$

For real roots $D \ge 0$

$\Rightarrow 4p^3 + 9p^2 + 6p + 1 \ge 0$

$(p + 1)^2(4p + 1) \ge 0$

$p \ge -1/4.$

EXAMPLE 1.9 Solve the equation

$$\sqrt[4]{1 - x} + \sqrt[4]{15 + x} = 2$$

SOLUTION $\sqrt[4]{1 - x} + \sqrt[4]{15 + x} = 2$ ...(1)

Let us assume $\begin{cases} u = \sqrt[4]{1 - x} \\ v = \sqrt[4]{15 + x} \end{cases}$

Then equation (1) takes the form : $u + v = 2$. But to find the values of the new variables, one equation is not sufficient. Raising both sides of each equation to the fourth power, we get

$$\begin{cases} u^4 = 1 - x \\ v^4 = 15 + x \end{cases}$$

We then add together the equations of the last system:

$u^4 + v^4 = 16.$

Thus, for finding u, v we have the following symmetric system of equations:

$$\begin{cases} u + v = 2 \\ u^4 + v^4 = 16 \end{cases}$$

Solving this system we find (confining ourselves to real solutions):

$$\begin{cases} u_1 = 0 \\ v_1 = 2 \end{cases}; \begin{cases} u_2 = 2 \\ v_2 = 0 \end{cases}$$

The problem has been reduced to solving the collection of the systems:

$$\begin{cases} \sqrt{1-x} = 0 \\ \sqrt{15+x} = 2 \end{cases}; \begin{cases} \sqrt[4]{1-x} = 2 \\ \sqrt{15+x} = 0 \end{cases}$$

Solving this collection, we find : $x_1 = 1$, $x_2 = -15$.

EXAMPLE 1.10 Solve the equation

$$\sqrt[3]{x+1} = \sqrt{x-3}$$

SOLUTION We introduce the designation

$\sqrt[3]{x+1} = u$, $\sqrt{x-3} = v$.

Eliminating x in the equations

$u^3 = x + 1$ and $v^2 = x - 3$, we arrive at a system of equations

$u = v$, $\quad u^3 - v^2 = 4$.

Its solution reduces to that of the equation $v^3 - v^2 - 4 = 0$, whose only real root is $v = 2$. Returning to the initial unknown, we get a linear equation $4 = x - 3$, whose root is the only root of the initial equation. Hence, $x = 7$.

Irrational Inequalities

1. An inequation of the form $\sqrt[2n]{f(x)} < g(x)$, $n \in N$ is equivalent to the system

$$\begin{cases} f(x) \ge 0 \\ g(x) > 0 \\ f(x) < g^{2n}(x) \end{cases}$$

and inequation of the form $\sqrt[2n+1]{f(x)} < g(x)$, $n \in N$ is equivalent to the inequation $f(x) < g^{2n+1}(x)$.

EXAMPLE 1.11 Solve the inequation

$$\sqrt{(x+14)} < (x+2).$$

SOLUTION This inequation is equivalent to the system

$$\begin{cases} x + 14 \ge 0 \\ x + 2 > 0 \\ x + 14 < (x+2)^2 \end{cases}$$

$$\Rightarrow \begin{cases} x \ge -14 \\ x > -2 \\ x^2 + 3x - 10 > 0 \end{cases}$$

$$\Rightarrow \begin{cases} x \ge -14 \\ x > -2 \\ (x+5)(x-2) > 0 \end{cases}$$

$$\Rightarrow \begin{cases} x \ge -14 \\ x > -2 \\ x < -5 \text{ and } x > 2 \end{cases}$$

Taking intersection, we get $x > 2$ i.e. $x \in (2, \infty)$.

2. An inequation of the form $\sqrt[2n]{f(x)} > g(x)$, $n \in N$ is equivalent to the collection of two systems of inequations

$$\begin{cases} g(x) \ge 0 \\ f(x) > g^{2n}(x) \end{cases} \text{ and } \begin{cases} g(x) < 0 \\ f(x) \ge 0 \end{cases}$$

and inequation of the form $\sqrt[2n+1]{f(x)} > g(x)$, $n \in N$ is equivalent to the inequation $f(x) > g^{2n+1}(x)$.

EXAMPLE 1.12 Solve $\sqrt{x^2 + 4x - 5} > x - 3$

SOLUTION For solving $\sqrt{f(x)} > g(x)$, we take two cases:

Case 1: $g(x) \ge 0$ and $f(x) > g^2(x)$

$x - 3 \ge 0 \quad \Rightarrow x \ge 3$

and $\quad x^2 + 4x - 5 > x^2 + 9 - 6x$

$\Rightarrow 10x - 14 > 0 \quad \Rightarrow x > 1.4$

Hence, $x \in [3, \infty)$.

Case 2: $g(x) < 0$ and $f(x) \geq 0$

$x - 3 < 0 \quad \Rightarrow x < 3$

and $\quad x^2 + 4x - 5 \geq 0$

$\Rightarrow x^2 - x + 5x - 5 \geq 0$

$\Rightarrow x(x - 1) + 5(x - 1) \geq 0$

$\Rightarrow x \in (-\infty, -5], \cup [1, \infty)$

Hence, $x \in (-\infty, -5), \cup [1, 3)$.

Finally taking union of the two cases, we have

$x \in (-\infty, -5), \cup [1, \infty)$.

EXAMPLE 1.13 Solve the inequality

$$\sqrt{x^2 + 3x + 2} < 1 + \sqrt{x^2 - x + 1}.$$

SOLUTION The domain of the variable here consists of two intervals: $x \leq -2$ and $x \geq -1$.

In the domain, both members of our inequality are non-negative and so squaring yields the equivalent inequality $2x < \sqrt{x^2 - x + 1}$.

Case 1: For $x \leq -2$ and $-1 \leq x < 0$, this inequality is true since for each of these values of x there is a negative number on the left and a positive number on the right. Thus, all these values of x are solutions to the original inequality.

Case 2: For $x \geq 0$, both members of the inequality

$2x < \sqrt{x^2 - x + 1}$

are non-negative and so squaring yields the equivalent inequality

$3x^2 + x - 1 < 0. \Rightarrow (-1 - \sqrt{13})/6 < x < (-1 + \sqrt{13})/6.$

Taking the condition into account, we find that in the latter case the solution of the original inequality will consist of all values of x in the interval $0 \leq x < (-1 + \sqrt{13})/6$.

Combining both cases we get the answer :

$x \leq -2$ and $-1 \leq x < (1 + \sqrt{13})/6$

CAUTION It should be noted that if we do not consider the cases 1 and 2 and square the inequality

$2x < \sqrt{x^2 - x + 1}$

from the startt, we will naturally loose some of the solutions. This is loss of "vigilance" while squaring.

EXAMPLE 1.14 Solve the inequality

$$\sqrt{x-5} - \sqrt{9-x} > 1. \quad (1)$$

SOLUTION The set of permissible value is $x \in [5, 9]$. Inequality (1) is equivalent to the inequality

$$\sqrt{x-5} - \sqrt{9-x} + 1 \quad (2)$$

whose both sides are nonnegative. Squaring both sides of inequality (2), we get an equivalent inequality

$$2x - 15 > 2\sqrt{9-x}. \quad (3)$$

Case 1: If $2x - 15 \leq 0$, i.e. $x \leq 15/2$, then the left-hand side of the inequality is negative or equal to zero and the right-hand side is positive. Therefore, for any value of x on the interval $[5, 15/2]$ inequality (3) is not satisfied.

Case 2: If $2x - 15 > 0$, i.e. $x > 15/2$, then both sides of the inequality are non-negative, and after squaring we get an inequality which is equivalent to inequality (3) :

$(2x - 15)^2 > 4(9 - x)$.

Thus the set of solutions of inequality (1) is obtained as the set of solutions of the system of inequalities

$5 \leq x \leq 9,\ 2x - 15 > 0,\ (2x - 15)^2 > 4(9 - x),$

Hence we get $x \in \left(\dfrac{14+\sqrt{7}}{2}, 9\right]$.

EXAMPLE 1.15 Solve the inequality

$$\sqrt{3x} - \sqrt{12x+1} \geq 1\cdot \quad (1)$$

SOLUTION Inequality (1) is equivalent to the following system:

$$\begin{cases} 3x \geq 0 \\ 2x + 1 \geq 0 \\ \sqrt{3x} - \sqrt{2x+1} \geq 1. \end{cases} \quad (2)$$

It is advisable to rewrite the last inequality of system (2) in the form $\sqrt{3x} \geq 1 + \sqrt{12x+1}$, where both

sides are non-negative, and therefore the squaring of both sides of this inequality is an equivalent transformation. Thus, from system (2) we pass to the following system which is equivalent to (2):

$$\begin{cases} x \geq 0 \\ (\sqrt{3x})^2 \geq (1+\sqrt{2x+1})^2 \end{cases} \Rightarrow \begin{cases} x \geq 0 \\ \sqrt{2x+1} \leq \dfrac{x}{2} - 1 \end{cases}$$

Further, we have $\begin{cases} x \geq 0 \\ \dfrac{x}{2} - 1 \geq 0 \\ 2x+1 \leq \left(\dfrac{x}{2} - 1\right)^2 \end{cases}$,

from where we get $[12, \infty)$ which is the solution of the last system and, at the same time, of inequality (1).

EXAMPLE 1.16 Solve the inequality

$$\sqrt{2x+5} + \sqrt{x-1} > 8. \quad (1)$$

SOLUTION Inequality (1) is equivalent to the system

$$\begin{cases} 2x+5 \geq 0 \\ x-1 \geq 0 \\ (\sqrt{2x+5} + \sqrt{x-1} > 8) \end{cases} \quad (2)$$

Since both sides of the last inequality of system (2) take on only non-negative values, system (2) is equivalent to the following system:

$$\begin{cases} 2x+5 \geq 0 \\ x-1 \geq 0 \\ (\sqrt{2x+5} + \sqrt{x-1})^2 > 64 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 1 \\ 2\sqrt{2x^2+3x-5} > 60-3x \end{cases} \quad (3)$$

System (3) is equivalent to the following collection of systems:

$$\begin{cases} x \geq 1 \\ 60-3x \geq 0 \\ x^2 - 372x + 3620 < 0 \end{cases}; \quad \begin{cases} x \geq 1 \\ 60-3x < 0 \end{cases}.$$

Note that for $x \geq 1$ the inequality $2x^2 + 3x - 5 \geq 0$ is true (since $2x^2 + 3x - 5 = (2x + 5)(x - 1)$), therefore the last collection of systems of inequalities is equivalent to the collection

$$\begin{cases} x \geq 1 \\ x \leq 20 \\ (x-10)(x-362) < 0 \end{cases}; \quad \begin{cases} x \geq 1 \\ x > 20 \end{cases}.$$

Solving this collection, we get: $10 < x \leq 20$; $x > 20$. Combining these solutions, we get $(10, \infty)$ which is the solution of inequality (1).

PRACTICE PROBLEMS [E]

1. Express with rational denominator $\dfrac{2\sqrt{x+1}}{\sqrt{x-1} - \sqrt{2x} + \sqrt{x+1}}$.
2. Find the square root of
   (a) $3/2\,(x-1) + \sqrt{2x^2-7x-4}$ (b) $2a - \sqrt{3a^2-2ax-x^2}$.
3. Find the cube root of
   (i) $72 - 32\sqrt{5}$ (ii) $99 - 70\sqrt{2}$
4. Solve the following equations:
   (i) $\sqrt{x+1} = a$ (ii) $\sqrt{x+3} = \sqrt{a-x}$
   (iii) $\sqrt{25-x} = 2 - \sqrt{9+x}$
   (iv) $(x+4)(x+1) - 3\sqrt{x^2+5x+2} = 6.$

5. Solve the following inequations:

(i) $(x^2-1)\sqrt{x^2-x-2} \ge 0$

(ii) $\sqrt{\dfrac{x-2}{1-2x}} > -1$ (iii) $\dfrac{\sqrt{x-3}}{x-2} > 0.$

6. Solve the following inequations:

(i) $\sqrt{1-\dfrac{x+2}{x^2}} < \dfrac{2}{3}$ (ii) $\dfrac{3}{\sqrt{2-x}} - \sqrt{2-x} < 2$

(iii) $\dfrac{x^2-13x+40}{\sqrt{19x-x^2-78}} \le 0.$

7. Find the domain of the function $f(x) = \sqrt[3]{\dfrac{x}{1-x^2}}$.

8. Show that $\sqrt{\dfrac{1-x}{x^2}}$ is not equivalent to

(i) $\dfrac{\sqrt{1-x}}{x}$ for $x \in (-\infty, 1] - \{0\}$, or,

(ii) $\dfrac{\sqrt{1-x}}{-|x|}$ for $x \in (-\infty, 0)$, or,

(iii) $\dfrac{\sqrt{1-x}}{-x}$ for $x \in (-\infty, 1] - \{0\}$

but is equivalent to $\dfrac{\sqrt{1-x}}{-x}$ for $x \in (-\infty, 0)$.

9. Consider the functions $f(x) = \sqrt{x^3+5x^2+4x}$ and $g(x) = 2\sqrt{x}$.
   (i) Find the natural domain of f,
   (ii) Find the x and y-intercepts of the curve $y = f(x)$,
   (iii) If $f(4) = a$, express $f(1)$ in terms of a,
   (iv) Solve equation $f(x) = g(x)$.

10. Solve the following equations:

(i) $\dfrac{8}{\sqrt{10-2x}} - \sqrt{10-2x} = 2$ (ii) $\sqrt{6x-x^2-5} = 2x-6$

(iii) $\dfrac{2+\sqrt{19-2x}}{x} = 1.$ (iv) $\sqrt[3]{16-x^3} = 4-x.$

(v) $\sqrt{2x-4} - \sqrt{x+5} = 1.$ (vi) $x^3+1 = 2\sqrt[3]{2x-1}$

11. Solve the following inequations:

(i) $\sqrt{2x-x^2} < 5-x.$ (ii) $\sqrt{x^2-3x-10} < 8-x.$

(iii) $3-x > 3\sqrt{1-x^2}$

12. Solve the following inequations:
    (i) $\sqrt{8+2x-x^2} > 6-3x$. (ii) $\sqrt{-x^2+6x-5} > 8-2x$.
    (iii) $x-3\sqrt{x-3}-1>0$
13. Solve the following inequations:
    (i) $\dfrac{\sqrt{2x-1}}{x-2} < 1$ (ii) $\dfrac{1-\sqrt{21-4x-x^2}}{x+1} \ge 0$
    (iii) $\dfrac{4-\sqrt{x+1}}{1-\sqrt{x+3}} \le 3$
14. Solve the following inequations:
    (i) $\sqrt{x+3} + \sqrt{x+15} < 6$. (ii) $\sqrt{x-6} - \sqrt{10-x} \ge 1$
15. Find the domain of the functions:
    (i) $y = \sqrt{\dfrac{1}{2x^2-5x-3}}$ (ii) $y = \dfrac{\sqrt{12+x-x^2}}{x(x-2)}$
    (iii) $y = \sqrt{\dfrac{x^2-7x+12}{x^2-2x-3}}$ (iv) $y = \sqrt{x-x^2} + \sqrt{3x-x^2-2}$
16. Solve for x in terms of a:
    (i) $\sqrt{x+3} = \sqrt{a-x}$ (ii) $\sqrt{7-x} + \sqrt{x-3} = a$

1.8 | MODULUS FUNCTION

The modulus function returns a non-negative value of a variable or an expression. For this reason, this function is also referred as absolute value function.

The modulus of a real number x is defined as

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}.$$

When the value of x is a non-negative number, then function is "x"; otherwise "–x". The negative sign in the second interval ensures that the function returns a positive value when variable is negative. Thus,

$|5| = 5$, $|\pi| = \pi$, $\left|\dfrac{1}{3}\right| = \dfrac{1}{3}$,

$|0| = 0$, $|-3| = -(-3) = 3$.

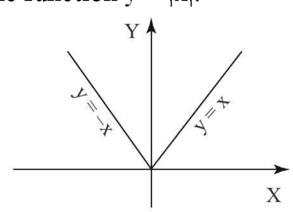
Also, $|f(x)| = \begin{cases} f(x) & \text{if } f(x) \ge 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$.

Let us write $f(x) = |2x-1|$ as a piecewise defined function.

We have $f(x) = 2x-1$ for $2x-1 \ge 0$ and $f(x) = -(2x-1)$ for $2x-1 < 0$. This gives

$$f(x) = \begin{cases} 2x-1 & \text{if } x \le \dfrac{1}{2} \\ 1-2x & \text{if } x > \dfrac{1}{2} \end{cases}.$$

Graph of the function $y = |x|$.



The graph of modulus function is continuous having a corner at x = 0. Since graph is symmetric about y-axis, modulus function is even function. We also see that there are pair of x values for positive values of y.

It is clear from the graph that the domain of modulus function is R. However, the function values are only positive values, including zero. Hence, Domain = R and Range = [0, ∞).

The modulus of a variable can also be thought as the square root of the square of the variable. We must emphasise that the square root is a non-negative value same as modulus value. Hence, $|x| = \sqrt{x^2}$.

EXAMPLE 1.1 Prove that

$$\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}$$

is equal to 2, if $1 \le x \le 2$, and to $2\sqrt{x-1}$, if $x > 2$.

SOLUTION $\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}$

$$= \sqrt{x-1+2\sqrt{x-1}+1} + \sqrt{x-1-2\sqrt{x-1}+1}$$

$$= \left|\sqrt{x-1}+1\right| + \left|\sqrt{x-1}-1\right|$$

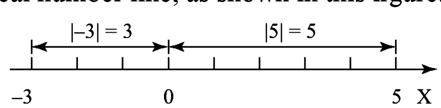
$$= \sqrt{x-1}+1+\left|\sqrt{x-1}-1\right|$$

$$= \begin{cases} 2\sqrt{x-1} & \text{if} \quad \sqrt{x-1} > 1, \\ 2 & \text{if} \quad \sqrt{x-1} \le 1. \end{cases}$$

$$= \begin{cases} 2\sqrt{x-1} & \text{if} \quad x > 2, \\ 2 & \text{if} \quad 1 \le x \le 2. \end{cases}$$

Modulus as Distance

Geometrically, the absolute value of x is the distance between the real number x and the origin on the real number line, as shown in this figure.



Absolute value as the distance from 0

The modulus of an expression "x – a" is interpreted to represent distance between "x" and "a" on the real number line. For example :

$|x - 2| = 5$

This means that the variable "x" is at a distance "5" from "2". We see here that the values of "x" satisfying this equation are:

$x - 2 = \pm 5$

Either $x = 2 + 5 = 7$

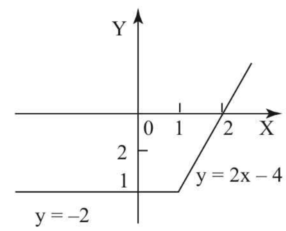
or, $x = 2 - 5 = -3$

"x = 7" is indeed at a distance "5" from "2" and "x = –3" is indeed at a distance "5" from "2".

EXAMPLE 1.2 Simplify the function y = x + |x – 1| – 3 and then draw its graph.

SOLUTION $y = \begin{cases} x + (x-1) - 3 & \text{if } x - 1 \ge 0 \\ x - (x-1) - 3 & \text{if } x - 1 < 0 \end{cases}$

$$= \begin{cases} 2x - 4 \ , & x \ge 1 \\ -2 \ , & x < 1 \end{cases}$$

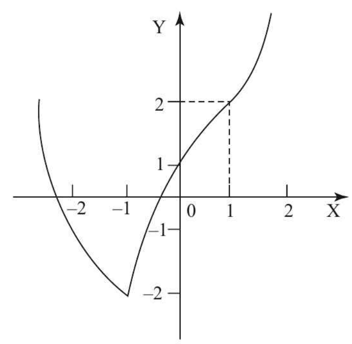


EXAMPLE 1.3 Simplify the function

$y = |x^2 - 1| + 2x$ and then draw its graph.

SOLUTION $y = \begin{cases} x^2 - 1 + 2x & \text{if } x^2 - 1 \ge 0 \\ -(x^2 - 1) + 2x & \text{if } x^2 - 1 < 0 \end{cases}$

$$= \begin{cases} x^2 + 2x - 1 \ ; & x \le -1 \ , \ x \ge 1 \\ -x^2 + 2x + 1 \ ; & -1 < x < 1 \end{cases}$$



EXAMPLE 1.4 Simplify the function

$f(x) = 2\,|\,x - 1\,| - 3\,|\,2 - x\,| + 5\,|\,x - 7\,|$

SOLUTION This problem requires considering a total of 2 × 2 × 2 = 8 combinations of signs, but we can manage things so as to consider only four. This is achieved by a special technique called the "method of intervals".

Mark on the number line those values of x for which each of the expressions under the absolute value sign vanishes: the points 1, 2 and 7. Thus, the entire number line is divided into four intervals:

$x < 1,\ 1 \le x < 2,\ 2 \le x < 7,\ x \ge 7.$

$$f(x) = \begin{cases} -2(x-1)-3(2-x)-5(x-7), & x<1 \\ 2(x-1)-3(2-x)-5(x-7), & 1\le x<2 \\ 2(x-1)+3(2-x)-5(x-7), & 2\le x<7 \\ 2(x-1)+3(2-x)+5(x-7), & x\ge 7 \end{cases}$$

$$\text{Hence } f(x) = \begin{cases} -4x+31, & x<1 \\ 27, & 1\le x<2 \\ -6x+39, & 2\le x<7 \\ 4x-31, & x\ge 7 \end{cases}.$$

EXAMPLE 1.5 Simplify the function $y = |x-1| + |x-3|$ and then draw its graph.

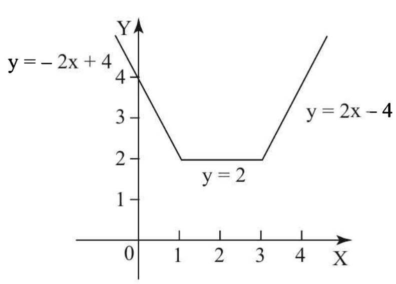
SOLUTION We use method of intervals. The points of change in de finition are obtained by equating to zero, each term within modulus sign.

$$\left.\begin{matrix} x-1=0 \\ x-3=0 \end{matrix}\right\} \Rightarrow x = 1, 3$$

$$\underset{1}{x<1}\ \Big|\ 1\le x<3\ \Big|\ \underset{3}{x\ge 3}$$

$$y = \begin{cases} -(x-1)-(x-3), & x<1 \\ (x-1)-(x-3), & 1\le x<3 \\ (x-1)+(x-3), & x\ge 3 \end{cases}$$

$$\Rightarrow \quad y = \begin{cases} -2x+4, & x<1 \\ 2, & 1\le x<3 \\ 2x-4, & x\ge 3 \end{cases}$$



EXAMPLE 1.6 If $f(x) = \begin{cases} -2, & -2\le x\le 0 \\ x-2, & 0<x\le 2 \end{cases}$, determine $g(x) = f(|x|) + |f(x)|$.

SOLUTION We have

$$f(|x|) = \begin{cases} -2, & -2\le |x| \le 0 \\ |x|-2, & 0<|x|\le 2 \end{cases}$$

$$\Rightarrow \quad f(|x|) = \begin{cases} -2, & x=0 \\ |x|-2, & 0<|x|\le 2 \end{cases}$$

[∵ |x| cannot be –ve]

$$\Rightarrow \quad f(|x|) = \begin{cases} -x-2, & -2\le x<0 \\ -2, & x=0 \\ x-2, & x<x\le 2 \end{cases} \quad ...(1)$$

$$\text{and} \quad |f(x)| = \begin{cases} |-2|, & -2\le x\le 0 \\ |x-2|, & 0<x\le 2 \end{cases}$$

$$\Rightarrow \quad |f(x)| = \begin{cases} 2, & -2\le x\le 0 \\ 2-x, & 0<x\le 2 \end{cases} \quad ...(2)$$

Hence, we have

$$g(x) = f(|x|) + |f(x)|$$

$$= \begin{cases} (-x-2)+2, & -2\le x<0 \\ 2+(-2), & x=0 \\ (x-2)+(2-x), & 0<x\le 2 \end{cases}$$

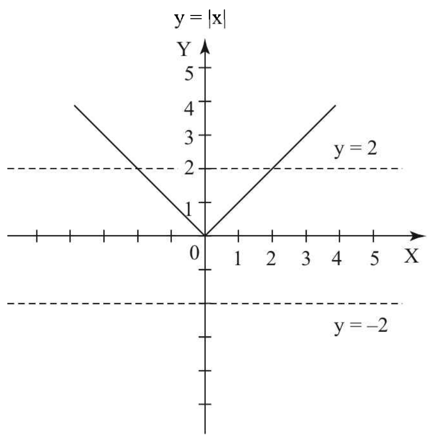
$$\Rightarrow \quad g(x) = \begin{cases} -x, & -2\le x<0 \\ 0, & 0\le x\le 2 \end{cases}$$

Modulus Equation

For understanding modulus equations, we consider a non-negative number 2 equated to modulus of independent variable x like : $|x| = 2$.

Then, the values of x satisfying this equation is : $x = \pm 2$.

It is intuitive to note that values of x satisfying the above equation is actually the intersection of graphs of modulus function $y = |x|$ and $y = 2$ as shown in the figure.



Further, it is easy to realise that equating a modulus function to a negative number is meaningless. Observe in the figure above that line $y = -2$ does not intersect modulus plot at all.

We express these results in general form, using an expression f(x) in place of x as :

$|f(x)| = a;\quad a > 0 \quad \Rightarrow \quad f(x) = \pm a$

$|f(x)| = a;\quad a = 0 \quad \Rightarrow \quad f(x) = 0$

$|f(x)| = a;\quad a < 0 \quad \Rightarrow$ There is no solution of this equality.

EXAMPLE 1.7 Solve $|x - 3| = 2$

SOLUTION Since right hand side is positive,

$$x - 3 = \pm 2$$
$$x = 3 \pm 2$$
$$x = 5, 1$$

Note that the equation $|x - 3| = -4$ has no solution.

EXAMPLE 1.8 Solve $|3x - 1| = |x + 5|$

SOLUTION $|3x - 1| = |x + 5|$

$3x - 1 = x + 5 \quad$ or $\quad 3x - 1 = -(x + 5)$

$3x - 1 = x + 5 \quad$ or $\quad 3x - 1 = -x - 5$

$2x - 1 = 5 \qquad\qquad 4x - 1 = -5$

$2x = 6 \qquad\qquad 4x = -4$

$x = 3 \qquad\qquad x = -1$

The solution set is $\{-1, 3\}$.

EXAMPLE 1.9 Solve $|x - 3| = 2x + 1$

SOLUTION

Let $x < 3$, $-(x - 3) = 2x + 1$

$\Rightarrow \qquad x = 2/3$

We check that $x = 2/3$ satisfies $x < 3$.

Now, let $x \geq 3$

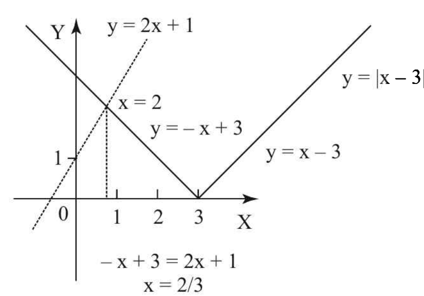
$$x - 3 = 2x + 1$$

$x = -4$. This does not satisfy $x \geq 3$.

Hence, the solution is $x = 2/3$.

Graphical Method

We draw $y = |x - 3|$ and $y = 2x + 1$ and find the points of intersection.



From the figure, we see that there is only one solution, which can be found by equating the respective functions.

EXAMPLE 1.10 Solve the equation $|x^2 - x - 6| = x + 2$

SOLUTION

Case 1 : Let $x^2 - x - 6 < 0 \Rightarrow -2 < x < 3$

We have the equation $-x^2 + x + 6 = x + 2$ with roots

$$x_1 = 2,\ x_2 = -2.$$

Now check to see whether x_1 and x_2 satisfy the above condition. The first is valid, the second is not; and so only 2 is a root of the original equation.

Case 2 : Let $x^2 - x - 6 \geq 0$

$\Rightarrow$ $x \leq -2$ or $x \geq 3$

We have the equation $x^2 - x - 6 = x + 2$ whose roots are $x_1 = 4$, $x_2 = -2$.

Since both of these values of x satisfy the above condition, both 4 and – 2 are roots of the original equation. Thus the original equation has three roots –2, 2 , 4.

Study Tip

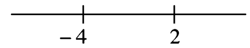
Let us examine this solution more carefully; first we rejected the value $x = -2$ and then found it again, so that in the end this value was a root of the original equation. How is this to be explained? The point is that in the first case we rejected $x = -2$ but we did not assert that is not a root of the original equation. The only thing we maintained was that this value is discarded due to the restrictions imposed on x by the condition of case 1. Quite naturally, there is nothing to stop this value from satisfying the condition of another case and thus to become a root of the original equation.

EXAMPLE 1.11 Solve the equation

$$2\,|\,x - 2\,| - 3\,|\,x + 4\,| = 1$$

SOLUTION We first equate each quantity in the modulus signs to zero and find the roots. These roots are used to partition the number line into several parts so that it becomes convenient to decide the sign of each quantity in the modulus sign.

Here the number line should be partitioned at – 4 and 2.



Let $-\infty < x < -4$

The given equation becomes

$$2(2 - x) - (3)\{-(x + 4)\} = 1$$

$\Rightarrow$ $x = -15$ and this agrees with the initial condition

$$-\infty < x < -4$$

$\therefore$ $x = -15$ is a solution.

Let $-4 \leq x < 2$

The given equation becomes

$2(2 - x) - 3(x + 4) = 1$

$\Rightarrow x = \dfrac{9}{5}$ which agrees with the initial condition.

Let $2 \leq x < \infty$

The given equation becomes

$$2(x - 2) - 3(x + 4) = 1$$

$\Rightarrow x = -17$ and this does not belong to the interval $[2, \infty)$.

Hence, $x = -17$ is not acceptable.

Thus, the solution set is $\left\{-15, -\dfrac{9}{5}\right\}$

EXAMPLE 1.12 Solve the equation $|x^2 - 9| + |x^2 - 4| = 5$

SOLUTION We consider three cases:

(a) $x^2 < 4$, (b) $4 \leq x^2 \leq 9$

(c) $9 < x^2$

In the first case, $|x^2 - 9| = 9 - x^2$,

$|x^2 - 4| = 4 - x^2$ or

$9 - x^2 + 4 - x^2 = 5$, $x^2 = 4$,

$x = \pm\, 2$ are unsuitable and the given equation has no roots in this case.

In the second case, $|x^2 - 9| = 9 - x^2$, $|x^2 - 4| = 5$ or $5 = 5$.

At this point some students think the equation has "disappeared". Actually, the original equation is equivalent to the identity $5 = 5$ when $4 \le x^2 \le 9$, which is to say it is satisfied for all values of x.

This means that any value of x which satisfied the condition $4 \le x^2 \le 9$ is a solution of the equation. It now remains to solve this double inequality. We then get $-3 \le x \le -2$, $2 \le x \le 3$.

In the third case, $x^2 - 9 + x^2 - 4 = 5$, $x^2 = 9$,

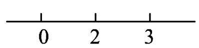
$x = \pm 3$ are unsuitable and the given equation has no roots in this case.

Finally, we have the solution $-3 \le x \le -2$, $2 \le x \le 3$.

EXAMPLE 1.13 Solve $|x^2 - 2x| + |x - 3| = 4$.

SOLUTION

$$\left.\begin{array}{l} x^2 - 2x = 0 \\ x - 3 = 0 \end{array}\right\} \Rightarrow x = 0, 2, 3$$



Case 1: $x < 0$

$$\Rightarrow \quad (x^2 - 2x) - (x - 3) = 4$$

$$\Rightarrow \quad x^2 - 3x - 1 = 0$$

$$\Rightarrow \quad x = \frac{3 \pm \sqrt{13}}{2}$$

Under the condition $x < 0$, we have $x = \dfrac{3 - \sqrt{13}}{2}$.

Case 2: $0 \le x < 2$

$$-(x^2 - 2x) - (x - 3) = 4$$

$$x^2 - x + 1 = 0$$

No solution since $D < 0$.

Case 3: $2 \le x < 3$

$$(x^2 - 2x) - (x - 3) = 4$$

$$x^2 - x - 1 = 0$$

This yield no solution in $2 \le x < 3$.

Case 4: $x \ge 3$

$$x^2 - 2x + x - 3 = 4$$

$$x^2 - x - 7 = 0$$

$$x = \frac{1 \pm \sqrt{29}}{2} \Rightarrow x = \frac{1 \pm \sqrt{29}}{2}$$

Under the condition, we have $x = \dfrac{1 + \sqrt{29}}{2}$.

Finally, the solutions are $x = \dfrac{3 - \sqrt{13}}{2}, \dfrac{1 + \sqrt{29}}{2}$.

Note

The equation of the form $h(|f(x)|) = g(x)$

(when h is the function of $|f(x)|$, is equivalent to the collection of systems

$$\begin{cases} h(f(x)) = g(x), \text{ if } f(x) \ge 0 \\ h(-f(x)) = g(x), \text{ if } f(x) < 0 \end{cases}$$

EXAMPLE 1.14 Solve the equation

$$|x - |4 - x|| - 2x = 4$$

SOLUTION This equation is equivalent to the collection of systems

$$\begin{cases} |x - (4 - x)| - 2x = 4, \text{ if } 4 - x \ge 0 \\ |x + (4 - x)| - 2x = 4, \text{ if } 4 - x < 0 \end{cases}$$

$$\Rightarrow \begin{cases} |2x - 4| - 2x = 4, \text{ if } x \le 0 \\ \quad 4 - 2x = 4, \text{ if } x > 4 \end{cases}$$

The second system of this collection gives

$x = 4$ but $x > 4$.

Hence the second system has no solution.

The first system of collection (1) is equivalent to the system

$$\begin{cases} 2x - 4 - 2x = 4, \text{ if } 2x \ge 4 \\ -2x + 4 - 2x = 4, \text{ if } 2x < 4 \end{cases}$$

$$\Rightarrow \begin{cases} -4 = 4, \text{ if } x \ge 2 \\ -4x = 0, \text{ if } x < 2 \end{cases}$$

The first equation has no solution

and second one gives $x = 0$

Hence x = 0 is unique solution of the given equation.

Study Tip

Sometimes $ax^2 + bx + c$ in $|ax^2 + bx + c|$ cannot be factorised in real factors. It happens when $ax^2 + bx + c = 0$ has imaginary roots, i.e., D < 0. In this case, if a > 0, $ax^2 + bx + c$ will be always positive and so $|ax^2 + bx + c|$ can be replaced by $ax^2 + bx + c$, If a < 0, $ax^2 + bx + c$ will be always negative and so $|ax^2 + bx + c|$ will be replaced by $-(ax^2 + bx + c)$.

EXAMPLE 1.15 Find the value of x satisfying the equation

$$\left|\left|\left|x^2 - x + 4\right| - 2\right| - 3\right| = x^2 + x - 12.$$

SOLUTION $\left|\left|\left|x^2 - x + 4\right| - 2\right| - 3\right| = x^2 + x - 12$

$x^2 - x + 4 > 0$ for all x as D < 0

$\Rightarrow \quad |x^2 - x + 4| = x^2 - x + 4$

$\left|\left|x^2 - x + 2\right| - 3\right| = x^2 + x - 12$

$x^2 - x + 2 > 0$ for all x as D < 0

$\Rightarrow \quad |x^2 - x + 2| = x^2 - x + 2$

$\left|x^2 - x - 1\right| = x^2 + x - 12$

For solution, $x^2 + x - 12 \geq 0$

$\Rightarrow \quad x \leq -4, x \geq 3.$

For such values of x, $x^2 - x - 1 > 0$

Hence, $x^2 - x - 1 = x^2 + x - 12$

$\Rightarrow \quad 2x = 11 \Rightarrow x = 11/2.$

EXAMPLE 1.16 Solve for x,

$$|x-4|^{\frac{(x^2-10x+24)}{x-3}} = 1$$

SOLUTION Taking log on both sides, the given equation reduces to

$$\frac{(x-4)(x-6)}{(x-3)} \ln|x-4| = 0 \qquad ...(1)$$

Solving equation (1), we have

either $\dfrac{(x-4)(x-6)}{(x-3)} = 0$ or $|x-4| = 1$

$\Rightarrow x = 4, 6 \quad \Rightarrow x = 3, 5$

But x = 4 is not acceptable, since ln |x – 4| is not defined for x = 4,

and x = 3 is not acceptable, since $\dfrac{1}{x-3}$ is not defined for x = 3. Thus the required solutions are x = 5, 6.

Modulus Inequalities

Let x be a variable or an algebraic expression and let a be a positive real number i.e., a > 0.

1. The solutions of |x| < a are all values of x that lie between –a and a. That is,

   |x| < a if and only if –a < x < a.

2. The solution of |x| > a are all values of x that are less than –a or greater than a. That is,

   |x| > a if and only if x < –a or x < a.

   In short, assuming a > 0

   $|x| < a \Rightarrow \quad -a < x < a$

   $|x| \leq a \Rightarrow \quad -a \leq x \leq a$

   $|x| > a \Rightarrow x < -a$ and $x > a$

and $|x| \geq a \Rightarrow x \leq -a$ and $x \geq a$

Some simple inequations:

$|x| < 2 \Rightarrow -2 < x < 2$

$|x| > 2 \Rightarrow x < -2$ or $x > 2$

$|x| < -2 \Rightarrow \phi$

$|x| < 0 \Rightarrow \phi$

$|x| \leq 0 \Rightarrow x = 0$

An important aspect of these inequalities is that they can be used to express intervals in compact form. For example, range of trigonometric function cosecant is $y \in (-\infty, -1] \cup [1, \infty\}$. Equivalently, we can write this interval as $|y| \geq 1$.

EXAMPLE 1.17 Solve $|x - 5| < 2$

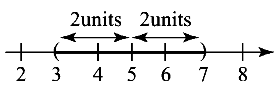
SOLUTION $|x - 5| < 2$

$\Rightarrow -2 < x - 5 < 2$

$\Rightarrow -2 + 5 < x - 5 + 5 < 2 + 5$

$\Rightarrow 3 < x < 7$

The solution set consists of all real numbers that are greater than 3 and less than 7. The interval notation, for this solution set is (3, 7). The graph of this solution set is shown below:



EXAMPLE 1.18 Solve $|x + 3| \geq 7$

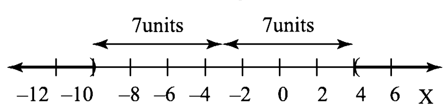
SOLUTION $|x + 3| \geq 7$

$x + 3 \leq -7$ or $x + 3 \geq 7$

$x + 3 - 3 \leq -7 - 3$ $x + 3 - 3 \geq 7 - 3$

$x \leq -10$ $x \geq 4$

The solution set consists of all real numbers that are less than or equal to –10 or greater than or equal to 4. The interval notation for this solution set is $(-\infty, -10] \cup [4, \infty)$. The graphs of this solution set is shown in this figure.



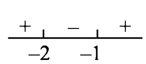
EXAMPLE 1.19 Solve $|x^2 + x| < 2$

SOLUTION $|x^2 + x| < 2$

$\Rightarrow -2x < x^2 + x < 2$

$x^2 + x + 2 > 0$ and $x^2 + x - 2 < 0$

$x \in R$ ...(1)



$x \in (-2, 1)$ ...(2)

Taking intersection of (1) and (2), we get $x \in (-2, 1)$.

EXAMPLE 1.20 Solve $\left|\dfrac{x-1}{x+2}\right| \geq 2$

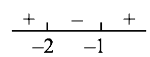
SOLUTION We have

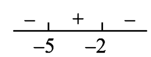
$$\frac{x-1}{x+2} \leq -2 \quad \text{or} \quad \frac{x-1}{x+2} \geq 2$$

$$\frac{x-1}{x+2} + 2 \leq 0 \qquad \frac{x-1}{x+2} - 2 \geq 0$$

$$\frac{x-1+2x+4}{x+4} \leq 0 \qquad \frac{x-1-2x-4}{x+2} \geq 0$$

$$\frac{3x+3}{x+2} \leq 0 \qquad \frac{-(x+5)}{x+2} \geq 0$$





$x \in (-2, 1]$ $x \in [-5, -2)$

Taking union, we get $x \in [-5, -2) \cup (-2, -1]$

i.e. $x \in [-5, -1] - \{-2\}$.

Forms of the Inequations Containing Absolute Values:

1. The inequation of the form $f(|x|) < g(x)$ is equivalent to the collection of systems
$$\begin{cases} f(x) < g(x), \text{ if } x \geq 0 \\ f(-x) < g(x), \text{ if } x < 0 \end{cases}$$

2. The inequation of the form $|f(x)| < g(x)$ is equivalent to $\begin{cases} -g(x) < f(x) < g(x) \\ g(x) > 0 \end{cases}$
In particular $|f(x)| < a$ has no solution for $a \leq 0$ and for $a > 0$ it is equivalent to $-a < f(x) < a$

3. The inequation of the form $|f(x)| > g(x)$ is equivalent to the
$$\begin{cases} f(x) > g(x), \text{ if } g(x) < 0 \\ -f(x) > g(x), \text{ if } g(x) < 0 \end{cases}$$
$$\Rightarrow \begin{cases} f(x) < -g(x) \text{ or } f(x) > g(x), \text{ if } g(x) > 0 \\ \text{always true, if } g(x) < 0 \end{cases}$$

In particular $|f(x)| > a$ has no solution for $a \le 0$ and for $a > 0$, is equivalent to $f(x) < -a$, $f(x) > a$

4. The inequation of the form $|f(x)| \ge |g(x)|$ is equivalent to the system $f^2(x) \ge g^2(x)$

$|f(x)| \ge |g(x)|$ (1)

Squaring both sides,

$f^2(x) \ge g^2(x)$

$f^2(x) - g^2(x) \ge 0$

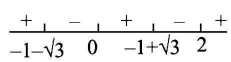
$(f(x) + g(x))\ (f(x) - g(x)) \ge 0$ (2)

Inequality (1) is equivalent to (2).

EXAMPLE 1.21 Solve $|x^2 - 1| \ge |2x - 1|$

SOLUTION $(x^2 - 1)^2 \ge (2x - 2)^2 \ge 0$

$(x^2 + 2x - 2)\ (x^2 - 2x) \ge 0$



$x \in (-\infty, -1-\sqrt{3}) \cup [0, -1+\sqrt{3}] \cup [2, \infty)$

EXAMPLE 1.22 Find the solution set of the inequality $\left|\dfrac{3|x|-2}{|x|-1}\right| \ge 2$.

SOLUTION Let $|x| = y$ (note $|x| \ne 1$)

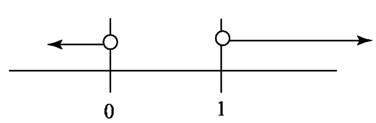
$$\left|\frac{3y-2}{y-1}\right| \ge 2$$

Hence, $\dfrac{3y-2}{y-1} \ge 2$ ...(1)

or $\dfrac{3y-2}{y-1} \ge -2$ ...(2)

From (1), $\dfrac{3y-2}{y-1} - 2 \ge 0$

$$\frac{y}{y-1} \ge 0$$



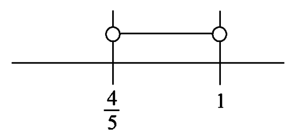
$y \in (-\infty, 0] \cup (1, \infty)$

But $|x| \ge 0$, hence $|x| \in \{0\} \cup (1, \infty)$

$\Rightarrow$ $x \in \{0\} \cup (-\infty, -1) \cup (1, \infty)$ (3)

From (2), $\dfrac{3y-2}{y-1} + 2 \le 0 \Rightarrow \dfrac{5y-4}{y-1} \le 0$

$\therefore$ $y \in \left[\dfrac{4}{5}, 1\right) \Rightarrow |x| \in \left[\dfrac{4}{5}, 1\right)$



$\therefore$ $x \in \left[-1, -\dfrac{4}{5}\right) \cup \left[\dfrac{4}{5}, 1\right)$ ...(4)

For (3) and (4),

$x \in (-\infty, -1) \cup \left(-1, -\dfrac{4}{5}\right) \cup \{0\} \cup \left(\dfrac{4}{5}, 1\right) \cup (1, \infty)$.

EXAMPLE 1.23 Solve the inequality $x^2 - |5x + 6| > 0$.

SOLUTION Let us divide the entire number line into two subintervals

$(-\infty, -6/5)$ and $[-6/5, \infty)$

on each of which the inequality can be written without the modulus sign.

The equality $|5x + 6| = -5x - 6$ is true for the subinterval $(-\infty, -6/5)$ and consequently, the inequality assumes the form

$x^2 + 5x + 6 > 0 \Rightarrow (x + 3)\ (x + 2) > 0$

$\Rightarrow x \in (-\infty, -3) \cup (-2, \infty)$.

Taking intersection with the subinterval we get

$(-\infty, -3) \cup (-2, -6/5)$.

On the second subinterval, we have $|5x + 6| = 5x + 6$

$[-6/5, \infty)$,

and consequently, we can write the inequality as

$$x^2 - 5x - 6 > 0 \Rightarrow (x + 1)\ x - 6) > 0$$

$\Rightarrow\ x \in (-\infty, -1) \cup (6, \infty)$.

Taking into account that the variable belongs to the subinterval $[-6/5, \infty)$, we get the set of solutions of the inequality on that subinterval

$[-6/5, -1] \cup (6, \infty)$

The final answer is $(-\infty, -3) \cup (-2, -1) \cup (6, \infty)$.

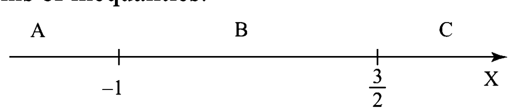
EXAMPLE 1.24 Solve the inequality

$$x + |3 - 2x| > |x + 1| - 1.$$

SOLUTION Let us write the given inequality in the form $x + 2\left|x - \frac{3}{2}\right| > |x + 1| - 1$ and mark on the number axis the points $x = -1$ and $x = \frac{3}{2}$ at which the expressions standing inside the modulus sign vanish. These points break the number axis into three intervals;

$$x < -1 \text{ (A)},\ -1 \le x < \frac{3}{2} \text{ (B)},\ x \ge \frac{3}{2} \text{ (C)}.$$

Considering x consecutively on each of these intervals, we obtain that the original inequality is equivalent to the set of the following three systems of inequalities:



$$\begin{cases} x < -1 \\ x - 2\left(x - \frac{3}{2}\right) > -(x + 1) - 1; \end{cases}$$

$$\begin{cases} -1 \le x < \frac{3}{2} \\ x - 2\left(x - \frac{3}{2}\right) > x + 1 - 1. \end{cases}$$

$$\begin{cases} x \ge \frac{3}{2} \\ x + 2\left(x - \frac{3}{2}\right) > x + 1 - 1; \end{cases}$$

Solving each of these systems, we find that the solution of the first system is the interval $x < -1$, the solution of the second system is the interval $-1 \le x < \frac{3}{2}$ and the solution of the third system is the interval $x > \frac{3}{2}$. Consequently, the given inequality is satisfied by all real numbers x, except for $x = \frac{3}{2}$.

Hence, the solution is $x \in R - \left\{\frac{3}{2}\right\}$.

EXAMPLE 1.25 Solve the inequality

$$|x^2 + 3x| + x^2 - 2 \ge 0$$

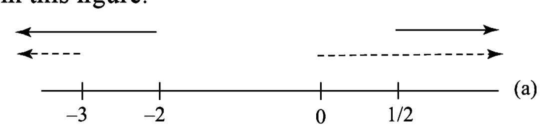
SOLUTION

Case 1: $x^2 + 3x \ge 0$

$\Rightarrow x \ge -3$ and $x \ge 0$ (1)

We get the inequality $2x^2 + 3x - 2 \ge 0$, the solutions of which are $x \le -2$ and $x \ge 1/2$.

We now have to choose from these solutions those which satisfy condition (1). This is shown in this figure.



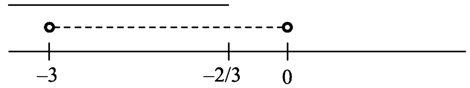
We get the solution for case 1 as $x \ge -3$ and $x \ge 1/2$.

Case 2: $x^2 + 3x < 0 \Rightarrow -3 < x < 0$ (2)

We get the inequality $-3x - 2 \le 0 \Rightarrow x \le -2/3$.

We now have to choose from these solutions

those which satisfy condition (2). This is shown in this figure.



(b)

We get the solution for case 2 as $-3 < x \le -2/3$.

Combining the solutions found in the two cases, we get the answer : $x \le -2/3$ and $x \ge 1/2$.

EXAMPLE 1.26 Solve the inequality

$$|x^2 - 3x - 3| > |x^2 + 7x - 13|.$$

SOLUTION Since both sides are non-negative we can square both sides of the inequality.

$$|x^2 - 3x - 3|^2 > |x^2 + 7x - 13|^2$$

But $|a|^2 = a^2$ so that this inequality can be re-written as

$$(x^2 - 3x - 3)^2 > (x^2 + 7x - 13)^2$$

Now, transposing all terms to the right side and using the formula for the difference of squares, we get

$$2(x^2 + 2x - 8) \,.\, 10(x - 1) < 0$$

$$\Rightarrow \quad (x + 4)(x - 2)(x - 1) < 0$$

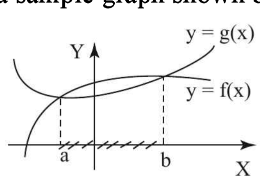
This inequality is readily solved by the method of intervals. Its solution, and consequently the solution of the original inequality is $x < -4$ and $1 < x < 2$.

Graphical Solution of Inequality

Assume that we need to solve $f(x) > g(x)$.

We draw the graphs of $y = f(x)$ and $y = g(x)$ and then find their points of intersection. The solution consists of all such values of x for which the y-coordinate of f lies above the y-coordinate of g.

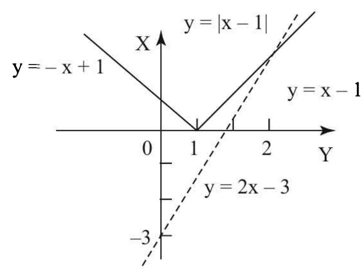
Consider a sample graph shown below:



From the figure, the solution set of $f(x) > g(x)$ is (a, b).

EXAMPLE 1.27 Solve $|x - 1| > 2x - 3$ graphically.

SOLUTION We draw $y = |x - 1|$ and $y = 2x - 3$.



We solve for point of intersection:

$$x - 1 = 2x - 3 \quad \Rightarrow x = 2$$

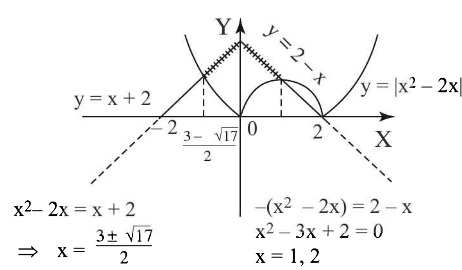
From the figure, the solution set of

$|x - 1| > 2x - 3$ is $(-\infty, 2)$.

EXAMPLE 1.28 Solve $|x^2 - 2x| + |x| \le 2$

SOLUTION $|x^2 - 2x| \le 2 - |x|$

We draw $y = |x^2 - 2x|$ and $y = 2 - |x|$



From the figure, the solution set is

$$x \in \left[\frac{3-\sqrt{17}}{2}, 1\right] \cup \{2\}.$$

PRACTICE PROBLEMS [F]

1. Express without absolute values:

   (i) $\left|\sqrt{3}-\sqrt{|2-\sqrt{5}|}\right|$

   (ii) $|1-|2-x||$ given that $x > 3$

   (iii) $\left|x-\sqrt{(x-1)^2}\right|$ given that $x < 0$

   (iv) $|x-1|-|x|+|x+1|$ given that $0 \le x \le 1$.

2. Simplify the expression

   $\sqrt{9-6a+a^2}+\sqrt{9+6a+a^2}$ if $a < -3$.

3. Find all solutions of the equation $(3|x|-3)^2 = |x| + 7$ belonging to the domain of $y = \sqrt{x(x-3)}$

4. Solve:

   (i) $(x-1)|x^2-4x+3| + 2x^2 + 3x - 5 = 0$

   (ii) $|x^3+1| + x^2 - x - 2 = 0$.

5. Solve :

   (i) $|x-|4-x|| = 2x + 4$

   (ii) $|x^2 - 3|x| + 2| = x^2 - 2x$.

6. Draw the plane region

   $\{(x, y) \in R^2 : |x-1| \le 1, |y| \ge 2\}$.

7. solve :

   (i) $|x-1| > 2$

   (ii) $\left|\frac{2x-1}{x-1}\right| > 2$

   (iii) $\left|\frac{x^2-3x-1}{x^2+x+1}\right| < 3$

   (iv) $\frac{2}{|x+2|} \le 1$.

8. Solve :

   (i) $|x+2|-|x-1| < x - 3/2$

   (ii) $(|x-1|-3)(|x+2|-5) < 0$

   (iii) $|2x+1|-|5x-2| \ge 1$.

9. Find the values of 'a' for which the equation

   $|x-1| + |x-2| + x - a = 0$ has two solutions.

10. Find the number of ordered pairs (x, y) satisfying the system of equations $|x-1| + 3y = 4$, $x - |y-1| = 2$.

11. Find the minimum value of the expression

    $|x-p| + |x-15| + |x-p-15|$

    for 'x' in the range $p \le x \le 15$ where $0 < p < 15$.

12. Find the number of integral solutions of the equations
$$\left|\frac{x^2-10x+21}{x^2-12x+32}\right| = -\left(\frac{x^2-10x+21}{x^2-12x+32}\right).$$
13. Solve:
   (i) $|2^{x+1}-1| + |2^{x+1}+1| = 2^{|x+1|}$
   (ii) $2^{|x+2|} - |2^{x+1}-1| = 2^{x+1}+1$
   (iii) $(x+4)\, 3^{1-|x-1|} - x = (x+1)\,|3^x - 1| + 3^{x-1} + 1$
14. Solve:
   (i) $|x^3 - x| \le x$
   (ii) $|x| \ge \dfrac{2x}{|x-3|}$
   (iii) $|x^2 - 2|x|| \ge x - 2$
   (iv) $|34 + 2|x - x^2| \le -1$
15. Solve for x: $(4a-15)x^2 + 2a|x| + 4 = 0$.
16. Solve:
   (i) $\dfrac{2}{|x-2|} > \left|\dfrac{-3}{2x-1}\right|$
   (ii) $\dfrac{x^2-4x+4}{x^2-6x+9} + \dfrac{|x-2|}{x-3} - 12 < 0$.
17. Consider the inequalities
   $||x-2| - x + 3| < 5$ (1)
   $|2x - |3-x| - 2| \le 4$ (2)
   Show that the solution set of inequality (1) contains the solution set of inequality (2).

1.9 | PROPERTIES OF MODULUS

Next, we list several basic properties of absolute value. Each of these properties can be derived from the definition of absolute value.

For all real numbers a and b, and natural number n,

(i) $|a| \ge 0$
(ii) $|-a| = |a|$
(iii) $-|a| \le a \le |a|$
(iv) $|ab| = |a|\,|b|$
(v) $|a_1a_2.....a_n| = |a_1|\,|a_2|.....|a_n|$
(vi) $|a^n| = |a|^n$
(vii) $|a^{2n}| = a^{2n}$
$|a^2| = |a|^2 = a^2$
(viii) $\sqrt[2n]{a^{2n}} = |a|$, $\sqrt[2n+1]{a^{2n+1}} = a$
(ix) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, $b \ne 0$
(x) $|a+b| \le |a| + |b|$
(xi) $|a+b| = |a| + |b|$ only if $ab \ge 0$
(xii) $|a-b| \ge ||a| - |b||$
(xiii) $|a-b| = ||a| - |b||$ only if $ab \ge 0$
(xiv) $|a+b+c| \le |a| + |b| + |c|$
(xv) $|a+b+c| = |a| + |b| + |c|$ only if $ab \ge 0$, $bc \ge 0$ and $ca \ge 0$

(xvi) $|a \pm b \pm c| \le |a| + |b| + |c|$

To prove the property $|a + b| \le |a| + |b|$

note that $|a + b|^2 = (a + b)^2 = a^2 + 2ab + b^2$

and $(|a| + |b|)^2 = |a|^2 + 2|a|.|b| + |b|^2 = a^2 + 2|ab| + b^2$

but $ab \le |ab|$, so that $|a + b|^2 \le (|a| + |b|)^2$

Here equality holds when $ab = |ab|$, which happens when $ab \ge 0$

EXAMPLE 1.1 Solve $|x^2 - 1| + \sqrt{x^2 - 3x + 2} = 0$.

SOLUTION We have $\underbrace{x^2 - 1}_{\ge 0} + \underbrace{\sqrt{x^2 - 3x + 2}}_{\ge 0} = 0$

Both the expressions must be zero simultaneously.

$$\left.\begin{array}{l} |x^2 - 1| = 0 \quad \Rightarrow x = \pm 1 \\ x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2 \end{array}\right\} \Rightarrow x = 1.$$

EXAMPLE 1.2 If $(x^2 - 1) f(x) = \sqrt{x^2 - 4} + |f(x)|$, then find the formula of f(x).

SOLUTION The values of x should satisfy $x^2 - 4 \ge 0$

$\Rightarrow x^2 \ge 4 \Rightarrow x^2 - 1 > 0$ ...(1)

$\Rightarrow (x^2 - 1) f(x) = \underbrace{\sqrt{x^2 - 4}}_{\ge 0} + \underbrace{|f(x)|}_{\ge 0}$.

Clearly RHS ≥ 0

$\therefore$ LHS ≥ 0

$(x^2 - 1)(f(x)) \ge 0 \Rightarrow f(x) \ge 0$ [using(1)]

$\therefore |f(x)| = f(x)$

$\Rightarrow (x^2 - 1) f(x) = \sqrt{x^2 - 4} + f(x)$

$\Rightarrow (x^2 - 2) f(x) = \sqrt{x^2 - 4}$

$\Rightarrow f(x) = \dfrac{\sqrt{x^2 - 4}}{(x^2 - 2)}$.

EXAMPLE 1.3 Prove that $-|x^3| \le x^3 \sin x \le |x|^3$

SOLUTION Since $-|x| \le x \le |x|$,

$-|x^3 \sin x| \le x^3 \sin x \le |x^3 \sin x|$

$-|x^3| \le x^3 \sin x \le |x^3|$.

EXAMPLE 1.4 Solve the equation

$$\left|\frac{x^2 - 8x + 12}{x^2 - 10x + 21}\right| = -\frac{x^2 - 8x + 12}{x^2 - 10x + 21}$$

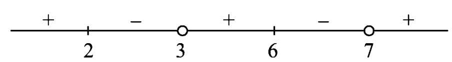
SOLUTION This equation has the form $|f(x)| = -f(x)$ where $f(x) = \dfrac{x^2 - 8x + 12}{x^2 - 10x + 21}$.

It is equivalent to $f(x) \le 0$.

$$\therefore \frac{x^2 - 8x + 12}{x^2 - 10x + 21} \le 0$$

$$\Rightarrow \frac{(x-2)(x-6)}{(x-3)(x-7)} \le 0$$

By method of intervals



$\therefore x \in [2, 3) \cup [6, 7)$.

Study Tip

The equation $|f(x) + g(x)| = |f(x)| + |g(x)|$ is equivalent to $f(x).g(x) \ge 0$

The equation $|f(x)| + |g(x)| = f(x) - g(x)$ is equivalent to $f(x) \ge 0$ and $g(x) \le 0$.

EXAMPLE 1.5 Solve the equation

$|x^2 - 1| + |2 - x^2| = 1$

SOLUTION Since $x^2 - 1 + 2 - x^2 = 1$, we can write the given equation as

$|x^2 - 1| + |2 - x^2| = |(x^2 - 1) + (2 - x^2)|$

|This is compared with $|a| + |b| = |a + b|$; which is true only when $ab \ge 0$

i.e. $(x^2 - 1)(2 - x^2) \ge 0$.

This is true when $1 \le x^2 \le 2$

$\Rightarrow x \in [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$.

EXAMPLE 1.6 Solve $|(x^2+2x+2)+(3x+7)| < |x^2+2x+2| + |3x+7|$.

SOLUTION We have $|a + b| < |a| + |b|$, if a, b are of opposite signs.

Now, $x^2 + 2x + 2 = (x + 1)^2 + 1 > 0\ \forall\ x \in R$.

Thus, the equation is satisfied for those values of x at which $3x + 7 < 0$ i.e. $x < \frac{-7}{3}$.

EXAMPLE 1.7 Solve

$$|(x^4-9)-(x^2+3)| = |x^4-9| - |x^2+3|.$$

SOLUTION We have $|a - b| = |a| - |b|$ if a, b have the same sign and $|a| \ge |b|$

i.e. $x^4 - 9$ and $x^2 + 3$ must be of same sign

i.e. $(x^4 - 9)(x^2 + 3) > 0$

i.e. $(x^2 - 3)(x^2 + 3)^2 > 0$

i.e. $x^2 - 3 > 0$ ...(1)

and $|x^4 - 9| \ge |x^2 + 3|$

i.e. $(x^2-3)(x^2+3) \ge x^2+3$ [using result (1)]

i.e. $(x^2 + 3)(x^2 - 4) \ge 0$

i.e. $x^2 \ge 4$

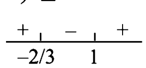
which gives $x \in (-\infty, -2] \cup [2, \infty)$.

EXAMPLE 1.8 Solve $|2x + 3| = |3x + 2| + |x - 1|$

SOLUTION We have $|x - y| \le |x| + |y|$

For equality $x(-y) \ge 0 \ge xy \le 0$

$(3x + 2)(x - 1) \le 0$



Hence, $x \in \left[-\frac{2}{3}, 1\right]$.

EXAMPLE 1.9 Solve

$$|x-1| + |4x-3x^2| = 3x^2 - 3x - 1$$

SOLUTION Here, we have $|x| + |y| = x - y$ (1)

One can guess that (1) holds when $\begin{cases} |x| = x \\ |y| = -y \end{cases}$

i.e., $x \ge 0, y \le 0$.

To establish this we square both sides of (1):

$$x^2 + y^2 + 2|x|\,|y| = x^2 + y^2 - 2xy$$

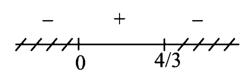
$$|x|\,|y| = -xy \Rightarrow xy \le 0$$

Also, the right hand side of (1) must be non-negative.

Hence, $x \ge 0,\ y \le 0$.

$\therefore$ $x - 1 \ge 0,\ 4x - 3x^2 \le 0$

$x \ge 1,$



Taking intersection, we get $x \in [4/3, \infty)$.

EXAMPLE 1.10 Solve $|x^2 - 2x| - |x^2 - 1| = 2x^2 - 2x - 1$.

SOLUTION Here, we have $|x| - |y| = x + y$.

Squaring both sides,

$$x^2 + y^2 - 2|x|\,|y| = x^2 + y^2 + 2xy$$

$\Rightarrow |x|\,|y| = -xy$

$\Rightarrow xy \le 0$

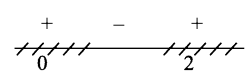
$\Rightarrow$ (i) $x \ge 0, y \le 0$

(ii) $x \le 0,\ y \ge 0$

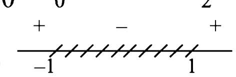
$-x - y = x + y$

$\Rightarrow y = -x$

(i) $x^2 - 2x \ge 0$

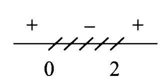


$x^2 - 1 \le 0$



$x \in [-1, 0]$

(ii) $x^2 - 2x \le 0$



$x^2 - 1 = -(x^2 - 2x)$

$2x^2 - 2x - 1 = 0$

$\Rightarrow x = \frac{1+\sqrt{3}}{2}$

Finally, $x \in [-1, 0] \cup \left\{\frac{1+\sqrt{3}}{2}\right\}$.

EXAMPLE 1.11 Solve $|f(x) + 6 - x^2| = |f(x)| + |4 - x^2| + 2$, given that $f(x) \geq 0$.

SOLUTION We have,

$$|x + y + z| \leq |x| + |y + z| \leq |x| + |y| + |z|$$

Now, equality $|x + y + z| = |x| + |y| + |z|$

holds when $\begin{cases} x(y+z) \geq 0 \\ yz \geq 0 \end{cases}$ or

$$xy \geq 0,\ yz \geq 0,\ zx \geq 0$$

$f(x) \cdot (4 - x^2) \geq 0 \Rightarrow 4 - x^2 \geq 0 \Rightarrow -2 \leq x \leq 2$

$(4 - x^2) \cdot 2 \geq 0 \Rightarrow -2 \leq x \leq 2$

$f(x) \cdot 2 \geq 0$, which is given to be true.

Hence, the solution is $-2 \leq x \leq 2$.

PRACTICE PROBLEMS [G]

1. Prove each of the following properties of modulus:
   (a) $|x| = 0$ if and only if $x = 0$ (b) $|x - y| = |y - x|$.
   (c) $|x - y| \leq |x| + |y|$. (d) $|x| - |y| \leq |x - y|$.
2. Prove that:
   (i) $|x^3| = |x|^3$ (ii) $|x^{1/2}| = |x|^{1/2}$ if $x \geq 0$
   (iii) $|x^{1/3}| = |x|^{1/3}$, $x \in R$ (iv) $\sqrt{x^2(x^2+1)} = |x|\sqrt{x^2+1}$
   (v) $x\sqrt{x^2+1} = \begin{cases} \sqrt{x^2(x^2+1)}\ ; x \geq 0 \\ -\sqrt{x^2(x^2+1)}\ ; x < 0 \end{cases}$
3. If $|a - b| < l$ and $|b - c| < m$ prove that $|a - c| < l + m$.
4. Prove that $|x + y + z| \geq |x| - |y| - |z|$, $x, y, z \in R$
5. Prove that $|p - s| \leq |p - q| + |q - r| + |r - s|$.
6. Show that the following inequalities hold for all $x \in R$:
   (i) $|x - 1| + 2|x| \geq |3x - 1|$ (ii) $|2x + 1| - |x - 1| \leq |x + 2|$
7. Prove that if the numbers x, y are of one sign, then
   $$\left|\frac{x+y}{2} - \sqrt{xy}\right| + \left|\frac{x+y}{2} + \sqrt{xy}\right| = |x| + |y|.$$
8. Solve $|x^2 - x| + 2|x - 1| = |x^2 + x - 2|$.
9. Solve $|x^2 + x - 1| = \left||x^2 + 2x| - |x + 1|\right|$.
10. Solve $|x + 2| + |x^2 - 5x + 1| \leq |x^2 - 4x + 3|$.
11. Solve:
    (i) $\left|\frac{x}{x-1}\right| + |x| = \frac{x^2}{|x-1|}$ (ii) $\left|\frac{1-y^2}{y}\right| + |y| = \left|\frac{1}{y}\right|$
    (iii) $|2^x - 1| + |4 - 2^x| \leq 3$.

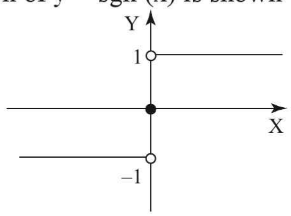
12. Solve:
    (i) $|x^2 - 5x + 7| + |x^2 - 5x - 14| = 21$ (ii) $|x^2 + 8x + 7| = |x^2 + 4x + 4| + |4x + 3|$.
13. Prove that the smallest possible value of $|x - 1| + |y - x| + |y - 2010|$ for any real number x and y is 2009.

1.10 | SIGNUM FUNCTION

The signum function, denoted as y = sgn (x), is defined as follows:

$$y = \text{sgn}(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

The graph of y = sgn (x) is shown below:

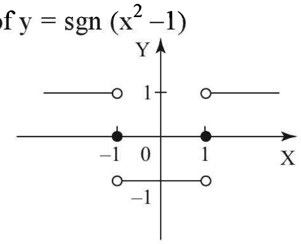


EXAMPLE 1.1 Simply $y = \text{sgn}(x^2-1)$ and draw its graph.

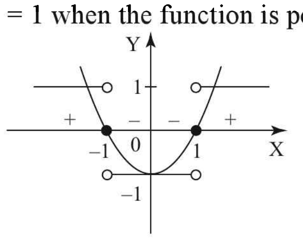
SOLUTION
$$y = \begin{cases} -1, & \text{if } x^2 - 1 < 0 \\ 0, & \text{if } x^2 - 1 = 0 \\ 1 & \text{if } x^2 - 1 > 0 \end{cases}$$

$$= \begin{cases} -1, & -1 < x < 1 \\ 0, & x = \pm 1 \\ 1, & x < -1 \text{ or } x > 1 \end{cases}$$

Graph of $y = \text{sgn}(x^2 - 1)$

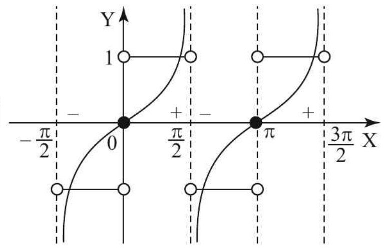


Alternatively, we first draw the curve $y = x^2 - 1$, and draw y = 1 when the function is positive; draw y = –1 when the function is negative; and draw y = 0 when the function is zero.



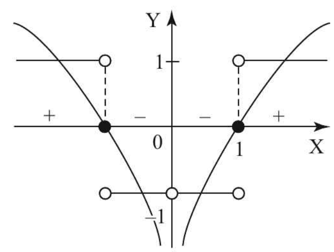
EXAMPLE 1.2 Draw y = sgn (tan x).

SOLUTION



EXAMPLE 1.3 Draw y = sgn (*ln* |x|).

SOLUTION



EXAMPLE 1.4 Solve the equation
$x^2 - 5x \,\text{sgn}(x^2 - 4) + 6 = 0$.

SOLUTION We consider the sign scheme of $x^2 - 4$

$+ \quad - \quad +$
$-2 \quad 2$

Let $x < -2$, $x > 2$. Then
$x^2 - 5x \cdot 1 + 6 = 0 \quad x = 2, 3$
$\Rightarrow \quad x = 3$

Let $-2 < x < 2$. Then
$x^2 - 5x \cdot (-1) + 6 = 0$
$x = -2, -3$ No solution

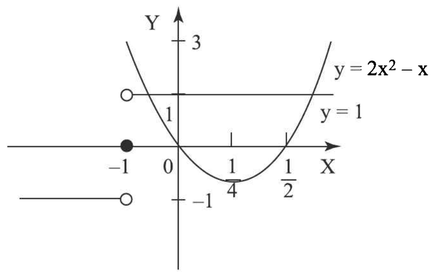
Let $x = \pm 2$. Then

$$x^2 - 0 + 6 = 0 \quad \text{No solution}$$

Finally, the solution is $x = 3$.

EXAMPLE 1.5 Solve $\text{sgn}(x + 1) = 2x^2 - x$

SOLUTION We draw $y = \text{sgn}(x + 1)$ and $y = 2x^2 - x$.



From the graph, we see that there are two solutions, which are obtained by solving $2x^2 - x = 1$.

$$\Rightarrow \quad x = -1/2,\ 1.$$

EXAMPLE 1.6 Solve $\text{sgn}\left(\dfrac{x-2}{x+1}\right) \le \dfrac{x-1}{2}$.

SOLUTION We consider the sign scheme of $\dfrac{x-2}{x+1}$.

$$\underset{}{+}\ \ \underset{-1}{|}\ \ -\ \ \underset{2}{|}\ \ +$$

Let $x < -1$ and $x > 2$.

The given inequality becomes

$$2 \le x - 1$$

$$\Rightarrow \quad x \ge 3$$

$$\Rightarrow \quad x \in [3, \infty)$$

Let $\quad -1 < x < 2$. Then

$$-2 \le x - 1$$

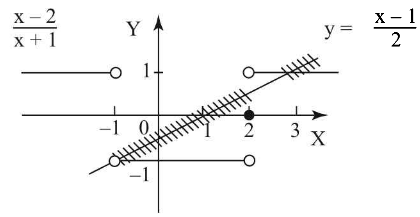
$$\Rightarrow \quad x \ge -1$$

$$\Rightarrow \quad x \in (-1, 2)$$

Now $x = 2$ also satisfies the inequality.

Hence $x \in (-1, 2] \cup [3, \infty)$

Graphical Method



From the figure, we can see that $x \in (-1, 2] \cup [3, \infty)$.

Max-min Functions

Consider the function represented as

$$y = \max.\ \{f(x),\ g(x)\}$$

Let $\quad h(x) = \max.\ \{x^2,\ 2 - x\}$

Here, $h(3) = \text{maximum of } \{9, 1\} = 9$

$h(1) = \text{maximum of } \{1, 1\} = 1$.

The meaning is:

$$y = \begin{cases} f(x) & , f(x) \ge g(x) \\ g(x) & , f(x) < g(x) \end{cases}$$

This can also be represented as

$$y = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}.$$

***Proof*:**

If $f(x) > g(x)$,

then $\quad |f(x) - g(x)| = f(x) - g(x) \quad$ ...(1)

$$\therefore \quad \max\ \{f(x),\ g(x)\} = f(x)$$

$$= \frac{f(x)+g(x)}{2} + \frac{f(x)-g(x)}{2}$$

$$= \frac{f(x)+g(x)}{2} + \left|\frac{f(x)-g(x)}{2}\right|$$

{from (1) }

and if $f(x) < g(x)$,

then $|f(x) - g(x)| = -(f(x) - g(x))$ ...(2)

$$\therefore \max\ \{f(x),\ g(x)\} = g(x)$$

$$= \frac{f(x)+g(x)}{2} + \frac{g(x)-f(x)}{2}$$

$$= \frac{f(x)+g(x)}{2} + \left|\frac{f(x)-g(x)}{2}\right|$$

{from (2)}

Similarly, y = min. {f(x), g(x)} is conceived as

$$y = \begin{cases} g(x)\ ,\ f(x) \geq g(x) \\ f(x)\ ,\ f(x) < g(x) \end{cases}$$

or, $$y = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$$

Proof:

If f(x) > g(x), then | f(x) – g(x) | = f(x) – g(x) (3)

∴ min {f(x), g(x)} = g(x)

$$= \frac{f(x)+g(x)}{2} - \left(\frac{f(x)-g(x)}{2}\right)$$

$$= \frac{f(x)+g(x)}{2} - \left|\frac{f(x)-g(x)}{2}\right|$$

{from (3)}

and if f(x) < g(x), then

|f(x) – g(x) | = – (f(x) – g(x)) = g(x) – f(x) (4)

∴ min {f(x), g(x)} = f(x)

$$= \frac{f(x)+g(x)}{2} - \left(\frac{g(x)-f(x)}{2}\right)$$

$$= \frac{f(x)+g(x)}{2} - \left|\frac{f(x)-g(x)}{2}\right|$$

{from (4)}

Method to Draw the Graphs of Functions

y = max {f(x), g(x)} and y = min {f(x), g(x)}

First we draw the graphs of f(x) and g(x).

Then we locate their points of intersection.

Between any two consecutive points of intersection, if the graph of f(x) is above the graph of g(x) i.e., f(x) > g(x).

then max {f(x), g(x)} = f(x)

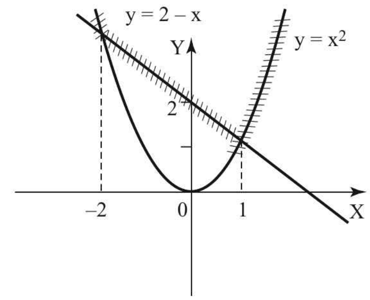
and min {f(x), g(x)} = g(x)

and if the graph of f(x) is below the graph of g(x) i.e., f(x) < g(x)

then max {f(x), g(x)} = g(x)

and min {f(x), g(x)} = f(x).

For example we can simplify the function h(x) = max. {x^2, 2 – x}using graph as follows:



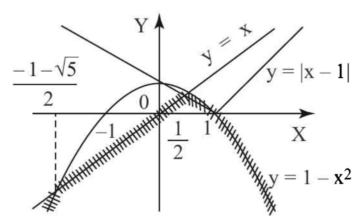
The sketched part of the graph represents y = h(x).

From the graph, we can express h as

$$h(x) = \begin{cases} x^2\ ,\ x \leq 2, x \geq 1 \\ 2-x\ ,\ -2 < x < 1 \end{cases}.$$

EXAMPLE 1.7 Draw y = min.{|x –1|, 1 – x^2, x}. Give a simplified representation of the function and solve the inequality min. {|x – 1|, 1 – x^2, x} ≥ 0.

SOLUTION



$$y = \begin{cases} 1-x^2\ x \leq \frac{-1-\sqrt{5}}{2}, x \geq 1 \\ -x, \quad \frac{-1-\sqrt{5}}{2} < x \\ |x-1|, \quad \frac{1}{2} \leq x < 1 \end{cases}$$

We see that the graph lies above the x-axis when x ∈ [0, 1]. Hence, the solution of

min. {|x – 1|, 1 – x^2 , x} ≥ 0 is x ∈ [0, 1].

EXAMPLE 1.8 If $f(x, y) = (\max(x, y))^{\min(x, y)}$ and $g(x, y) = \max(x, y) - \min(x, y)$, then find the value of $f\left(g\left(-1, -\frac{3}{2}\right), g(-4, -1.75)\right)$.

SOLUTION

$$g\left(-1, -\frac{3}{2}\right) = \max\left(-1, -\frac{3}{2}\right) - \min\left(-1, -\frac{3}{2}\right)$$

$$= -1 - \left(-\frac{3}{2}\right) = \frac{1}{2}$$

and $g(-4, -1.75)$

$$= \max(-4, -1.75) - \min(-4, -1.75)$$

$$= -1.75 - (-4) = 2.25 = \frac{9}{4}.$$

Then $f\left(\frac{1}{2}, \frac{9}{4}\right) = \left(\max\left(\frac{1}{2}, \frac{9}{4}\right)\right)^{\min\left(\frac{1}{2}, \frac{9}{4}\right)}$

$$= \left(\frac{9}{4}\right)^{\frac{1}{2}} = \frac{3}{2}.$$

EXAMPLE 1.9 Let $a, b, c \in R$. Prove that $\min.[(a-b)^2, (b-c)^2, (c-a)^2)] \le \frac{1}{2}(a^2 + b^2 + c^2)$.

SOLUTION Without loss of generality, let us assume

$a \le b \le c$.

Let $b - a = x$, $c - b = y$

Let $m = \min.(x, y)$.

Then $\frac{1}{2}(a^2 + b^2 + c^2)$

$$= \frac{1}{2}[a^2 + (a + x)^2 + (a + x + y)^2]$$

$$= \frac{1}{2}[a^2 + 2(a + x)^2 + 2(a + x)y + y^2]$$

$$= \frac{1}{2}[(a + y)^2 + 2(a + x)^2 + 2xy] \ge xy \ge m^2$$

Equality occurs when $b - a = c - b = -a$.

EXAMPLE 1.10 Find the smallest number k for which

$\min.[(a-b)^2, (b-c)^2, (a-d)^2, (b-c)^2,$
$(b-d)^2, (c-d)^2] \le k[a^2 + b^2 + c^2 + d^2]$

SOLUTION Without loss of generality, let us assume

$a \le b \le c \le d$

Let $m = \min.\{b - a, c - b, d - c\}$

$$20m^2 = 3m^2 + 2(2m)^2 + (3m)^2$$
$$\le [(b-a)^2 + (c-b)^2 + (d-c)^2] + [(c-a)^2 + (d-b)^2] + (d-a)^2$$
$$= 3(a^2 + b^2 + c^2 + d^2) - (ab + ac + ad + bc + bd + cd)$$
$$= 4(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2$$
$$\le 4(a^2 + b^2 + c^2 + d^2)$$

$$\Rightarrow m^2 \le \frac{1}{5}[a^2 + b^2 + c^2 + d^2]$$

Equality occurs if and only if $b - a = c - b = d - c$ and $a + b + c + d = 0$. $\therefore$ The smallest number k is 1/5.

EXAMPLE 1.11 Find polynomials f(x) . g(x), and h(x), if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \le x \le 0 \\ -2x + 2 & \text{if } x > 0 \end{cases}$$

SOLUTION Note that if r(x) and s(x) are any two functions. Then

$\max.(r, s) = (r + s + |r - s|)/2$.

Therefore, if F(x) is the given function, we have

$$F(x) = \max.\{-3x - 3, 0\} - \max.\{5x, 0\} + 3x + 2$$
$$= (-3x - 3 + |3x - 3|)/2 - (5x + |5x|)/2 + 3x + 2$$
$$= |(3x - 3)/2| - |5x/2| - x + \frac{1}{2}.$$

so we may set

$f(x) = (3x - 3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

1.11 | EXPONENTIAL AND LOGARITHMIC FUNCTION

Exponential and Logarithmic Functions

Logarithmic and exponential functions are closely related functions. Logarithmic functions are useful in interpreting expressions/equations, in which exponents are unknown. On the other hand, exponential functions are representation of natural process of mathematical relations, having exponential growth or decay. We shall encounter many expressions, involving these two functions in mathematics, while analysing or describing processes.

Symbolically, exponential and logarithmic functions, respectively, are:

$f(x) = a^x$ and $f(x) = \log_a x$.

Exponential Function

An exponential function associates every real number x to a function given by $f(x) = y = a^x$

where

(i) The base a is positive real number, but excluding 1, i.e., $a > 0$, $a \neq 1$.

(ii) The exponent x is a real number.

(iii) The number y represents the result of exponentiation, a^x and is a positive real number.

i.e., $y > 0$.

The exponential function is designed to be invertible. This means that f(x) and x be uniquely related.

The exponential function is considered only for $a > 0$, since for $a < 0$ and $a = 0$ the expression a^x loses sense for some values of the variable x. The base a cannot be zero, because 0^x is not uniquely defined as required for invertible function.

For instance, for $a = -2$ the expression $(-2)x$ has no sense for $x = 1/4$. Also, if a is a negative number like -2, then $(-2)x$ may evaluate to positive or negative value depending on whether x is even or odd integer. Clearly sign of function depends on the nature of integer value of x. In case x is not an integer, then sign of $(-2)^x$ cannot be interpreted for all values of x.

Further if $a = 1$, then 1^x evaluates to 1 for all values of x. Again function is not uniquely defined. In a nutshell, we conclude that base a is a positive number, but not equal to 1.

In particular, the exponential function corresponding to base e, which is nearly equal to 2.718281828, is called natural exponential function.

Once nature of a is decided, it is easy to find the nature of y. Consider simplified exponents like 10^2, 10^1, 10^{-2}, 10^{-100}. All these numbers are greater than zero. This is true even if exponent is not integer. We conclude that y is positive number. Note that neither a nor y are zero. On the other hand, x by definition is a real number.

Clearly, the domain and range of exponential function are:

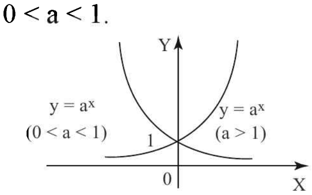
Domain: $x \in R$

Range: $y \in (0, \infty)$.

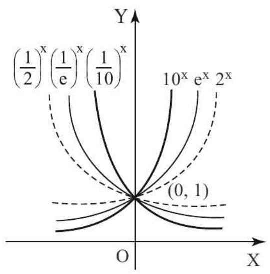
We shall see subsequently that roles of x and f(x) are exchanged for logarithmic function. Here, x is the exponent, i.e., the logarithmic value of a positive number and f(x) is the result of exponentiation, which is argument of logarithmic function. For this reason, we say that exponential and logarithmic functions are inverse to each other. As expected for inverse functions, we shall also see that domain and range of exponential and logarithmic function are exchanged.

Note

(i) The exponential function $y = a^x$ is monotonous, it increases for $a > 1$ and decreases for $0 < a < 1$.



(ii) For a > 1 we have $a^x > 1$ for x > 0 and $a^x < 1$ for x < 0; for 0 < a < 1 we have $0 < a^x < 1$ for x > 0 and $a^{xs} > 1$ for x < 0. The following figure helps in comparing exponential functions with different bases.



Law of Indices

(i) $a^0 = 1$, $(a \neq 0)$

(ii) $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$

(iii) $a^{m+n} = a^m \cdot a^n$, where m and n are rational numbers.

(iv) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$.

(v) $(a^m)^n = a^{mn}$

(vi) $a^{p/q} = \sqrt[q]{a^p}$

A **composite exponential function** is a function in which both the base and the exponent are functions of x, for instance, $(\sin x)x^2$, $x^{\tan x}$, x^x, $(\ln x)^x$.

Generally, any function of the form

$y = [u(x)]^{v(x)} \equiv u^v$ is a composite exponential function.

This function is also called an exponential-power function or a power-exponential function. In calculus, the domain consists of such values of x for which u(x) and v(x) are defined and u(x) > 0.
For example, the domain of the function

$y = (x+2)^{\frac{1}{x-3}}$ is obtained by the conditions:

$$\left.\begin{array}{l} x+2>0 \\ x \neq 3 \end{array}\right\} \Rightarrow x \in (-2, 3) \cup (3, \infty)$$

Note that the function $g(x) = \sqrt[x]{f(x)}$

has the same meaning as $h(x) = (f(x))^{\frac{1}{x}}$, but their domains are different.
g(x) is defined under the conditions
$f(x) > 0$ and $x \in \{2, 3, 4, ...\}$

while h(x) is defined under the conditions
$f(x) > 0$ and $x \neq 0$.

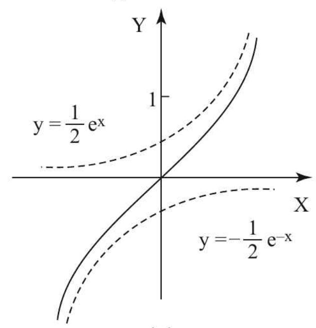
For example, to write the domain of $y = \sqrt[x]{4-x}$, we have $4 - x > 0 \Rightarrow x < 4$,
and $x \in \{2, 3......\}$.
Hence the domain is $x \in \{2, 3\}$,

while the domain of $y = (4-x)^{\frac{1}{x}}$
is $x \in (-\infty, 4) - \{0\}$.

Now, consider the following hyperbolic functions and their graphs:

Hyperbolic sine

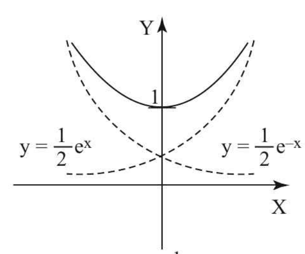
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



y = sinh x

Hyperbolic cosine

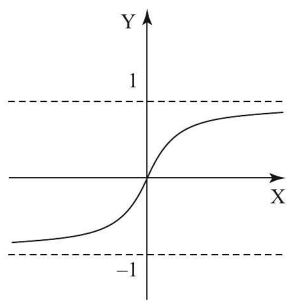
$$\cosh x = \frac{e^x + e^{-x}}{2}$$



y = cosh x

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



y = tanh x

Logarithmic functions

A logarithmic function gives exponent of an expression in terms of a base, a, and a number, x. The following two representations, in this context, are equivalent:

$$a^y = x \text{ and } f(x) = y = \log_a x$$

where

(i) The base a is positive real number, but excluding 1. Symbolically, $a > 0$, $a \neq 1$.

(ii) The number x represents result of exponentiation, a^y and is also a positive real number symbolically, $x > 0$.

(iii) The exponent y, i.e., logarithm of x is a real number.

Note that neither a nor x equals to zero.

Following the earlier discussion in the case of exponential function, we exclude a = 1, as logarithmic function is not relevant to this base.

$$1^y = 1$$

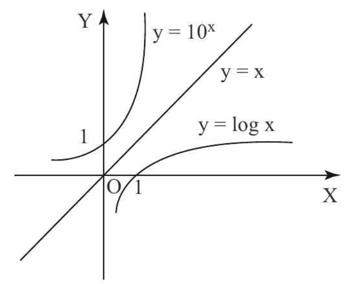
We can easily see here that whatever be the exponent, the value of logarithmic function is 1. Hence, base 1 is irrelevant as exponent y is not uniquely associated with x.

The domain and range of logarithmic function is:

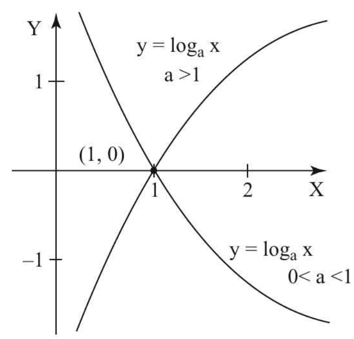
Domain : $x \in (0, \infty)$

Range : $y \in R$.

Note that domain and range of logarithmic function is exchanged with respect to domain and range or exponential function. A logarithmic function is an inverse of the exponential function. Therefore, their graphs are symmetric about the straight line y = x. Knowing the graph of an exponential function, we obtain the graph of a logarithmic function.



Graph of $y = \log_a x$



Note

Domain $(0, \infty)$

Range : $(-\infty, \infty)$

Intercept : (1, 0)

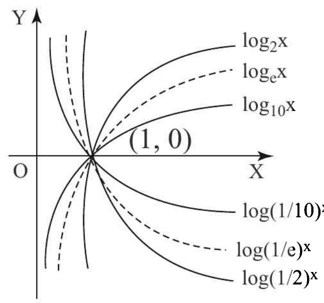
Reflection of graph of $y = a^x$ about the line $y = x$.

Increasing if $a > 1$ and decreasing if $0 < a < 1$.

At x = 0, $\log_a x$ is undefined and as x gets close to 0, the value of $\log_a x$ approaches infinity.

The base of the logarithmic function can be any positive number. However, 10 and e are two common base that we often use.

If we use e as the base, then the corresponding logarithmic function is called natural logarithmic function. The plots, here, show some logarithmic functions. Note that all graphs intersects the x–axis at the point (1,0)



Note

1. $\log_a 1 = 0$ because 0 is the power to which a must be raised to obtain 1.
2. $\log_a a = 1$ because 1 is the power to which a must be raised to obtain a.
3. $\log_a a^N = N$ because N is the power to which a must be raised to obtain a^N.
4. $a^{\log_a b} = b$ because $\log_a b$ is the power to which a must be raised to obtain b.

EXAMPLE 1.12 Find the domain of the function given by

$$f(x) = \log_x 2.$$

SOLUTION By definition of logarithmic function, we know that base of logarithmic function is a positive number excluding x = 1.

$$x > 0, x \neq 1$$

Hence, domain of the given function is:

Domain = $(0, \infty) - \{1\}$.

EXAMPLE 1.13 Find the domain of the function given by $f(x) = \log_{10} \dfrac{x^2 - 5x + 6}{x^2 + 5x + 9}$.

SOLUTION The argument (input to the function) of logarithmic function is a rational function. We need to find values of "x" such that the argument of the function evaluates to a positive number. Hence,

$$\frac{x^2 - 5x + 6}{x^2 + 5x + 9} > 0$$

We see that discriminant $D = b^2 - 4ac = -11$, of the quadratic expression in the denominator, is negative the quadratic expression in denominator is positive for all value of x as coefficient of squared term is positive.

Clearly, sign of rational function is same as that of quadratic expression in the numerator.

$\Rightarrow x^2 - 2x - 3x + 6 > 0$

$\Rightarrow (x - 2)(x - 3) > 0.$

Hence, the domain is $x \in (-\infty, 2) \cup (3, \infty)$.

1.12 | LAWS OF LOGARITHM

Let a be a positive number such that a ≠ 1, and let n be a real number. If u and v are positive real numbers, then the following laws hold good:

1. To find the logarithm of a product.

   $\log_a MN = \log_a M + \log_a N$

2. To find the logarithm of a fraction.

   $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

3. To find the logarithm of a number raised to any power

   $\log_a (M^p) = p \log_a M$

4. To find the logarithm of a number with base raised to any power

   $\log_{a^q} M = \dfrac{1}{q} \log_a M$

5. To find the logarithm to a different base b.

   (i) $\log_b M = \dfrac{\log_a M}{\log_a b}$

   (ii) $\log_b a = \dfrac{1}{\log_a b}$

6. Fundamental logarithmic identity

   $a^{\log_a x} = x$

7. $a^{\log_b x} = x^{\log_b a}$.

EXAMPLE 1.1 Given that $\log_{a^2}(a^2 + 1) = 16$, find the value of $\log_{a^{32}}\left(a + \dfrac{1}{a}\right)$.

SOLUTION Given $\log_{a^2}(a^2 + 1) = 16$

Now, $\log_{a^{32}}\left(\frac{a^2+1}{a}\right)$

$$= \frac{\log_{a^2}\left(\frac{a^2+1}{a}\right)}{\log_{a^2}(a^{32})}$$

$$= \frac{1}{16}\log_{a^2}\left(\frac{a^2+1}{a}\right)$$

$$= \frac{1}{16}\left[\log_{a^2}(a^2+1) - \log_{a^2} a\right]$$

$$= \frac{1}{16}\left[16 - \frac{1}{2}\right] = \frac{1}{16}\left[\frac{31}{2}\right] = \frac{31}{32}.$$

EXAMPLE 1.2 If $\log_{12}27 = a$ find the value of $\log_6 16$ in term of a.

SOLUTION $\frac{3\log_2 3}{2+\log_2 3} = a$

$\Rightarrow \quad 3\log_2 3 = 2a + a\log_2 3$

$\Rightarrow (3 - a)\log_2 3 = 2a$

$\Rightarrow \quad \log_2 3 = \frac{2a}{3-a}$

Now $\quad \log_6 16 = \frac{\log_2 16}{\log_2 6} = \frac{4}{1+\log_2 3}$

But $\quad 1 + \log_2 3 = 1 + \frac{2a}{3-a} = \frac{3+a}{3-a}$

$\therefore \quad \log_6 16 = \frac{4(3-a)}{3+a}.$

EXAMPLE 1.3 The non-negative integers smaller than 10^n are split into two subsets A and B. The subset A contains all those integers whose decimal expansion does not contain a 5, and the set B contains all those integers whose decimal expansion contains atleast one 5. Given n, which subset, A or B is the larger set ? One may use the fact that $\log_{10} 2 = .3010$ and the $\log_{10} 3 = .4771$.

SOLUTION The set B contains $10^n - 9^n$ elements and the set A contains 9^n elements.

Now if $10^n - 9^n > 9^m$ then $10^n > 2 \cdot 9^n$ and taking logarithms base 10 we deduce

$n > \log_{10} 2 + 2n \log_{10} 3.$

Thus,

$$n > \frac{\log_{10} 2}{1 - 2\log_{10} 3} = 6.57....$$

Therefore, if $n \le 6$, A has more elements than B and if $m > 6$, B has more elements than A.

EXAMPLE 1.4 If $\sum_{r=0}^{n-1}\log_2\left(\frac{r+2}{r+1}\right) = \prod_{r=10}^{99}\log_r(r+1)$, then find n.

SOLUTION L.H.S. $= \sum_{r=0}^{n-1}\log_2\left(\frac{r+2}{r+1}\right)$

$$= \log_2\left(\frac{2}{1}\right) + \log_2\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \ldots + \log_2\left(\frac{n}{n-1}\right) + \log_2\left(\frac{n+1}{n}\right)$$

$$= \log_2\left(\frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times\ldots\times\frac{n}{n-1}\times\frac{n+1}{n}\right)$$

$$= \log_2(n + 1) \qquad (1)$$

Now R.H.S. $= \prod_{r=10}^{99}\log_r(r+1)$

$$= \log_{10}11 \times \log_{11}12 \times \log_{12}13 \times \ldots \times \log_{98}99 \times \log_{99}100$$

$$= \frac{\log 11}{\log 10} \times \frac{\log 12}{\log 11} \times \frac{\log 13}{\log 12} \times \ldots\ldots \times \frac{\log 99}{\log 98} \times \frac{\log 100}{\log 99}$$

$$= \log_{10}100 = 2 \qquad (2)$$

$\therefore$ From (1) and (2), we get

$\log_2 (n + 1) = 2$

$\Rightarrow \quad n + 1 = 4$

$\Rightarrow \quad n = 3.$

EXAMPLE 1.5 If the equation

$$\ln\left(\frac{k^{1/k}}{(k+1)^{1/(k+1)}}\right)$$

$$= F(k)\cdot\left[\ln\left(1-\frac{1}{k+1}\right)+\frac{1}{k}\ln k\right] \text{ is true for all } k$$

wherever defined, then find $F(100)$.

SOLUTION L.H.S. $= \frac{1}{k}\ln k - \frac{1}{k+1}\ln(k+1)$

$$= \frac{(k+1)\ln k - k\ln(k+1)}{k(k+1)}$$

$$\text{R.H.S.} = \frac{F(k)}{k}[k\ln k - k\ln(k+1) + \ln k]$$

$$= \frac{F(k)}{k}[(k+1)\ln k - k\ln(k+1)]$$

$\Rightarrow \quad$ L.H.S. = R.H.S.

$$\Rightarrow \quad \frac{(k+1)\ln k - k\ln(k+1)}{k(k+1)}$$

$$= \frac{F(k)}{k}[(k+1)\ln k - k\ln(k+1)]$$

$$\therefore \quad F(k) = \frac{1}{k+1}$$

$$F(100) = \frac{1}{101}.$$

EXAMPLE 1.6 Given that $\log_2 a = s$, $\log_4 b = s^2$ and $\log_{c^2}(8) = \frac{2}{s^3+1}$. Write $\log_2 \frac{a^2b^5}{c^4}$ as a function of 's' $(a, b, c > 0, c \neq 1)$.

SOLUTION Given $\log_2 a = s$ ...(1)

$\log_2 b = 2s^2$ ...(2)

$\log_8 c^2 = \frac{s^3+1}{2}$ ...(3)

$$\Rightarrow \quad \frac{2\log c}{3\log 2} = \frac{s^3+1}{2}$$

$\Rightarrow \quad 4\log_2 c = 3(s^3 + 1)$ ...(4)

Now, $2\log_2 a + 5\log_2 b - 4\log_2 c$

$= 2s + 10s^2 - 3(s^3 + 1).$

EXAMPLE 1.7 Suppose that a and b are positive real numbers such that $\log_{27}a + \log_9 b = \frac{7}{2}$ and $\log_{27}b + \log_9 a = \frac{2}{3}$. Find the value of the ab.

SOLUTION $\log_{27}a+\log_9 b=\frac{7}{2}$; $\log_{27}b+\log_9 a=\frac{2}{3}$

$$\frac{1}{3}\log_3 a + \frac{1}{2}\log_3 b = \frac{7}{2};$$

$$\frac{1}{3}\log_3 b + \frac{1}{2}\log_3 a = \frac{2}{3}$$

Adding the equations:

$$\frac{1}{3}\log_3(ab) + \frac{1}{2}\log_3(ab)$$

$$= \frac{7}{2} + \frac{2}{3} = \frac{25}{6}$$

$$\Rightarrow \frac{5}{6}\log_3(ab) = \frac{25}{6} \Rightarrow \log_3(ab) = 5$$

$\Rightarrow \quad ab = 3^5 = 243.$

EXAMPLE 1.8 If $a = \log_6 15$ and $b = \log_{12} 18$, then find $\log_{25} 24$ in terms of a, b.

SOLUTION $\log_{25}24 = \frac{3}{2}\log_5 2 + \frac{1}{2}\log_5 3$

$$a = \log_6 15 = \log_6 3 + \log_6 5$$

$$= \frac{1}{1+\log_3 2} + \frac{1}{\log_5 2 + \log_5 3}$$

$$= \frac{1+\log_5 3}{\log_5 2 + \log_5 3}$$

$$b = \log_{12}18 = \log_{12}2 + 2\log_{12}3$$

$$= \frac{1}{2+\log_2 3} + \frac{2}{1+2\log_3 2}$$

$$= \frac{\log_5 2 + 2\log_5 3}{\log_5 3 + 2\log_5 2}$$

$\Rightarrow$ a $\log_5 2 + (a-1)\log_5 3 = 1$ (1)

$(2b-1)\log_5 2 + (b-2)\log_5 3 = 0$ (2)

Solving simultaneously (1) and (2) for $\log_5 2$,and $\log_5 3$ we get

$$\log_5 2 = \frac{2-b}{a+ab-2b+1}, \text{ and}$$

$$\log_5 3 = \frac{2b-1}{a+ab-2b+1}.$$

$$\Rightarrow \quad \log_{25} 24 = \frac{5-b}{2a+2ab-4b+2}.$$

EXAMPLE 1.9 Given $\log_2(\log_8 x) = \log_8(\log_2 x)$ then find the value of $(\log_2 x)^2$.

SOLUTION Given $\log_2\left(\frac{1}{3}\log_2 x\right) = \frac{1}{3}\log_2(\log_2 x)$

Let $\log_2 x = 3y$

$$\Rightarrow \quad \log_2 y = \frac{1}{3}\log_2 3y \Rightarrow 3\log_2 y = \log_2 3 + \log_2 y$$

$$\Rightarrow \quad 2\log_2 y = \log_2 3$$

$$\Rightarrow \quad y_2 = 3 \Rightarrow \frac{1}{9}(\log_2 x)^2 = 3$$

$$\therefore \quad (\log_2 x)^2 = 27.$$

EXAMPLE 1.10 For $x \ge 0$, what is the smallest possible value of the expression

$$\log(x^3 - 4x^2 + x + 26) - \log(x+2)?$$

SOLUTION $\log\dfrac{(x^3-4x^2+x+26)}{(x+2)}$

$$= \log\frac{(x^2-6x^2+13)(x+2)}{(x+2)}$$

$$= \log(x^2 - 6x + 13) \quad [\because \quad x \ne -2]$$

$$= \log((x-3)^2 + 4)$$

$\therefore$ The minimum value is log 4 when x = 3.

EXAMPLE 1.11 If $x = \log_d(abc)$; $y = \log_b(acd)$; $z = \log_c(abd)$ and $t = \log_a(bcd)$ then find the value of

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{t+1}$$

SOLUTION $x + 1 = \log_d(abcd)$

$y + 1 = \log_b(abcd)$

$z + 1 = \log_c(abcd)$

$t + 1 = \log_a(abcd)$

$\Rightarrow \dfrac{1}{x+1} + \dfrac{1}{y+1} + \dfrac{1}{z+1} + \dfrac{1}{t+1}$ is equal to

$$\frac{1}{\log_a(abcd)} + \frac{1}{\log_b(abcd)} + \frac{1}{\log_c(abcd)} + \frac{1}{\log_a(abcd)}$$

$$= \log_{abcd}(abcd) = 1.$$

EXAMPLE 1.12 Find the greatest value of the expression

$$\log_2^4 x + 12\log_2^2 x \log_2\frac{8}{x}$$

when x varies between 1 and 64.

SOLUTION Denote the expression in the question by y and transform it in the following way:

$$y = \log_2^4 x + 12\log_2^2 x(\log_2 8 - \log_2 x)$$

$$= \log_2^2 x(\log_2^2 x - 12\log_2 x + 36)$$

$$= = \log_2^2 x(6 - \log_2 x)^2.$$

Let us put $\log_2 x = z$, then $0 \le z \le 6$. The problem is thus reduced to finding the greatest value of the variable $y = z^2(6-z)^2$.

It is sufficient to find the greatest value of $z(6-z)$ for $0 \le z \le 6$ because the greater a positive number, the greater its square.

The quadratic trinomial $z(6-z) = -(z-3)^2 + 9$ attains its greatest value for z = 3. Thus, the sought for greatest value is attained for z = 3 and is equal to 81.

EXAMPLE 1.13 Given a and b are positive numbers satisfying $4(\log_{10}a)^2 + (\log_2 b)^2 = 1$ then find the possible range of values of a and b.

SOLUTION $(\log_2 b)^2 = 1 - (2\log_{10}a)^2 \geq 0$

$\Rightarrow (2\log_{10}a)^2 - 1 \leq 0$

$\Rightarrow (2\log_{10}a - 1)(2\log_{10}a + 1) \leq 0$

$\Rightarrow \log_{10}a \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

$\Rightarrow a \in \left[-\frac{1}{\sqrt{10}}, \sqrt{10}\right]$.

Also, $(\log 10^a)^2 = \dfrac{1 - (\log_{10} b)^2}{4} \geq 0$

$\Rightarrow (\log_{10} b)^2 - 1 \leq 0$

$\Rightarrow (\log_{10} b - 1)(\log_{10} b + 1) \leq 0$

$\Rightarrow \log_{10}b \in [-1, 1]$

$\Rightarrow b \in \left[\frac{1}{10}, 10\right]$.

CAUTION It should be noted that the logarithmic formulae given earlier also have drawbacks. Their left and right members are meaningful for different restrictions on the values of the letters that enter into them.

For example, $\log_a MN$ has meaning when M and N are both positive or both negative. But $\log_a M + \log_a N$ has meaning for only positive M and N. For this reason, replacing $\log_a MN$ by $\log_a M + \log_a N$ when M and N are negative is meaningless. When solving equations this can lead to loss of solutions. Now consider:

1*. $\log_a MN = \log_a |M| + \log_a|N| \quad (MN > 0)$

2*. $\log_a \dfrac{M}{N} = \log_a|M| - \log_a|N| \quad (MN > 0)$

3*. $\log_a N^{2k} = 2k\log_a|N| \quad (N \neq 0,\ k \text{ an integer})$

4*. $\log_{x^{2k}} N = \dfrac{1}{2k} \log_{|x|}N$

$(N > 0, k \neq 0 \text{ an integer}, x \neq 0, |x| \neq 1)$

It should be noted that formulae 1* and 2* also have drawbacks; their left and right members are meaningful for different restrictions on the values of the letters that enter into them. Namely the right side has meaning for arbitrary M and N different from zero, while the left hand side is only meaningful for M and N having the same sign, which means that they are subject to more stringent restrictions. For this reason, replacing $\log_a MN$ by $\log_a |M| + \log_a |N|$ when solving equations can lead to extraneous solutions but not to the loss of solutions, as can happen when using formulae 1 – 4. Since acquiring extraneous solutions of an equation is preferable to losing solutions (superfluous solutions may be discarded by verification, but lost solutions cannot be found), one should use formulae 1* to 4* when manipulating literal expressions.

Consider some examples:

$$\log_a x^2 = 2\log_a|x|$$

$$\log_a x^3 = 3\log_a x$$

$$\log_a x^{1/3} = \frac{1}{3}\log_a x$$

$$\log_a x^{1/2} = \frac{1}{2}\log_a x$$

$$\log_a x^{2/3} = \frac{2}{3}\log_a|x|$$

$$\log_{x^2}(x+3) = \frac{1}{2}\log_{|x|}(x+3)$$

$$\log_{x^3}(x+3) = \frac{1}{3}\log_x(x+3)$$

$$\log_a x = \begin{cases} \dfrac{1}{\log_x a}, & x \neq 1 \\ 0, & x = 1 \end{cases}$$

Here is a problem which illustrate the importance of utilising these properties.

EXAMPLE 1.14 Simplify the expression

$\log_4 \dfrac{x^2}{4} - 2\log_4 4x^4$ and then compute its value for $x = -2$.

SOLUTION It is quite evident here that computations by formulae 1 and 3, that is

$$\log_4 \frac{x^2}{4} - 2\log_4 4x^4 = 2\log_4 x - \log_4 4$$

$-2\log_4 4 - 8\log_4 x = -3 - 6\log_4 x$

are erroneous because the latter expression for $x = -2$ is meaningless, whereas the original one is meaningful and is equal to -6.

This paradoxical result is due to the fact that formulae 1 and 3 are only applicable to positive values of the letters. Now if we use formulae 1* and 3* in which the values of the letters may be negative as well, we get

$$\log_4 \frac{x^2}{4} - 2\log_4 4x^4 = 2\log_4|x| - 1 - 2 - 8\log_4|x|$$
$$= -3 - 6\log_4|x|$$

It is clear that for $x = -2$ this expression is equal to -6.

Common Logarithms

The integral part of a logarithm is called the characteristic and the decimal part is called the mantissa.

The characteristic of the logarithm of any number to the base 10 can be written down by inspection.

To determine the characteristic of the logarithm of any number greater than unity

Since
$$10^1 = 10,$$
$$10^2 = 100,$$
$$10^3 = 1000,$$
....

it follows that a number with two digits in its integral part lies between 10^1 and 10^2 ; a number with three digits in its integral part lies between 10^2 and 10^3 ; and so on. Hence a number with n digits in its integral part lies between 10^{n-1} and 10^n.

Let N be a number whose integral part contains n digits; then

$$N = 10^{(n-1) + \text{a fraction}}$$

$$\therefore \quad \log N = (n-1) + \text{a fraction}.$$

Hence the characteristic is $n - 1$; that is, the characteristic of the logarithm of a number greater than unity is less by one than the number of digits in its integral part, and is positive.

To determine the characteristic of the logarithm of a decimal fraction.

Since
$$10^0 = 1,$$
$$10^{-1} = \frac{1}{10} = 0.1,$$
$$10^{-2} = \frac{1}{100} = 0.01,$$
$$10^{-3} = \frac{1}{1000} = 0.001,$$
....

It follows that a decimal with one zero immediately after the decimal point, such as 0.0324, being greater than 0.01 and less than 0.1, lies between 10^{-2} and 10^{-1}; a number with two zeros after the decimal point lies between 10^{-3} and 10^{-2}; and so on. Hence a decimal fraction with n zeros immediately after the decimal point lies between $10^{-(n+1)}$ and 10^{-n}.

Let D be a decimal beginning with n zeros; then

$$D = 10^{-(n+1) + \text{a fraction}}$$
$$\log D = -(n+1) + \text{a fraction}.$$

Hence the characteristic is $-(n+1)$; that is, the characteristic of the logarithm of a decimal fraction is greater by unity than the number of zeros immediately after the decimal point, and is negative.

Let N be any number, then since multiplying or dividing by a power of 10 merely alters the position of the decimal point without changing the sequence of digits, it follows that $N \times 10^p$. and $N + 10^q$, where p and q are any integers, are numbers whose significant digits are the same as those of N.

Now
$$\log(N \times 10^p) = \log N + p\log 10$$
$$= \log N + p \qquad (1)$$

Again
$$\log(N \div 10^q) = \log N - q\log 10$$
$$= \log N - p \qquad (2)$$

In (1) an integer is added to log N, and in (2) an integer is subtracted from log N; that is, the mantissa or decimal portion of the logarithm remains unaltered.

Till now the mantissae have been supposed positive.

In the case of a negative logarithm the minus sign is written over the characteristic, and not before it, to indicate that the $\bar{4}.30103$, the logarithm of 0.0002, is equivalent to $-4 + 0.30103$, and must be distinguished from -4.30103, an expression in which both the integer and the decimal are negative. In working with negative logarithm it is necessary to make the mantissa positive. For instance, the logarithm -3.69897, in which the whole expression is negative, may be transformed by subtracting 1 from the characteristic and adding 1 to the mantissa. Thus,

$$-3.69897 = -4 + (1 - .69897) = \bar{4}.30103.$$

EXAMPLE 1.15 Let $x = (0.15)^{20}$. Find the characteristic and mantissa in the logarithm of x, to the base 10.

Assume $\log_{10}2 = 0.301$ and $\log_{10}3 = 0.477$.

SOLUTION $\log x = \log(0.15)^{20} = 20\log\left(\frac{15}{100}\right)$

$= 20[\log 15 - 2] = 20[\log 3 + \log 5 - 2]$

$= 20[\log 3 + 1 - \log 2 - 2]$

$= 20[-1 + \log 3 - \log 2]$

$= 20[-1 + 0.477 - 0.301]$

$= -20 \times 0.824 = -16.48 = \overline{17}.52$

Hence characteristic $= -17$ and mantissa $= 0.52$.

EXAMPLE 1.16 Find the number of positive integers which have the characteristic 3, when the base of the logarithm is 7.

SOLUTION $\log_7 N = x$ where $3 \le x < 4$

$\Rightarrow 7^3 \le N < 7^4$

$\Rightarrow$ number of integers are 2058.

EXAMPLE 1.17 How many digits are contained in the number 2^{75} ?

SOLUTION Computing $\log 2^{75}$, we have $\log 2^{75} = 75.\ \log 2 \approx 75.\ 0.3010 = 22.5750$. Consequently, the characteristic of this common logarithm is equal to 22. Therefore, $2^{75} = a.\ 10^{22}$, where $1 \le a < 10$, a is an integer, and, hence the number 2^{75} has 23 digits.

EXAMPLE 1.18 Given $\log 3 = 0.4771$, find

$$\log\left\{(2.7)^3 \times (0.81)^{\frac{4}{5}} \div (90)^{\frac{5}{4}}\right\}$$

SOLUTION The required value

$= 3\log\frac{27}{10} + \frac{4}{5}\log\frac{81}{100} - \frac{5}{4}\log 90$

$= 3(\log 3^3 - 1) + \frac{4}{5}(\log 3^4 - 2) - \frac{5}{4}(\log 3^2 + 1)$

$= \left(9 + \frac{16}{5} - \frac{5}{2}\right)\log 3 - \left(3 + \frac{8}{5} + \frac{5}{4}\right)$

$= \frac{97}{10}\log 3 - 5\frac{17}{20} = 4.62787 - 5.85 = -1.2213$

We should notice that the logarithm of 5 and its powers can be obtained from log 2 as

$$\log 5 = \log\frac{10}{2} = \log 10 - \log 2 = 1 - \log 2.$$

EXAMPLE 1.19 Find the number of digits in 875^{16}, given $\log 2 = 0.3010$, $\log 7 = 0.8451$.

SOLUTION $\log(875^{16}) = 16(\log 7 + 3\log 5)$

$= 6(\log 7 + 3 - 3\log 2)$

$= 16 \times 2.9421$

$= 47.\ 0736$

Hence the number of digits is 48.

EXAMPLE 1.20 Let P denotes number of digits in $8^{12}\,5^{35}$ and Q denotes number of zeroes after decimal before a significant figure start in $\left(\frac{8}{27}\right)^{20}$ where $\log_{10}2 = 0.3010$ and $\log_{10}3 = 0.4771$.

Find the product of P and Q.

SOLUTION Let $N_1 = 8^{12}\,5^{35}$

$\Rightarrow \log_{10} N_1 = \log_{10}(2^{36} \cdot 5^{35}) = \log_{10}(10^{35} \cdot 2)$

$= 35 + \log_{10} 2 = 35 + 0.3010$

$= 35.3010$

$\therefore$ P = No. of digits in $N_1 = 36$

Now, let $N_2 = \left(\frac{8}{27}\right)^{20}$

$\Rightarrow \log_{10} N_2 = 20 \log_{10}\left(\frac{8}{27}\right)$

$= 60 \log_{10}\frac{2}{3} = 60(\log_{10}2 - \log_{10}3)$

$= 60 \times (0.3010 - 0.4771)$

$= -60 \times 0.1761$

$= -10.56 = \overline{11} + 0.44$

$\therefore$ Q = No. of zeroes after decimal before a significant figure start in $N_2 = 10$.

$\Rightarrow$ PQ = 36 × 10 = 360.

PRACTICE PROBLEMS [H]

1. Find the natural number whose square is the expression $(0.05)^{\log_{\sqrt{20}}(0.\bar{1})}$.
2. Simplify:
   (i) $\dfrac{\log_5 250}{\log_{50} 5} - \dfrac{\log_5 10}{\log_{1250} 5}$
   (ii) $\log_3 2 . \log_8 7 . \log_4 3 . \log_7 6 . \log_5 4$
   (iii) $\log_3 \log_2 \log_{\sqrt{3}} 81$
   (iv) Simplify $x^{\ln y - \ln z} . y^{\ln z - \ln x} . z^{\ln x - \ln y}$.
3. Prove that $\log_{a+b}m + \log_{a-b}m = 2 \log_{a+b}m \log_{a-b}m$ if it is known that $m^2 = a^2 - b^2$.
4. (i) Find the characteristics of the logarithms (base 10) of 23.8, 0.035.
   (ii) How many digits are there in the integral part of the numbers whose logarithms are 4.3010, 1.4771.
5. Given log 194.8445 = 2.2896883, find the eleventh root of $(39.2)^2$.
6. Show that $\left(\frac{21}{20}\right)^{100}$ is greater than 100.
7. If $\dfrac{\log_a \sqrt{a^2-1} . \log^2_{1/a} \sqrt{a^2-1}}{\log_{a^2}(a^2-1) . \log_{\sqrt[3]{a}} \sqrt[6]{a^2-1}} = \dfrac{1}{2}$, then find the value of 'a'.
8. If $\dfrac{\log_2(b^3/8)}{\log_3(27/a^2)} = 1$ and $\log_3\left(\frac{9}{a}\right) = \log_2\left(\frac{b}{4}\right)$ then find the value of $\frac{a}{b}$.
9. If $\log_{18} 36 = a$ & $\log_{24} 72 = b$, then find the value of 4 (a + b) − 5 ab.
10. Simplify
    $b . a^{\frac{2}{\log_b a}+1} - 2a^{\log_a b+1} . b^{\log_b a+1} + a . b^{\frac{2}{\log_a b}+1}$.
11. Simplify
    (i) $\dfrac{81^{1/\log_5 9} + 3^{3/\log_{\sqrt{6}} 3}}{409}\left((\sqrt{7})^{2/\log_{25} 7} - 125^{\log_{25} 6}\right)$

(ii) $\left(2^{\log_{\sqrt[4]{2}} a} - 3^{\log_{27}(a^2+1)^3} - 2a\right) / \left(7^{4\log_{49} a} - a - 1\right)$

(iii) $\dfrac{\log_a b + \log_a \left(b^{\frac{1}{2}\log_b a^2}\right)}{\log_a b - \log_{ab} b} \; \dfrac{\log_{ab} b \log_a b}{b^{2\log_b \log_a b} - 1}$

(iv) $5^{\log_{1/5}\left(\frac{1}{2}\right)} + \log_{\sqrt{2}} \dfrac{4}{\sqrt{7}+\sqrt{3}} + \log_{1/2} \dfrac{1}{10+2\sqrt{21}}$

12. If the number 5^{2000} is written out (in decimal notation), how many digits does it have ?
13. If $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$, then find the number of digits in the number $N = 3^{12} \times 28$.
14. (i)Find the seventh root of 0.00324, having given that log 4409 = 3.6443.

    (ii) How many zeros are there between the decimal point and the first significant digit in $\left(\dfrac{1}{2}\right)^{1000}$.

1.13 | EXPONENTIAL EQUATION

If we have an equation of the form $\mathbf{a^x = b}$ $(a > 0)$, then

(i) $x \in \phi$ if $b \le 0$

(ii) $x = \log_a b$ if $b > 0$ $a \ne 1$

(iii) $x \in \phi$ if $a = 1, b \ne 1$

(iv) $x \in R$ if $a = 1, b = 1$

(Since $1^x = 1 \Rightarrow 1 = 1, x \in R$)

Consider equations of the form $a^{f(x)} = a^{g(x)}$, where $a > 0$ and $a \ne 1$ and equations which can be reduced to them. The solution of such equations is based on the following result :

If $a > 0$ and $a \ne 1$, then the equation $a^{f(x)} = a^{g(x)}$ is equivalent to the equation $f(x) = g(x)$.

CAUTION A common mistake made by students is the incorrect use of the following statement: "if two exponential expressions are equal and if their bases are equal (different from 0 and ±1), then their exponents are equal as well." What is usually forgotten is the phrase "different from 0 and ±1". The result is a loss of roots, namely those for which the base is equal to 0 or ±1.

If we have an equation of the form $(a(x))^{f(x)} = (a(x))^{g(x)}$ called exponential-power equation, then

(i) $a(x) > 0$ and $a(x) \ne 1$, $f(x) = g(x)$, or

(ii) $a(x) = 1$, or

(iii) $a(x) = -1$, $(-1)^{f(x)} = (-1)^{g(x)}$, or

(iv) $a(x) = 0$, $f(x) > 0$, $g(x) > 0$.

Note that no value of x should lead to 0^0 form.

EXAMPLE 1.1 Solve the equation $2^{x^2-2x} = 2^{3x-6}$.

SOLUTION The given equation is equivalent to the equation $x^2 - 2x = 3x - 6$, and therefore the roots of the last equation $x_1 = 2$ and $x_2 = 3$ are also roots of the original equation.

EXAMPLE 1.2 Solve $16^{(x^2+3x-1)} = 8^{(x^2+3x+2)}$.

SOLUTION $16^{(x^2+3x-1)} = 8^{(x^2+3x+2)}$.

$\Rightarrow 2^{4(x^2+3x-1)} = 2^{3(x^2+3x+2)}$.

$\Rightarrow 2^{4x^2+12x-4} = 2^{3x^2+9x+6}$

$\Rightarrow 4x^2 + 12x - 4 = 3x^2 + 9x + 6$

$\Rightarrow x^2 + 3x - 10 = 0$

$\Rightarrow (x + 5)(x - 2) = 0$

$\Rightarrow x = -5$ or $x = 2$

EXAMPLE 1.3 Solve the equation $4^x + 2^{x+1} - 24 = 0$.

SOLUTION Let us apply the method of introducing new variables.

Since $4^x = (2^2)^x = (2^x)^2$ and $2^{x+1} = 2 \times 2^x$, the equation can be rewritten in the following way:

$(2^x)^2 + 2 \times 2^x - 24 = 0$.

Setting $u = 2^x$, we get the quadratic equation

$u^2 + 2u - 24 = 0$, whose roots are:

$u_1 = 4$ and $u_2 = -6$

$\Rightarrow 2^x = 4;\ 2^x = -6$.

From the first equation we get $x = 2$. The second equation has no solutions since $2^x > 0$ for any values of x. Thus, the root of the equation is $x = 2$.

EXAMPLE 1.4 Solve $8 + x \,.\, 2^x - 2^{x+3} - x = 0$.

SOLUTION $x(2^x - 1) + 8(1 - 2^x) = 0$

$\Rightarrow (2^x - 1)(x - 8) = 0$

$\Rightarrow 2^x = 1$, or $x = 8$

$\Rightarrow x = 0, 8$.

EXAMPLE 1.5 Solve $3.\sqrt[x]{81} - 10\sqrt[x]{9} + 3 = 0$

SOLUTION $3.9^{2/x} - 10.9^{1/x} + 3 = 0$

Let $9^{1/x} = t$.

$\Rightarrow 3t^2 - 10t + 3 = 0$

$\Rightarrow t = 3, 1/3$

$\Rightarrow 9^{1/x} = 3,\ 9^{1/x} = 1/3$

$\Rightarrow x = 2, -2$.

But x should be a natural number greater than 1.

Hence, $x = 2$.

EXAMPLE 1.6 Solve the equation

$$4^{\sqrt{x^2-2x+x}} - 5.2^{x-1} + \sqrt{x^2-2} = 6.$$

SOLUTION Assuming $2^{\sqrt{x^2-2+x}} = y$, we get a quadratic equation

$$y^2 - \frac{5}{2}y - 6 = 0$$

whose roots are $y_1 = 4$ and $y_2 = -3/2$.

$$\Rightarrow\ 2x + \sqrt{x^2-2} = 4,\quad 2x + \sqrt{x^2-2} = -3/2.$$

The second equation has no solutions since $2x + \sqrt{x^2-2} > 0$ for all values of x.

From the first equation we get $x + \sqrt{x^2-2} = 2$

Separating out the radical and squaring both sides of the equation, we obtain $x^2 - 2 = 4 - 4x + x^2$

Collecting terms, we get the only root $x = 3/2$.

Verification shows that this root satisfies the initial equation.

EXAMPLE 1.7 Solve $2^{2x} - 4.6^x + 3^{2x+1} = 0$.

SOLUTION We divide both sides by 3^{2x}

$$\Rightarrow \left(\frac{2}{3}\right)^{2x} - 4 - \left(\frac{2}{3}\right)^{x} + 3 = 0$$

$$\Rightarrow \left(\frac{2}{3}\right)^{2x} = 1, 3$$

$$\Rightarrow x = 0,\ \log_{\frac{2}{3}} 3$$

Alternative:

$u^2 - 4uv + 3v^2 = 0$, where $u = 2^x$ and $v = 3^x$.

$\Rightarrow (u - v)(u - 3v) = 0$

$\Rightarrow 2^x - 3^x = 0$ or $2^x - 3.3^x = 0$

$\Rightarrow x = 0$ or $x = \log_{\frac{2}{3}} 3$.

CAUTION In solving the equation $3^x.2^{\frac{3x}{x+2}} = 6$ the student equate the exponents of the respective bases as:

$$x = 1,\ \frac{3x}{x+2} = 1$$

The answer is x = 1.

But this solution is incorrect in the sense that only one root of the equation is found and nothing has been said about any other root. Actually, if the exponents on the appropriate bases are equal, then the products of these powers are equal, however the converse is simply incorrect.

For instance, the equation $3^1 . 2^1 = 3^2 . 2^{\log_2(2/3)}$ is valid, but $1 \neq 2$ and $1 \neq \log_2 (2/3)$. Therefore, the foregoing reasoning may lead to a loss of roots, and this is exactly what occurred in the equation at hand.

Taking logarithms of both members of the original equation to the base 3, we get

$$x + \frac{3x}{x+2}\log_3 2 = \log_3 6$$

or $\quad x^2 + x(3 \log_3 2 + 2 - \log_3 6) - 2 \log_3 6 = 0$

We now have to solve this quadratic equation. This can be done using a familiar formula, but we have already seen, by trial and error, that $x_1 = 1$ is a root of the original equation and, consequently, satisfies the quadratic equation. Hence by product of roots the second root of the quadratic equation is $x_2 = -2 \log_3 6$ and so the original equation has two roots:

$x_1 = 1,\ x_2 = -2 \log_3 6$

Thus, it is useful to be able to guess a root, but never consider the guessing as the whole solution.

EXAMPLE 1.8 If $(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$ then find x.

SOLUTION

$$(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$$

Here we see that $(5 + 2\sqrt{6}) = \dfrac{1}{5 - 2\sqrt{6}}$

So, let $(5 + 2\sqrt{6})^{x^2-3} = y$

$\Rightarrow y + \dfrac{1}{y} = 10$

$\Rightarrow y^2 - 10y + 1 = 0$

$\Rightarrow y = 5 + 2\sqrt{6}$

or $\quad 5 - 2\sqrt{6}$

So, $x^2 - 3 = 1 \quad$ or $\quad x^2 - 3 = -1$

$\Rightarrow x = \pm 2 \quad$ or $\quad x = \pm\sqrt{2}$.

EXAMPLE 1.9 Solve the equation $1 + 3^{x/2} = 2x$.

SOLUTION We have $1 + 3^{x/2} = 2x$

Dividing both sides by 2x, then we obtain

$$\frac{1}{2^x} + \frac{3^{x/2}}{2^x} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^x + \left(\frac{3}{4}\right)^{x/2} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^x + \left(\frac{\sqrt{3}}{2}\right)^x = 1$$

$\Rightarrow (\cos \pi/3)x + (\sin \pi/3)x = 1,$

which is possible only when x = 2.

Hence x = 2 is only solution of the equation.

EXAMPLE 1.10 Find the solution of the equation

$$2^{\frac{x}{2}} + \left(\sqrt{2}+1\right)^x = \left(5+2\sqrt{2}\right)^{\frac{x}{2}}.$$

SOLUTION The given equation can be written as

$$\left(\frac{\sqrt{2}}{\sqrt{5+2\sqrt{2}}}\right)^x + \left(\frac{\sqrt{2}+1}{\sqrt{5+2\sqrt{2}}}\right)^x = 1.$$

Now $\quad$ if $\cos\alpha = \dfrac{\sqrt{2}}{\sqrt{5+2\sqrt{2}}}$

Then $\quad \sin\alpha = \dfrac{\sqrt{2}+1}{\sqrt{5+2\sqrt{2}}}$

So, $\quad (\cos\alpha)x + (\sin\alpha)x = 1.$

[we have $\cos^2\alpha + \sin^2\alpha = 1$]

$\Rightarrow x = 2.$

EXAMPLE 1.11 Solve

$$\left(\sqrt{2+\sqrt{2}}\right)^x + \left(\sqrt{2-\sqrt{2}}\right)^x = 2.2^{x/4}$$

SOLUTION

$$\left(\sqrt{2+\sqrt{2}}\right)^x + \left(\sqrt{2-\sqrt{2}}\right)^x = 2.2^{x/4} \qquad (1)$$

Now, A.M. of the numbers $\left(\sqrt{2+\sqrt{2}}\right)^x$ and $\left(\sqrt{2-\sqrt{2}}\right)^x$ is $\dfrac{\left(\sqrt{2+\sqrt{2}}\right)^x + \left(\sqrt{2-\sqrt{2}}\right)^x}{2}$

$$\Rightarrow \left(\sqrt{2+\sqrt{2}}\right)^x + \left(\sqrt{2-\sqrt{2}}\right)^x = 2\text{ A.M.}$$

Now, G.M. of $\left(\sqrt{2+\sqrt{2}}\right)^x$ and $\left(\sqrt{2-\sqrt{2}}\right)^x$

$$\text{G.M.} = \left(\sqrt{2+\sqrt{2}}\right)^x \left(\sqrt{2-\sqrt{2}}\right)^x = 2^{\frac{x}{2}}.$$

Substituting respective values in (1), we get

$$2\text{ A.M.} = 2\text{ G.M.} \quad \Rightarrow \quad \text{A.M.} = \text{G.M.}$$

Now, $\quad$ A.M. = G.M. if and only if

$$\left(\sqrt{2+\sqrt{2}}\right)^x = \left(\sqrt{2-\sqrt{2}}\right)^x \quad \Rightarrow x = 0.$$

EXAMPLE 1.12 Solve the equation

$$x^{\sqrt{x}} = \sqrt{x^x}.$$

SOLUTION This equation may be rewritten as

$$x^{\sqrt{x}} = x^{x/2}$$

We see that the bases are equal and $x \geq 0$.

So as not to lose any roots, let us see whether the base can be 0 or 1.

Since the expression 0^0 is meaningless, the number 0 is not an element in the domain set and therefore $x = 0$ is not a root of the equation. However, $x = 1$ is a root.

Now let us seek roots that are different from 0 and 1.

Equating the exponents, we obtain $\sqrt{x} = x/2$, whereby we find the second root of the equation, $x = 4$

CAUTION One sometimes hears the erroneous assertion: " If an exponential expression is 1, then the exponent is equal to zero." This is only true if the base is different from ± 1 and 0.

EXAMPLE 1.13 Solve $x^{2x+3} = x$

SOLUTION Since bases are equal we equate the exponents

$$2x + 3 = 1 \quad \Rightarrow \quad x = -1$$

Now, we assume base x to be –1, 0 and 1

$x = -1$ has been found as a solution.

Put $x = 0$ in the given equation, $0^3 = 0$.

Hence, $x = 0$ is a solution.

Now with $x = 1$ we get $1^5 = 1$.

Hence, the solutions are $x = -1, 0, 1$.

EXAMPLE 1.14 Solve the equation

$$(x^2 + x - 57)^{3x^2+3} = (x^2 + x - 57)^{10x}. \qquad \ldots(1)$$

SOLUTION When solving the given exponential-power equation, we have to consider four cases:

Case 1 $\quad x^2 + x - 57 = 1$, that is, $x^2 + x - 58 = 0$. In this case equation (1) takes the form

$1^{3x^2+3} = 1^{10x}$, i.e. $1 = 1$.

Hence, the roots of the equation $x^2 + x - 58 = 0$ are also roots of equation (1).

From the equation $x^2 + x - 58 = 0$

we find $x_{1,2} = \dfrac{-1 \pm \sqrt{233}}{2}$.

Case 2: $\quad x^2 + x - 57 = -1$, that is, $x^2 + x - 56 = 0$. In this case equation (1) takes the form

$$(-1)^{3x^2+3} = (-1)^{10x}. \qquad (2)$$

Equation (2) can be satisfied only by those values of x for which $3x^2 + 3$ and $10x$ are integers (since the negative number (–1) can be raised only to an integer power of equal parity (both even or both odd).

From the equation $x^2 + x - 56 = 0$ we find : $x_1 = -8$, $x_2 = 7$. The value $x_1 = -8$ does not satisfy equation (2), while the value $x_2 = 7$ satisfies this equation. Hence, x = 7 is a root of equation (1).

Case 3: $x^2 + x - 57 = 0$. In this case equation (1) takes the from $0^{3x^2+3} = 0^{10x}$ (3)

Equation (3) can be satisfied only by those values of x for which $3x^2 + 3 > 0$ (this is true for all x's) and $10x > 0$; in this case equation (3) takes the form 0 = 0 (let us recall that the expression 0^r has sense only for r > 0).

From the equation $x^2 + x - 57 = 0$

we find $x_{1,2} = \dfrac{-1 \pm \sqrt{229}}{2}$

The value $x_1 = \dfrac{-1-\sqrt{229}}{2}$ does not satisfy the condition $10x > 0$, while $x_2 = \dfrac{-1-\sqrt{229}}{2}$ does.

Hence, $x = \dfrac{-1+\sqrt{229}}{2}$ is a root of equation (1).

Case 4: If $x^2 + x - 57 > 0$ and $x^2 + x - 57 \neq 1$, then from equation (1) we conclude that

$3x^2 + 3 = 10x$, we find : $x_1 = 3$, $x_2 = \dfrac{1}{3}$. Both of these values must be checked by substituting into equation (1). For x = 3 we get : $(-45)^{30} = (-45)^{30}$ which is a true equality.

Summing up, we conclude that equation (1) has six roots:

$$x_{1,2} = \frac{-1 \pm \sqrt{233}}{2},\ x_3 = 7,\ x_4 = \frac{-1-\sqrt{229}}{2},$$

$$x_5 = 3,\ x_6 = \frac{1}{3}.$$

EXAMPLE 1.15 Solve the equation

$$|x-2|^{10x^2-1} = |x-2|^{3x}$$

SOLUTION $x - 2 \neq 0 \quad \therefore \quad x \neq 2$

Case 1: $x - 2 > 0 \quad \Rightarrow \quad x > 2$

Taking log on both sides :

$(10x^2 - 1) \log(x-2) = 3x \log(x-2)$

$\Rightarrow \log(x-2)(10x^2 - 3x - 1) = 0$

$\Rightarrow x - 2 = 1$ or $10x^2 - 5x + 2x - 1 = 0$

$\Rightarrow (5x + 1)(2x - 1) = 0$

$\Rightarrow x = 3$ or $x = -1/5$ (reject), $x = 1/2$ (reject).

Case 2: $x - 2 < 0 \quad \Rightarrow \quad x < 2$

$(10x^2 - 1) \log(2-x) = 3x \log(2-x)$

$\Rightarrow \log(2-x)(10x^2 - 3x - 1) = 0$

$\Rightarrow 2 - x = 1$ or $x = -1/5$, $x = 1/2$

$\Rightarrow \quad x = 1.$

$\Rightarrow x \in \left\{-\dfrac{1}{5}, \dfrac{1}{2}, 1, 3\right\}$

EXAMPLE 1.16 Find the general solution, of the equation $x^{3\sin 2x + 2} = \sqrt{x}$.

SOLUTION $x^{3\sin 2x + 2} = x^{1/2}$

$\Rightarrow \left[(3\sin 2x + 2) - \dfrac{1}{2}\right] \log x = 0$

$\Rightarrow 3\sin 2x + 2 = -\dfrac{1}{2}$

$\Rightarrow \sin 2x = -\dfrac{1}{2}$

$\Rightarrow 2x = n\pi + (-1)^n\left(-\dfrac{\pi}{6}\right)$

$\Rightarrow x = \dfrac{n\pi}{2} - (-1)^n \dfrac{\pi}{12}$, where $n \in I$.

EXAMPLE 1.17 Solve the equation

$$(0.4)^{\log^2 x + 1} = (6.25)^{2 - \log x^3}.$$

SOLUTION Using the fact that

$$0.4 = \frac{2}{5} \text{ and } 6.25 = \left(\frac{5}{2}\right)^2$$

we reduce the original equation to the form

$$\left(\frac{2}{5}\right)^{\log^2 x+1} = \left(\frac{2}{5}\right)^{2(\log x^3-2)}$$

Equating the exponents, we pass to the equation $\log^2 x - 6\log x + 5 = 0$.

After solving it we find

$\log x = 1$, $x = 10$ and $\log x = 5$, $x = 10^5$.

The solutions are $10, 10^5$.

EXAMPLE 1.18 Solve the equation

$$x^{0.5\log_{\sqrt{x}}(x^2-x)} = 3^{\log_9 4}.$$

SOLUTION $x^{\frac{1}{2}\log\frac{(x^2-x)}{(1/2)\log x}} = 3^{\log 32}$

$\Rightarrow x^{\log_x(x^2-x)} = 2$

$\Rightarrow x^2 - x = 2 \quad \Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = 2$ or $x = -1$ (rejected)

$\therefore x = 2$.

EXAMPLE 1.19 Find all real numbers t satisfying the equation $(3^t - 9)^3 + (9^t - 3)^3 = (9^t + 3^t - 12)^3$.

SOLUTION Let $3^t = a$

$(a-9)^3 + (a^2-3)^3 = (a^2+a-12)^3$

$\Rightarrow (a-9)^3 + (a^2-3)^3 = [(a^2-3)+(a-9)]^3$

$\Rightarrow 3(a^2-3)(a-9)[(a^2-3)+(a-9)] = 0$

$\Rightarrow (a^2-3)(a-9)(a^2+a-12) = 0$

This gives $a = 3, \sqrt{3}, -3, -\sqrt{3}, -4, 9$

The acceptable values of a are $a = 3, \sqrt{3}, 9$

$\therefore 3^t = 3, \quad 3^t = 3^{1/2}, 3^t = 3^2$

Hence, $t = 1, 1/2, 2$.

EXAMPLE 1.20 Solve the equation $7^{6-x} = x + 2$

SOLUTION By trial we find the root $x = 5$.

The equation has no other solutions since the function $f(x) = 7^{6-x}$ decreases and $g(x) = x + 2$ increases and consequently, the graphs of these functions cannot intersect more than once.

PRACTICE PROBLEMS [I]

Solve the following equations :

1. $7^{3x} + 9.5^{2x} = 5^{2x} + 9.7^{3x}$
2. $4^{x+1.5} + 9^x = 6^{x+1}$
3. $2^{2x^2} + 2^{x^2+2x+2} = 2^{5+4x}$
4. $4^{\sqrt{3x^2-2x}+1} + 2 = 9.2^{\sqrt{3x^2-2x}}$
5. $\left(\sqrt{5+\sqrt{24}}\right)^x + \left(\sqrt{5-\sqrt{24}}\right)^x = 10$
6. $9^{x^2-1} - 36 \times 3^{x^2-3} + 3 = 0$
7. $3^{3x+1} - 4 \times 27^{x-1} + 9^{1.5x-1} - 80 = 0$
8. $(2+\sqrt{3})^x + (2-\sqrt{3})^x = 4$
9. $(2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \dfrac{4}{2-\sqrt{3}}$

10. $5^{x-2} \times 2^{\frac{3x}{x+1}} = 4$

11. $(x^2 - x - 1)^{x^2-1} = 1$

12. $(x-2)^{x^2-x} = (x-2)^{12}$

13. $4x + (x-1)\,2x = 6 - 2x$

14. Find the range of the function $y = \left(\frac{1}{3}\right)^{-|x+2|}$.

15. Find the number of solution of the equation $\left(\frac{3}{5}\right)^{x} - 2x + \frac{7}{5} = 0$

16. Find the number of values of x which satisfies the equation $|x|^{x^2-2x} = 1$.

1.14 | EXPONENTIAL INEQUALITIES

The elementary exponential inequalities are inequalities of the form

$a^x > b$, $a^x < b$, (1)

where a and b are certain numbers ($a > 0, a \neq 1$).

Depending on the values of the parameters a and b, the set of solutions of the inequality $a^x > b$ can be in the following forms:

1. $x \in (\log_a b, \infty)$ for $a > 1, b > 0$;
2. $x \in (-\infty, \log_a b)$ for $0 < a < 1, b > 0$;
3. $x \in R$ for $a > 0, b < 0$;

Depending on the values of a and b, the set of solutions of the inequality $a^x < b$ can be in the following forms:

1. $x \in (-\infty, \log a\ b)$ for $a > 1, b > 0$;
2. $x \in (\log_a b, \infty)$ for for $0 < a < 1, b > 0$;
3. $x \in \phi$ for $a > 0, b < 0$ (i.e. the inequality has no solutions)

EXAMPLE 1.1 Solve the inequality

$$\sqrt[3]{2^{\frac{3x-1}{x-1}}} < 8^{\frac{x-3}{3x-7}}.$$

SOLUTION We transform the inequality

$$\sqrt[3]{2^{\frac{3x-1}{x-1}}} < 8^{\frac{x-3}{3x-7}} \qquad (1)$$

to $2^{\frac{3x-1}{3(x-1)}} < 2^{\frac{3(x-3)}{3x-7}}$

This is equivalent to the inequality

$$\frac{3x-1}{3(x-1)} < \frac{3(x-3)}{3x-7} \qquad (2)$$

$$\Rightarrow \frac{3x-1}{3x-3} - \frac{3x-9}{3x-7} < 0$$

$$\Rightarrow \frac{12x-20}{(3x-3)(3x-7)} < 0$$

$$\Rightarrow \frac{x-\frac{5}{3}}{(x-1)\left(x-\frac{7}{3}\right)} < 0$$

Solving the last inequality by the method of intervals,

we get $(-\infty, 1) \cup \left(\frac{5}{3}, \frac{7}{5}\right)$ which is the solution of inequality (1).

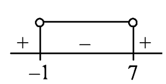
EXAMPLE 1.2 Find the number of integral values of x satisfying the inequality

$$\left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$$

SOLUTION $6x + 10 - x^2 > 3$

$\Rightarrow x^2 - 6x - 7 < 0$

$\Rightarrow (x+1)(x-7) < 0$



$\Rightarrow x = 0, 1, 2, 3, 4, 5, 6$

Hence, there are 7 integral values of x satisfying the inequality.

EXAMPLE 1.3 Solve the inequality

$$\left(\frac{1}{2}\right)^{(x^2-2x^2+1)} < \left(\frac{1}{2}\right)^{1-x}.$$

SOLUTION If we disregard the exponents, we can say that this is an elementary exponential inequality with base less than unity: $(1/2)^a < (1/2)^b$. Solving it, we find that the original inequality is equivalent to the inequality $(x^6 - 2x^3 + 1)^{1/2} > 1 - x$.

Since $(x^6 - 2x + 1)^{1/2} = \sqrt{(x^3-1)^2} = |x^3 - 1|$, it follows that we have yet to solve the inequality

$|x^3 - 1| > 1 - x$

Since the left member here is non negative, it is automatically satisfied for $1 - x < 0$, that is, when $x > 1$.

We now consider $x \le 1$. In this case, $x^3 \le 1$, and so $|x^3 - 1| = 1 - x^3$ and we have the inequality

$1 - x^3 > 1 - x$ or $x (x - 1) (x + 1) < 0$

Solving this inequality by the method of intervals, we find that it is true for $x < -1$ and for x located in the domain $x \le 1$ under consideration and so are solutions of the original inequality.

Thus, the original inequality is valid for

$x < -1, 0 < x < 1, x > 1$.

EXAMPLE 1.4 Solve the inequality

$$2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}.$$

SOLUTION We get

$$2^{x+2}(1 - 2 - 2^2) > 5^{x+2}(5^{-1} - 1),$$

$$\Rightarrow 2^{x+2}(-5) > 5^{x+2}\left(-\frac{4}{5}\right),$$

$$\Rightarrow \frac{2^{x+2}}{5^{x+2}} < \frac{4}{25} \quad \Rightarrow \quad \left(\frac{2}{5}\right)^{x+2} < \left(\frac{2}{5}\right)^2.$$

The last inequality is equivalent to the inequality $x + 2 > 2$,

from where we find $(0, \infty)$ which is the solution.

EXAMPLE 1.5 Solve $2^x + 2^{|x|} \ge 2\sqrt{2}$

SOLUTION Let $x < 0$

$2^x + 2^{-x} \ge 2\sqrt{2}$

$2^{2x} + 1 \ge 2\sqrt{2}\, 2^x$

$2^{2x} - 2\sqrt{2}\ 2x + 1 \ge 0$

$+ \quad - \quad +$; $\sqrt{2}-1 \quad \sqrt{2}+1$

$2^x \le \sqrt{2} - 1 \Rightarrow x \le \log 2(\sqrt{2} - 1)$

$2^x \ge \sqrt{2} + 1 \Rightarrow x > \log 2(\sqrt{2} + 1)$

Let $x \ge 0$

$2x + 2x \ge 2\sqrt{2} \Rightarrow 2.2x \ge 2\sqrt{2} \Rightarrow x \ge 1/2$

Hence, the solution is $x \in (-\infty, \log^2(\sqrt{2} - 1)] \cup [1/2, \infty)$

EXAMPLE 1.6 Solve $2^{2x} - 2x + 2 + 4.3^x \ge 6^x$

SOLUTION $2^x(2^x - 4) + 3^x (4 - 2^x) \ge 0$

$(2^x - 4)(2^x - 3^x) \ge 0$

The sign scheme of the expression is:

$- \quad + \quad -$; $0 \quad 2$

Thus the solution is $0 \le x \le 2$.

Alternative:

Case 1: $2^x - 4 \ge 0$ and $2^x - 3^x \ge 0$

$x \ge 2$ $x \le 0$

There is no solution in this case.

Case 2: $2^x - 4 \le 0$ and $2^x - 3^x \le 0$

$x \le 2$ $x \ge 0$

$\Rightarrow 0 \le x \le 2$.

Hence, the solution is $x \in [0, 2]$.

EXAMPLE 1.7 Solve the inequation

$$4^x \le 3.2^{x+\sqrt{x}} + 4^{1+\sqrt{x}}.$$

SOLUTION The inequation is defined if $x \ge 0$

Writing the inequation as $2^{2x} \le 3.2^{x+\sqrt{x}} + 4.2^{2\sqrt{x}}$ and dividing by 2^{2x} we get

$1 \le 3.2^{\sqrt{x}-x} + 4.2^{2(\sqrt{x}-x)}$

On putting $t = 2^{\sqrt{x}-x}$ we get $4t^2 + 3t - 1 \ge 0$

$\Rightarrow$ $(4t-1)(t+1) \geq 0$

$\Rightarrow$ $t \geq \dfrac{1}{4}$ $\quad (\because\ t+1>0)$

$\Rightarrow$ $2^{\sqrt{2}-2} \geq 2^{-2}$

$\Rightarrow$ $\sqrt{x} - x \geq -2$ ($\because$ 2α increases as α increases)

On putting $\sqrt{x} = u$, we get $u^2 - u - 2 \leq 0$

$\Rightarrow$ $(u-2)(u+1) \leq 0$

$\Rightarrow$ $u \leq 2$ $\quad (\sqrt{x}\ u = \sqrt{x} \geq 0)$

$\Rightarrow$ $\sqrt{x} \leq 2 \quad \Rightarrow \quad x \leq 4$

$\Rightarrow$ Solution set is $[0, 4]$.

EXAMPLE 1.8 Solve $|x|^{x^2-x-2} \geq 1$.

SOLUTION $|x|^{x^2-x-2} \geq |x|^0$

Case 1: $0 < |x| < 1 \Rightarrow -1 < x < 1,\ x \neq 0$

$x^2 - x - 2 \leq 0$.

$x^2 + x - 2x - 2 \leq 0$

$+ \quad - \quad +$; $-1 \quad 2$

$x(x+1) - 2(x+1) \leq 0$

$x \Rightarrow [-1, 2]$

Under the given conditions $-1 < x < 1,\ x \neq 0$.

Case 2: $|x| > 1 \Rightarrow x < -1$ and $x > 1$

$x^2 - x - 2 \geq 0$

$+ \quad - \quad +$; $-1 \quad 2$

$x \leq -1,\ x \geq 2$

Under the given conditions $x < -1,\ x \geq 2$

Case 3: $|x| = 1$

$x = \pm 1$, the inequation holds true.

Finally, $x \in (-\infty, 1] \cup [2, \infty) - \{0\}$.

PRACTICE PROBLEMS [J]

Solve the following inequations:

1. $0.1^{4x^2-2x-2} \leq 0.1^{2x-3}$
2. $x^2 \cdot 5^x - 5^{2+x} < 0$.
3. $\left(\dfrac{1}{3}\right)^{\frac{|x|+1}{x-2}} > 9$
4. $4^x + 2^{x+1} - 6 \leq 0$
5. $\left(\dfrac{1}{3}\right)^{\sqrt{x+4}} > \left(\dfrac{1}{3}\right)^{\sqrt{x^2++3x+4}}$
6. $\dfrac{2^{1-x} - 2^x + 1}{2^x - 1} \leq 0$
7. $9^{\sqrt{x^2-3}} + 3 < 3^{\sqrt{x^2-3}-1} \cdot 28$
8. $\sqrt{9^x + 3^x - 2} \geq 9 - 3^x$
9. $3^{\sqrt{x}} > 2^a$
10. $8 - x.2^x + 2^{3-x} - x > 0$.
11. $|x|^{x^2-x-2} < 1$
12. $3^{x^2-x} < 2^{1-(\sqrt{x})^2}$.

1.15 | LOGARITHMIC EQUATION

1. Consider logarithmic equations of the form

$\log_a f(x) = \log_a g(x)$, where $a > 0$ and $a \neq 1$. (1)

The equation loga f(x) = logag(x) is equivalent to the system:

$$\begin{cases} f(x) = g(x) \\ f(x) > 0 \\ g(x) > 0 \end{cases} \quad (2)$$

Note that for solving equation (1) it is not necessary to solve system (2). We may proceed in a different way, namely, to solve the equation f(x) = g(x) and from the found solutions to choose those which satisfy the system of inequalities

$\begin{cases} f(x) > 0 \\ g(x) > 0, \end{cases}$ that is, those which belong to the domain of equation (1).

2. Now, consider equations of the form

$\log_{a(x)} f(x) = \log_{a(x)} g(x)$.

It is equivalent to the system:

$$\begin{cases} f(x) = g(x) \\ f(x) > 0 \\ g(x) > 0 \\ a(x) > 0 \\ a(x) \neq 1 \end{cases}.$$

In other words, the roots of equation are represented by those and only those roots of the equation f(x) = g(x) which simultaneously satisfy the conditions :

$f(x) > 0$, $g(x) > 0$, $a(x) > 0$, $a(x) \neq 1$

(these conditions specify the domain of equation).

If the given equation includes logarithms to different bases, it is first necessary to reduce all the logarithms to the same base.

3. When solving logarithmic equations, we use the properties of logarithms. Consider, for instance, the equation $\log_a f(x) + \log_a g(x) = \log_a h(x)$ ...(1)

It is transformed to:

$\log_a (f(x)\, g(x)) = \log_a h(x)$. ...(2)

but Equations (1) and (2) may be non-equivalent. Indeed, the domain of the expression

$\log_a f(x) + \log_a g(x)$

is given by the system of inequalities $\begin{cases} f(x) > 0 \\ g(x) > 0 \end{cases}$,

whereas the domain of the expression $\log_a f(x)\, g(x))$ is specified by the inequality $f(x)\, g(x) > 0$ which is, in turn, equivalent to the collection of systems of inequalities:

$$\begin{cases} f(x) > 0 \\ g(x) > 0 \end{cases}; \qquad \begin{cases} f(x) < 0 \\ g(x) < 0 \end{cases}$$

Thus, when passing from equation (1) to equation (2), we can encounter an extension of the domain of equation (1) (at the expense of the solutions of the last system of inequalities), and, hence, extraneous roots may appear. Therefore, on solving equation (2), we have to choose those of its roots which belong to the domain of the original equation (1), that is, which satisfy the system of inequalities $\begin{cases} f(x) > 0 \\ g(x) > 0 \\ h(x) > 0 \end{cases}$. This check is an essential part of the solution of a logarithmic equation.

It is clear that the check may also be realised by a direct substitution of the found solutions into the original equation.

EXAMPLE 1.1 Find the sum of all solutions of the equation

$$3^{(\log_9 x)^2 - \frac{9}{2}\log_9 x + 5} = 3\sqrt{3}.$$

SOLUTION We have

$$(\log_9 x)^2 - \frac{9}{2}\log_9 x + 5 = \frac{3}{2}$$

Put $\log_9 x = y$

$$\Rightarrow y^2 - \frac{9}{2}y + 5 = \frac{3}{2}$$

$$\Rightarrow 2y^2 - 9y + 10 = 3$$

$\Rightarrow 2y^2 - 9y + 7 = 0$

$\Rightarrow (2y - 7)(y - 1) = 0$

$\Rightarrow y = \frac{7}{2}, 1.$

$\Rightarrow \log_9 x = 1$ or $\log_9 x = \frac{7}{2}$

$\Rightarrow x = 9$ or $x = 9^{7/2} = 3^7$

$\Rightarrow$ The sum of all solutions = 9 + 2187 = 2196.

EXAMPLE 1.2 Find the number of real roots of the equation $\log_{10}(7x-9)^2 + \log_{10}(3x-4)^2 = 2.$

SOLUTION $\log_{10}(7x-9)^2 + \log_{10}(3x-4)^2 = 2.$

$\log_{10}[(7x-9)^2 . (3x-4)^2] = 2$

$(7x - 9)(3x - 4) = +10$

Taking positive sign,

$21x^2 - 55x + 26 = 0 \quad \Rightarrow \quad x = 13/21, 2$

Taking negative sign $21x^2 - 55x + 46 = 0$, $D < 0$ (imaginary roots). Hence, the number of real roots of the equation is 2.

EXAMPLE 1.3 Solve that equation

$$\frac{\log(\sqrt{x+1} + 1)}{\log \sqrt[3]{x - 40}} = 3.$$

SOLUTION We have

$\log(\sqrt{x+1} + 1) = \log(x - 40).$

Putting $\sqrt{x+1} = t$ and raising we get the equation

$t^2 - t - 42 = 0,$

whose roots are $t_1 = 7$ and $t_2 = -6$.

Since $t = \sqrt{x+1} \geq 0$, the root t_2 is discarded. The value of x corresponding to the root t_1 is equal to 48. By substitution we check that it satisfies the original equation. Thus, the equation has the unique root x = 48.

EXAMPLE 1.4 Solve the equation $x + \log_{10}(1 + 2^x)$

$= x \log_{10} 5 + \log_{10} 6.$

SOLUTION $x + \log_{10}(1 + 2^x)$

$= x \log_{10} 5 + \log_{10} 6.$

$\Rightarrow x + \log_{10}(1 + 2^x) - x \log_{10} 5 = \log_{10} 6$

$\Rightarrow x \log_{10} 10 + \log_{10}(1 + 2^x) - x \log_{10} 5$

$= \log_{10} 6$

$\Rightarrow \log_{10} 10^x + \log_{10}(1 + 2^x) - \log_{10} 5^x$

$= \log_{10} 6$

$\Rightarrow \log_{10} \frac{10^x(1+2^x)}{5^x} = \log_{10} 6$

$\Rightarrow 2^x(1 + 2^x) = 6 \quad \Rightarrow \quad (2^x)^2 + 2x - 6 = 0$

Let $2^x = y$

$\Rightarrow y^2 + y - 6 = 0 \quad \Rightarrow \quad (y + 3)(y - 2) = 0$

$\Rightarrow y = -3$ or $y = 2 \quad \Rightarrow \quad 2x = -3$ or $2^x = 2$

But $2^x > 0$ for all values of x. So $2^x = -3$ is impossible. Now from $2^x = 2$

x = 1 is the solution of the equation.

EXAMPLE 1.5 Solve $\log_{(2x+3)}(6x^2+23x+21)$

$= 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$

SOLUTION Here, $\log_{(2x+3)}(2x + 3)(3x + 7)$

$= 4 - \log_{(3x+7)}(2x+3)2$

$\Rightarrow \log_{(2x+3)}(2x + 3) + \log_{(2x+3)}(3x + 7)$

$= 4 - 2\log_{(3x+7)}(2x + 3)$

$\Rightarrow 1 + \log_{(2x+3)}(3x + 7) = 4$

$- 2\log_{(3x+7)}(2x+3)$

Let $\log_{(2x+3)}(3x + 7) = y$

$\Rightarrow \log_{(3x+7)}(2x + 3) = \frac{1}{y}$

$\Rightarrow 1 + y = 4 - \frac{2}{y} \quad \Rightarrow \quad y^2 - 3y + 2 = 0$

$\Rightarrow (y - 2)(y - 1) = 0 \quad \Rightarrow \quad y = 1$ or 2

Now $\log_{(2x+3)}(3x + 7)$

$= 1$ or, $\log_{(2x+3)}(3x + 7) = 2$

$\Rightarrow 3x + 7 = 2x + 3$ or $3x + 7 = (2x + 3)^2$

$\Rightarrow x = -4 \quad \Rightarrow \quad 4x^2 + 9x + 2 = 0$

$\Rightarrow (4x + 1)(x + 2) = 0$

$\Rightarrow x = -\frac{1}{4}$, or -2.

But for the logarithm to be defined

$2x + 3 > 0,\ 3x + 7 > 0,\ 2x + 3 \neq 1$

and $3x + 7 \neq 1$

Now for $x = -4$

$2x + 3 = 2(-4) + 3 = -5 < 0$,

hence $x = -4$ is not solution.

Now for $x = -2$

$2x + 3 = 2(-2) + 3 = -1 < 0$,

hence $x = -2$ is not solution.

Now for $x = -\frac{1}{4}$

$$2x + 3 = 2\left(-\frac{1}{4}\right) + 3 = \frac{5}{2} > 0 \text{ and } \neq 1,$$

$$3x + 7 = 3\left(-\frac{1}{4}\right) + 7 = \frac{25}{4} > 0 \text{ and } \neq 1,$$

Therefore, $x = -\frac{1}{4}$ is the only solution.

EXAMPLE 1.6 Solve the equation

$$\sqrt{1 + \log_x \sqrt{27}}\ \log_3 x + 1 = 0.$$

SOLUTION $\left(\sqrt{1 + \frac{3}{2\log_3 x}}\right)\log_3 x + 1 = 0$

Let $\log_3 x = y$

$$\Rightarrow \left(\sqrt{1 + \frac{3}{2y}}\right)y = -1 \qquad \ldots(1)$$

$$\Rightarrow \left(1 + \frac{3}{2y}\right) = \frac{1}{y^2} \Rightarrow \frac{2y+3}{2y} = \frac{1}{y^2}$$

$\Rightarrow 2y^2 + 3y - 2 = 0 \Rightarrow 2y^2 + 4y - y - 2 = 0$

$\Rightarrow (y + 2)(2y - 1) = 0$

$\Rightarrow y = 1/2$ or $y = -2$

From (1), y is negative.

Hence, $y = 1/2 \Rightarrow x = 1/9$.

EXAMPLE 1.7 Find the value of x that satisfies the equation $\log\left(\frac{x^{1/x}}{x^{1/(x+1)}}\right) = \frac{1}{5050}$, where base of the logarithm is 10.

SOLUTION $\log_{10} x^{\left(\frac{1}{x} - \frac{1}{x+1}\right)} = \frac{1}{5050}$

$$\Rightarrow x^{\frac{1}{x(x+1)}} = 10^{\frac{1}{5050}}$$

$$\Rightarrow \log 10\, x^{\frac{1}{x(x+1)}} = \frac{1}{5050} \Rightarrow x = 100 \text{ (by trial).}$$

EXAMPLE 1.8 It is known that $x = 9$ is a root of the equation $\log_\pi (x^2 + 15a^2) - \log_\pi(a - 2) = \log_\pi \frac{8ax}{a-2}$. Find the other root(s) of this equation.

SOLUTION $\log_\pi \frac{x^2 + 15a^2}{a-2} = \log_\pi \frac{8ax}{a-2} \qquad (1)$

$\Rightarrow x^2 + 15a^2 = 8ax$

$x^2 - 8ax + 15a^2 = 0 \qquad (2)$

$(x - 5a)(x - 3a) = 0 \Rightarrow a = \frac{x}{5}$ or $a = \frac{x}{3}$

$\therefore x = 9$ satisfies (1) hence $a = \frac{9}{5}$ or $a = 3$

But $a = \frac{9}{5}$ is not possible since $a > 2$.

$\Rightarrow a = 3$

Substituting $a = 3$ in (2),

$x^2 - 24x + 225 = 0 \Rightarrow x = 9$ or $x = 15$

$\Rightarrow$ Hence, the other root is $x = 15$.

EXAMPLE 1.9 Solve the equation

$\log^2(4 - x) + \log(4 - x) \cdot \log\left(x + \frac{1}{2}\right) - 2\log^2\left(x + \frac{1}{2}\right) = 0.$

SOLUTION

$$\log^2(4 - x) + \log(4 - x) \cdot \log\left(x + \frac{1}{2}\right)$$

$$-2\log^2\left(x + \frac{1}{2}\right) = 0$$

Let $\log(4 - x) = A$ and $\log\left(x + \frac{1}{2}\right) = B$

$\Rightarrow A^2 + AB - 2B^2 = 0$

$\Rightarrow A^2 + 2AB - AB - 2B^2 = 0$

$\Rightarrow$ A(A + 2B) – B(A + 2B) = 0

$\Rightarrow$ A = B or A = – 2B

$\therefore \log(4 - x) = \log\left(x + \frac{1}{2}\right)$

$\therefore 4 - x = x + \frac{1}{2} \Rightarrow 2x = \frac{7}{4}$

$\Rightarrow x = \frac{7}{4}$.

Now $\log(4 - x) = -2\log\left(x + \frac{1}{2}\right)$

$\Rightarrow 4 - x = \frac{1}{(x + (1/2))^2}$

$\Rightarrow (4 - x)(x^2 + \frac{1}{4} + x) = 1$

$\Rightarrow 4x^2 - x^3 + 1 - \frac{x}{4} + 4x - x^2 - 1 = 0$

$\Rightarrow x^3 - 3x^2 - \frac{15x}{4} = 0$

$\Rightarrow x(4x^2 - 12x - 15) = 0$

$\Rightarrow x = 0$ or $4x^2 - 12x - 15 = 0$

$$x = \frac{12 \pm \sqrt{144 + 240}}{8} = \frac{12 \pm \sqrt{384}}{8}$$

$$= \frac{12 \pm 4\sqrt{24}}{8} = \frac{3 \pm \sqrt{24}}{2}$$

Rejecting $x = \frac{3 - \sqrt{24}}{2}$, we get

$$x = \frac{3 + \sqrt{24}}{2}.$$

$$\therefore \quad x = \left\{0, \frac{7}{4}, \frac{3 + \sqrt{24}}{2}\right\}.$$

EXAMPLE 1.10 Find the real numbers x which satisfy the equation

$$2\log_2\log_2 x + \log_{1/2}\log_2\left(2\sqrt{2}x\right) = 1$$

SOLUTION $2\log_2\log_2 x + \log_{1/2}\log_2\left(2\sqrt{2}x\right) = 1$

$\Rightarrow 2\log_2\log_2 x + \log_{(2)^{-1}}\log_2 2\sqrt{2}x = 1$

$\Rightarrow \log_2(\log_2 x)^2 - \log_2\log_2 2\sqrt{2}x = 1$

$\Rightarrow \log_2 \frac{(\log_2 x)^2}{\log_2 2\sqrt{2}x} = 1$

$\Rightarrow \frac{(\log_2 x)^2}{\log_2 2\sqrt{2}x} = 2$

$\Rightarrow (\log_2 x)^2 = 2\log_2 2\sqrt{2}x$

$\Rightarrow (\log_2 x)^2 = 2(\log_2 2\sqrt{2} + \log_2 x)$

$\Rightarrow (\log_2 x)^2 = 2\left(\frac{3}{2} + \log_2 x\right)$

$\Rightarrow (\log_2 x)^2 = 3 + 2\log_2 x$

Put $\log_2 x = y$, then

$\Rightarrow y^2 - 2y - 3 = 0$

$\Rightarrow (y - 3)(y + 1) = 1$

$\Rightarrow y = -1$ or 3

$\Rightarrow \log_2 x = -1$ or $\log_2 x = 3$

$\Rightarrow x = \frac{1}{2}$ or $x = 8$

Hence $x = \frac{1}{2}$, 8 are the solution of the given equation.

EXAMPLE 1.11 Solve the system of equations:

$$5(\log xy + \log_y x) = 26 \text{ and } xy = 64.$$

SOLUTION Let $\log_x y = t$.

Then the first equation becomes

$$5\left(t + \frac{1}{t}\right) = 26$$

$\Rightarrow 5t^2 - 26t + 5 = 0$

$\Rightarrow 5t^2 - 25t - t + 5 = 0$

$\Rightarrow 5t(t - 5) - (t - 5) = 0$

$\Rightarrow (t - 5)(5t - 1) = 0$

Hence $t = 5$ or $t = 1/5$

Now $\log_x y = 5$

$\Rightarrow x^5 = y = \frac{64}{x}$ (using second equation)

$\Rightarrow x^6 = 64$

$\Rightarrow x = 2$ and $y = 32$

When $\log_x y = 1/5$

$$y = x^{1/5} \Rightarrow x = y^5 \Rightarrow \frac{64}{y} = y^5$$

$\Rightarrow y^6 = 64 \Rightarrow y = 2$ and $x = 32$

Hence the solutions are (32, 2) or (2, 32).

EXAMPLE 1.12 Solve the equation

$$1 + \log_x \frac{4-x}{10} = (\log\log n - 1)\log_x 10.$$

How many roots has the equation for a given value of n ?

SOLUTION Passing over to logarithms to the base 10 we obtain

$$1 + \frac{\log\left(\frac{4-x}{10}\right)}{\log x} = (\log\log n - 1)\frac{1}{\log x}$$

After simple transformations this leads to the equation

$$\log\left(x \cdot \frac{4-x}{10}\right) = \log\frac{\log n}{10}.$$

Taking antilogarithms we obtain

$$x^2 - 4x + \log n = 0,$$

$$x_{1,2} = \pm\sqrt{4 - \log n}.$$

A simple argument now leads to the following results

(a) If $0 < n < 10^4$ and $n \neq 10^3$, the equation has two different roots, namely

$x_1 = +\sqrt{4-\log n}$ and $x_2 = -\sqrt{4 - \log n}$

(b) If $n = 10^3$, there is only one root $x = 3$ ($x = 1$ should be discarded) ; for $n = 10^4$ we also get one root $x = 2$.

(c) If $n > 10^4$ there are no roots.

EXAMPLE 1.13 Find the real solutions to the system of equations

$$\log_{10}(2000xy) - \log_{10}x \cdot \log_{10}y = 4$$

$$\log_{10}(2yz) - \log_{10}y \cdot \log_{10}z = 1$$

and $\log_{10}(zx) - \log_{10}z \cdot \log_{10}x = 0$

SOLUTION From the first equation,

$$3 + \log_{10}(2xy) - \log_{10}x \cdot \log_{10}y = 4 \quad ...(1)$$

$$\log_{10}(xy) - \log_{10}x \cdot \log_{10}y = 1 - \log_{10}(2)$$

From the second equation,

$$\log_{10}(yz) - \log_{10}y \cdot \log_{10}z = 1 - \log_{10}(2)$$

Hence, $\log x + \log y - \log x \cdot \log y$

$= \log y + \log z - \log y \cdot \log z$

$\Rightarrow \log x\,(1 - \log y) = \log z\,(1 - \log y)$

$\Rightarrow (\log x - \log z)(1 - \log y) = 0$

Hence either,

$\log x = \log z$ or $\log_{10}y = 1$

$\Rightarrow y = 10$

But $y = 10$ does not satisfy equation (1) hence rejected.

$\therefore \log x = \log z$

From the third equation,

$(\log_{10}x)^2 = 2(\log_{10}x)$

$\Rightarrow \log_{10}x\,[\log_{10}x - 2] = 0$

$\Rightarrow x = 1$ or $x = 100$

If $x = z = 1$ then $y = 5$

If $x = z = 100$ then $y = 20$.

Hence, the solutions are $x = 1, y = 5, z = 1$

or $x = 100, y = 20, z = 100$.

EXAMPLE 1.14 If the solutions of the system of equations

$$\log x + \frac{\log(xy^8)}{\log^2 x + \log^2 y} = 2$$

$$\log y + \frac{\log(x^{8/y})}{\log^2 x + \log^2 y} = 0$$

are (x_1, y_1) and (x_2, y_2), (the logarithms are taken to base 10), then find the value of $y_1 y_2$.

SOLUTION Let $u = \log x$, $v = \log y$

$$u + \frac{u + 8v}{u^2 + v^2} = 2 \quad ...(1)$$

$$v + \frac{8u - v}{u^2 + v^2} = 0 \quad ...(2)$$

$(1) \times u + (2) \times (v)$

$\Rightarrow 2uv + 8 = 2v \quad \Rightarrow \quad u = 1 - \dfrac{u}{v}$

Put this in (2) to get $v^2 - 16 = 0$

$v = \pm 4 \quad \Rightarrow \quad \log y = \pm 4$

$\Rightarrow y = 10^4, 10^{-4}$

$y_1 y_2 = 1.$

EXAMPLE 1.15 Solve

$$(2^{\log_5 x} + 3)^{\log_5 2} = x - 3.$$

SOLUTION Let $f(x) = 2^{\log_5 x} + 3$

The given equation is $f(f(x)) = x$.

Since f is increasing, if $f(x) < x$ then $f(f(x)) < f(x)$

If $f(x) > x$ then $f(f(x)) > f(x)$.

Since for $f(f(x)) = x$ to be have, we must have $f(x) = x$.

Let $t = \log_5 x$

$2^{\log_5 x} + 3 = x$

$2t + 3 = 5t$

Using trial, the solution is $t = 1$.

Hence, $x = 5$.

EXAMPLE 1.16 Solve for x : $\log_2(x - 1)$

$$- \log_{\sqrt{2}} \sqrt{x+3}$$

$$= \log_8(x - a)^3 + \log_{\frac{1}{2}}(x - 3).$$

SOLUTION Domain : $x - 1 > 0, \Rightarrow \quad x > 1$

$\sqrt{x+3} > 0 \quad \Rightarrow \quad x > -3$

$(x - a)^3 > 0 \quad \Rightarrow \quad x > a$

$x - 3 > 0 \quad \Rightarrow \quad x > 3$

Finally, $x > 3$ and $x > a$.

$\log_2(x - 1) - \log_2(x + 3) = \log_2(x - a) - \log_2(x - 3)$

$$\log_2 \frac{(x-1)}{(x+3)} = \log_2 \frac{(x-a)}{(x-3)}$$

$x_2 - x - 3x + 3 = x^2 - ax + 3x - 3a$

$x(7 - a) = 3 + 3a$

$\Rightarrow x = \dfrac{3+3a}{7-a}.$

Now we check for domain $\dfrac{3+3a}{7-a} > 3,$

$\Rightarrow \dfrac{3+3a}{7-a} - 3 > 0$

$\dfrac{3+3a-21+3a}{7-a} > 0 \quad \Rightarrow \quad a \in (3, 7).$

Also, $\dfrac{3+3a}{7-a} > a \quad \Rightarrow \quad \dfrac{3+3a}{7-a} - a > 0$

$\dfrac{3+3a-7a+a^2}{7-a} > 0$

$\Rightarrow \dfrac{a^2 - 4a + 3}{a - 7} < 0 \quad \dfrac{(a-3)(a-1)}{a-7} < 0$

$$\underset{}{-}\ \ 1\ \ \underset{}{+}\ \ 3\ \ \underset{}{-}\ \ 7\ \ \underset{}{+}$$

$a \in (-\infty, 1) \cup (3, 7).$

Finally, $x = \dfrac{3(1+a)}{7-a}$ if $a \in (3, 7).$

EXAMPLE 1.17 Locate the real root of the equation $2 - x - \ln x = 0$.

SOLUTION We write the left member of the equation in the following way :

$$f(x) = (2 - x) + (-\ln x),$$

We see that the function f(x) is a sum of two

strictly decreasing functions, and therefore it is also strictly decreasing. Consequently, the given equation has a single root.

Now, $f(1) > 0$, $f(2) < 0$

This shows that the root lies in the interval (1, 2).

EXAMPLE 1.18 Solve graphically the equation

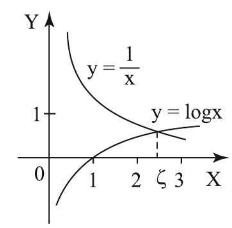
$$x \ln x - 1 = 0.$$

SOLUTION Let us rewrite the equations in the form

$$\ln x = \frac{1}{x}$$

Here $y = \ln x,\ y = \frac{1}{x}$.

Constructing the graphs $y = \ln x$ and $y = \frac{1}{x}$ (See figure), we find the approximate value of the only root $\zeta \approx 2.5$.



PRACTICE PROBLEMS [K]

1. $\frac{1}{10}\log\sqrt[3]{x^2 - 4x + 4} - \frac{1}{2}\log x - \log\frac{1}{\sqrt{x}} = 0$
2. $|1 - \log_{1/5} x| + 2 = |3 - \log_{1/5} x|$
3. $x^{\log x + 5} = 10^{15 + 3\log x}$
4. $\frac{x^{\frac{1}{2}\log_2 x}}{4} = 2^{\frac{\log_2^2 x}{4}}$
5. $9^{\log_3(\log_2 x)} = \log_2 x - (\log_2 x)^2 + 1$
6. $\log_{10}^2 x + \log_{10} x^2 = \log_{10}^2 2 - 1$
7. $3^x = 10 - \log_2 x$.
8. $(1 + x/2)\log_2 3 - \log_2(3^x - 13) = 3\log_{\sqrt{5}/25} 5 + 4$
9. $\log_{\frac{1}{2+|x|}}(5 + x^2) = \log_{3+x^2}(15 + \sqrt{x})$
10. $20\log_{4x}\sqrt{x} + 7\log_{16x} x^3 - 3\log_{x/2} x^2 = 0$
11. $\log_2\log_3(x^2 - 16) - \log_{1/2}\log_{1/3}\frac{1}{x^2 - 16} = 2$
12. $|\log_2(3x - 1) - \log_2 3| = |\log_2(5 - 2x) - 1|$
13. $|3 - x|^{\log^2 x - \log x^2} = |3 - x|^3$

Solve the following equations:

14. $|x - 1|^{(\ell n|x|)^2 - \ell n x^2} = |x - 1|^3$.

15. $20 \log_{4x} \sqrt{x} + 7\log_{16x} x^3 - 3 \log_{x/2} x^2 = 0$
16. $(x + 1) \log_3^2 x + 4x \log_3 x - 16 = 0$
17. $(\log_{1/\sqrt{1+x}} 10) \log_{10}(x^2 - 3x + 2) = (\log_{10}(x - 3)) \log_{1/\sqrt{1+x}} 10 - 2$
18. $\log_x(x^2 + 1) = \sqrt{\log_{\sqrt{x}}(x^2(1+x^2)) + 4}$;
19. $\log_4\left(6+\sqrt{x}-\left|\sqrt{x}-2\right|\right) = \frac{1}{2} + \log_2\left|\sqrt{x}-\left|\sqrt{x}-2\right|\right|$
20. Show that the equation $\log_2(2x^2) + \log_2 x \,.\, x^{\log_x(\log_2 x+1)} + \frac{1}{2}\log_4{}^2(x^4) + 2^{-3\log_{1/2}(\log_2 x)} = 1$ has no solution.
21. α and β are roots of
$\log^2\left(1+\frac{4}{x}\right) + \log^2\left(1-\frac{4}{x+4}\right) = 2\log^2\left(\frac{2}{x-1}-1\right)$, then find the value of $\alpha^2 + \beta^2$.
22. Solve the system of equations:
$\log_{12}x\left(\frac{1}{\log_x 2} + \log_2 y\right) = \log_2 x$ and $\log_2 x \,.\, (\log_3(x + y)) = 3 \log_3 x$
23. Find the sum of the roots of the equation $(x + 1) = 2 \log_2(2^x + 3) - 2 \log_4(1980 - 2^{-x})$.
24. Prove that solution of the equation, $2\log_9\left(2\left(\frac{1}{2}\right)^x - 1\right) = \log_{27}\left(\left(\frac{1}{4}\right)^x - 4\right)^3$ is an irrational number.
25. Find the sum of all integral solution of the equation, $4\log_{x/2}(\sqrt{x}) + 2\log_{4x}(x^2) = 3\log_{2x}(x^3)$.

1.16 | LOGARITHMIC INEQUALITIES

The nature of logarithmic function is dependent on the base. Let us consider an equality:

$\log_a x > y$

What should we conclude: $x > a^y$ or $x < a^y$? It depends on the value of a. We can understand the same by considering LHS of the inequality equal to an exponent z:

$\log_a x = z$

If $a > 1$, then a^z will yield a greater x than a^y, because $z > y$ (it is given by the inequality). On the other hand, if $0 < a < 1$, then a^z will yield a smaller x than a^y, because $z > y$. We can understand this conclusion with the help of an example.

Let $\log_2 x > 3$

Let $\log_2 x = 4$

Clearly, $2^4 > 2^3$ as $16 > 8$

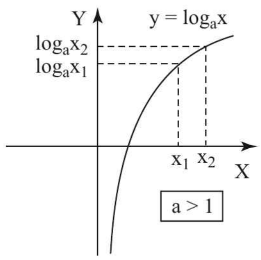
Let us now consider $a < 1$,

$\log_{0.5} x = 4$

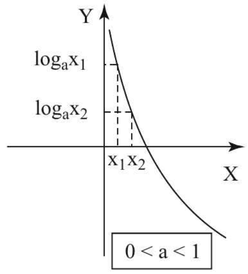
Clearly, $0.5^4 < 0.5^3$ as $0.0625 < 0.125$. Thus, we finally conclude:

$\log_a x > y \Rightarrow x > a^y;\ a > 1$

$\log_a x > y \Rightarrow x < a^y;\ 0 > a < 1$



If $0 < x_1 < x_2$ then
$\log_a x_1 < \log_a x_2$



If $0 < x_1 < x_2$ then
$\log_a x_1 > \log_a x_2$

Consider the following elementary logarithmic inequalities

(i) If $a > 1$, then the inequalities $0 < x_1 < x_2$ and $\log_a x_1 < \log_a x_2$ are equivalent

(ii) If $0 < a < 1$, then the inequalities $0 < x_1 < x_2$ and $\log_a x_1 < \log_a x_2$ are equivalent.

(iii) If $a > 1$, then the inequalities $\log_a x < k$ and $0 < x < a^k$

(iv) If $a > 1$, then the inequalities $\log_a x > k$ and $x > a^k$ are equivalent.

(v) If $0 < a < 1$, then the inequalities $\log_a x < k$ and $x > a^k$ are equivalent.

(vi) If $0 < a < 1$, then the inequalities $\log_a x > k$ and $0 < x < a^k$ are equivalent.

An inequality of the form $\log_a f(x) > b$ is equivalent to the following systems of inequalities:

(i) $f(x) > 0,\ f(x) > a^b$ for $a > 1$

(ii) $f(x) > 0.\ f(x) < a^b$ for $a < 1$,

and an inequality of the form $\log_a f(x) < b$ is equivalent to the following systems of inequalities:

(i) $f(x) > 0,\ f(x) < a^b$ for $a > 1$;

(ii) $f(x) > 0,\ f(x) > a^b$ for $0 < a < 1$.

EXAMPLE 1.1 Solve $\log_2(x^2 - 1) < 1$.

SOLUTION Domain

$x^2 - 1 > 0 \Rightarrow x < -1,\ x > 1$ ...(1)

Taking antilog, $x^2 - 1 < 2$

$\Rightarrow x^2 - 3 < 0,\ -\sqrt{3} < x < \sqrt{3}$ ...(2)

Taking intersection of (1) and (2), we get

$x \in (-\sqrt{3}, -1) \cup (1, \sqrt{3})$

EXAMPLE 1.2 Solve

$$\log_{\frac{1}{4}}(2 - x) > \log_{\frac{1}{4}}\left(\frac{2}{x+1}\right).$$

SOLUTION Domain: $2 - x > 0 \Rightarrow x < 2$

$$\frac{2}{x+1} > 0 \Rightarrow x > -1$$

Hence, $-1 < x < 2$ ...(1)

$$2 - x < \frac{2}{x+1}$$

$$\frac{2}{x+1} + x - 2 > 0$$

$$\frac{2 + x^2 - 2x + x - 2}{x+1} > 0$$

$$\Rightarrow \frac{x(x-1)}{x+1} > 0$$

$$\begin{array}{ccccccc} - & & + & & - & & + \\ & -1 & & 0 & & 1 & \end{array}$$

$x \in (-1, 0) \cup (1, \infty)$ ...(2)

Taking intersection of (1) and (2), we get

$x \in (-1, 0) \cup (1, 2)$

EXAMPLE 1.3 Solve $\log_{\frac{1}{3}}(\log_4(x^2 - 5)) > 0$

SOLUTION Domain: $x^2 - 5 > 0$

$x^2 > 5 \Rightarrow x \in (-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$ ...(1)

Also, $\log_4(x^2 - 5) > 0 \Rightarrow x^2 - 5 > 1 \Rightarrow x^2 > 6$

$\Rightarrow x \in (-\infty, -\sqrt{6}) \cup (\sqrt{6}, \infty)$ ...(2)

Now the inequality reduces to

$\log_4(x^2 - 5) < 1$

$x^2 - 5 < 4$

$x^2 < 9 \Rightarrow x \in (-3, 3)$ ...(3)

Taking intersection of (1),(2) and (3), we get

$x \in (-3, -\sqrt{6}) \cup (\sqrt{6}, 3)$

EXAMPLE 1.4 Find the value of x for which

$$\frac{(\log_3 x - 2)(\log_{1/2} x - 2)}{(\log_4 x - 1)} > 0$$

SOLUTION

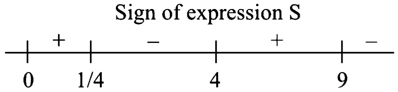
Let $S = \dfrac{(\log_3 x - 2)(\log_{1/2} x - 2)}{(\log_4 x - 1)} > 0$

The factor $\log_3 x - 2$ becomes zero at $x = 9$. Similarly the other two factors become zero at $x = 1/4$ and $x = 4$. Note that we must consider only positive values of x.

We now find the sign scheme of the expression by using some test points inside the various sub-intervals.

We also use the fact that logax > 0 for x > 1 if a > 1 and $\log_a x > 0$ for $0 < x < 1$ if $0 < a < 1$.

Sign of expression S



$\Rightarrow\ 0 < x < 1/4$ or $4 < x < 9$

$\Rightarrow\ x \in (0, 1/4) \cup (4, 9)$.

EXAMPLE 1.5 Solve $\log_x (2x - 3/4) < 2$.

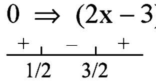
SOLUTION $2x - 3/4 > 0 \Rightarrow x > 3/8$ (1)

Case 1: $0 < x < 1$ (2)

Taking antilog,

$$2x - \frac{3}{4} < x^2 \Rightarrow x^2 - 2x + 3/4 > 0$$

$$4x^2 - 8x + 3 > 0 \Rightarrow (2x - 3)(2x - 1) > 0$$



$$x \in \left(+\infty, \frac{1}{2}\right) \cup \left(\frac{3}{2}, \infty\right) \qquad (3)$$

Taking intersection of (1), (2) and (3), we get

$$x \in \left(\frac{3}{8}, \frac{1}{2}\right) \qquad ...(4)$$

Case 2: $x > 1$ ...(5)

Taking antilog,

$$2x - 3/4 > x^2$$
$$4x^2 - 8x + 3 < 0$$

$$\frac{1}{2} < x < \frac{3}{2} \qquad (6)$$

Taking intersection of (1), (5) and (6), we get

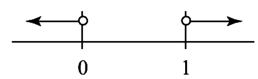
$$x \in \left(1, \frac{3}{2}\right) \qquad (7)$$

Taking union of (4) and (7), we get

$$x \in \left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$$

EXAMPLE 1.6 Find the solution set of inequality, $\log_{x+3}(x^2 - x) < 1$.

SOLUTION $x(x - 1) > 0$



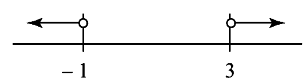
$\Rightarrow\ x < 0$ or $x > 1$ (1)

Case 1: $x + 3 > 1 \Rightarrow x > -2$ (2)

Taking antilog,

$\Rightarrow\ x^2 - x < x + 3 \Rightarrow x^2 - 2x - 3 < 0$

$\Rightarrow\ (x - 3)(x + 1) < 0$



$-1 < x < 3$. (3)

Taking intersection of (1), (2) and (3), we get

$x \in (-1, 0) \cup (1, 3)$. (4)

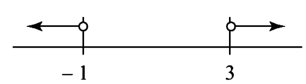
Case 2: $0 < x + 3 < 1$

$\Rightarrow\ -3 < x < -2$ (5)

Taking antilog,

$x^2 - x > x + 3 \Rightarrow x^2 - 2x - 3 > 0$

$\Rightarrow\ (x - 3)(x + 1) > 0$



$\Rightarrow\ x < -1,\ x > 3$ (6)

Taking intersection of (1), (5) and (6), we get

$x \in (-3, -2)$. (7)

Taking union of (4) and (7), we get

$x \in (-3, -2) \cup (-1, 0) \cup (1, 3)$.

EXAMPLE 1.7 Solve the inequality

$\log_{x-2}(2x-3) > \log_{x-2}(24-6x)$.

SOLUTION Since we do not know whether the base $(x-2)$ is greater or less than 1, we have to consider two cases: (i) $x-2>1$ (ii) $0<x-2<1$.

Thus, the problem has been reduced to solving the following collection of two systems of inequalities

$$\begin{cases} x-2>1 \\ 2x-3>0 \\ 24-6x>0 \\ 2x-3>24-6x \end{cases};\quad \begin{cases} 0<x-2<1 \\ 2x-3>0 \\ 24-6x>0 \\ 2x-3<24-6x \end{cases}$$

From the first system we get $\frac{27}{8} < x < 4$, from the second we get $2 < x < 3$.

Thus, the solution is $(2, 3) \cup \left(\frac{27}{8}, 4\right)$.

EXAMPLE 1.8 Solve $\log_{|x|}|x-1| \ge 0$

SOLUTION The function $\log_{|x|}|x-1|$ is not defined for $|x| = 0, 1$.

Case 1: Let $0 < |x| < 1$

The given inequality reduces to

$|x-1| \le |x|^0 = 1$

$\Rightarrow -1 \le x-1 \le 1$

$\Rightarrow 0 \le x \le 2$

Intersection with $0 < |x| < 1$ gives

$0 \le x \le 1$. ...(1)

Case 2: Let $|x| > 1$

The given inequality reduces to

$|x-1| \ge |x|^0 = 1$

$\Rightarrow x-1 \le -1$ or $x-1 \ge 1$

$\Rightarrow x \le 0$ or $x \ge 2$

Intersection with $|x| > 1$ gives

$x \le -1$ or $x \ge 2$. ...(2)

Taking union of intervals (1) and (2), gives the required solution set as

$x \in (-\infty, -1] \cup [0, 1] \cup [2, \infty)$.

EXAMPLE 1.9 Find the values of x smaller than 3 in absolute value which satisfy the inequality $\log_{(2a-x^2)}(x-2ax) > 1$ for all $a > 5$.

SOLUTION $\log_{(2a-x^2)}(x-2ax) > 1$, $a > 5$ and $-3 < x < 3$.

Note that $2a - x^2 > 1$ for all $a > 5$ and $x \in (-3, 3)$

Hence the above inequality will be true if

$(x-2ax) > 2a - x^2$

$\Rightarrow x - 2ax + x^2 - 2a > 0$

$\Rightarrow x(1+x) - 2a(1+x) > 0$

$\Rightarrow (x+1)(x-2a) > 0 \Rightarrow x \in (-3, -1)$.

EXAMPLE 1.10 Solve

$$\frac{1}{\log_4\left(\frac{x+1}{x+2}\right)} < \frac{1}{\log_4(x+3)}.$$

SOLUTION Domain $\frac{x+1}{x+2} > 0$, $x+3>0$

+ | − | +
−2 −1

$x \in (-\infty, -2) \cup (-1, \infty)$, $x > -3$

$\Rightarrow x \in (-3, -2) \cup (-1, \infty)$

Let $\log_4(x+3) > 0$

$\Rightarrow x+3 > 1$

$\Rightarrow x > -2$

Let $\log_4\left(\frac{x+1}{x+2}\right) > 0$

$\Rightarrow \frac{x+1}{x+2} > 1 \Rightarrow \frac{x+1-x-2}{x+2} > 0$

$\Rightarrow \frac{-1}{x+2} > 0 \Rightarrow x < -2$

Case 1: $x < -2$,

i.e. $x \in (-3, -2)$

The inequality becomes:

$\frac{1}{+ve} < \frac{1}{+ve}$

Hence, it has no solution.

Case 2: $x > -2$

i.e. $x \in (-1, \infty)$

The inequality becomes:

$\frac{1}{-ve} < \frac{1}{+ve}$ which is always true

Hence, the solution is $x \in (-1, \infty)$.

EXAMPLE 1.11 Solve $\log_{a+1}(x + a) < 1$.

SOLUTION $x + a > 0 \Rightarrow x > -a$

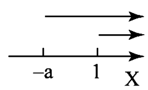
$a + 1 > 0 \Rightarrow a > -1$

$a + 1 \neq 1 \Rightarrow a \neq 0$

Case 1: $0 < a + 1 < 1 \Rightarrow -1 < a < 0$

$x + a > a + 1 \Rightarrow x > 1$

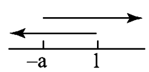
Since $-1 < a < 0$, we have $-a < 1$.



Hence, if $-1 < a < 0$ then $x \in (1, \infty)$.

Case 2: $a + 1 > 1 \Rightarrow a > 0$

$x + a < a + 1 \Rightarrow x < 1$

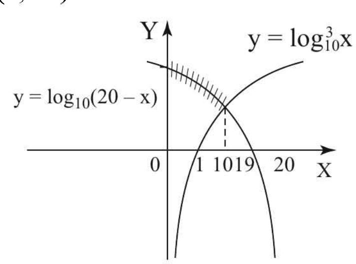


If $a > 0$, then $x \in (-a, 1)$.

EXAMPLE 1.12 Solve $\log_{10}(20 - x) \geq \log_{10}^3 x$.

SOLUTION Domain: $20 - x > 0$ and $x > 0$

i.e. $x \in (0, 20)$



The solution of the inequality is $x \in (0, 10]$.

EXAMPLE 1.13 Find the solution set in $[0, 2\pi]$ of the equation $\log_{|\sin x|}(x^2 - 8x + 23) > \log_{|\sin x|}(8)$.

SOLUTION Note that $x^2 - 8x + 23 > 0$

$$\forall\, x \in R,\ x \neq n\pi \text{ and } x \neq (2n + 1)\frac{\pi}{2}.$$

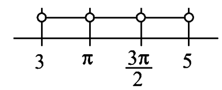
$$\Rightarrow x \neq \left\{0, \pi, 2\pi, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$$

Since $|\sin x| < 1$, the above inequality holds if

$x^2 - 8x + 23 < 8$

$\Rightarrow x^2 - 8x + 15 < 0$

$\Rightarrow (x - 3)(x - 5) < 0$



$$\Rightarrow x \in (3, \pi) \cup \left(\pi, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 5\right).$$

EXAMPLE 1.14 Find the domain of the function

$$f(x) = \sqrt{\log_{10} \frac{6x - x^2}{8}}.$$

SOLUTION We need to find the interval of x for which

$$\frac{6x - x^2}{8} > 0$$

$\Rightarrow 6x - x^2 > 0$

$\Rightarrow x^2 - 6x < 0$

$\Rightarrow 0 < x < 6$

Now, we interpret second condition according to which the whole logarithmic expression within the square root should be a non-negative number.

$$\Rightarrow \log_{10} \frac{6x - x^2}{8} \geq 0$$

$$\Rightarrow \frac{6x - x^2}{8} \geq 1$$

$\Rightarrow 6x - x^2 \geq 8 \quad \Rightarrow \quad 6x - x^2 - 8 \geq 0$

$\Rightarrow x^2 - 6x + 8 \leq 0 \quad \Rightarrow \quad x^2 - 2x - 4x + 8 \leq 0$

$\Rightarrow x(x - 2) - 4(x - 2) \leq 0$

$\Rightarrow (x - 2)(x - 4) \leq 0$

$\Rightarrow 2 \leq x \leq 4.$

Now the interval of x valid for real values of f(x) is the one which satisfies both conditions simultaneously, i.e., the interval common to two intervals determined.

$0 < x < 6 \cap 2 \leq x \leq 4$

Hence, the domain is $[2, 4]$.

EXAMPLE 1.15 Find the domain of the function

$$f(x) = \sqrt{\{(\log_{0.2} x)^3 + \log_{0.2} x^3 \log_{0.2} 0.0016x + 36\}}$$

SOLUTION $\log_{0.2}x^3 .\log_{0.2} 0.0016x = 3\log_{0.2}x .\log_{0.2} 0.2^4x$

$\Rightarrow \log_{0.2}x^3 . \log_{0.2}0.0016x$

$= 3\log_{0.2} x .(4\log_{0.2} 0.2 + \log_{0.2}x)$

We observe that all logarithmic functions have the base of 0.2. Let us consider that $z = \log_{0.2}x$, then logarithmic expression within square root is:

$z^3 + 3z (4 + z) + 36 = z^3 + 3z^2 + 12 z + 36$

$= z^2 (z + 3) + 12 (z + 3)$

Now, this expression is non-negative for square root to be real. Hence,

$(z^2 + 12) (z + 3) \geq 0$

But, we see that $z^2 + 12$ is a positive number. It means that:

$(z + 3) \geq 0 \quad \Rightarrow \quad \log_{0.2} x \geq -3$

$\Rightarrow x \leq 0.2^{-3} \quad \Rightarrow \quad \dfrac{1}{0.008} \quad \Rightarrow \quad x \leq 125.$

Note that we have reversed the inequality as the base is 0.2, which is less than 1.

Further, we have substituted $z = \log_{0.2}x$

This logarithmic function is valid by definition for all po sitive value of x.

Now, the domain of given function is $x \in (0, 125]$.

PRACTICE PROBLEMS [L]

Solve the following inequations:

1. $\log_{1/4}\left(\dfrac{35 - x^2}{x}\right) \geq -\dfrac{1}{2}$.
2. $\dfrac{\log^2 x - 3\log x + 3}{\log x - 1} < 1$.
3. $\dfrac{x - 1}{\log_3(9 - 3^x) - 3} \leq 1$.
4. $\dfrac{(x - 0.5)(3 - x)}{\log_2 |x - 1|} > 0$.
5. Find the domain of the function:

   (i) $y = \dfrac{\log_2(x^2 - 4)}{x - 3}$ (ii) $y = \log_{2x} (2x - x^2 + 3)$
6. Find the integral solutions of the inequality

   $\dfrac{\log_{0.3}(x - 1)}{\sqrt{2x - x^2 + 8}} \leq 0$.
7. $\log_{0.2} \log_6 \left(\dfrac{x^2 - x}{x^2 + 1}\right) > 0$.
8. $\dfrac{\log_2(4x^2 - x - 1)}{\log_2(x^2 + 1)} > 1$.
9. $\dfrac{(x - 1)^2 (x - 2) \log(1 + x)}{x^3 (x - 3)(x - 4)} \leq 0$.

10. $\log_{\frac{1}{x}}\left(\dfrac{2(x-2)}{(x+1)(x-5)}\right) \geq 1$.

11. $\sqrt{\log_3(9x-3)} \leq \log_3\left(x-\dfrac{1}{3}\right)$

12. $\log_{1/2}(\sqrt{5-x}-x+1) > -3$

13. $\dfrac{3\log_a x+6}{\log_a^2 x+2} > 1$

14. $\log_5 x + \log_x \dfrac{x}{3} > \dfrac{(\log_5 x)(2-\log_2 x)}{\log_3 x}$

15. $\log_{|x+6|} 2 \,.\, \log_2(x^2-x-2) \geq 1$.

16. $\log_{(x-3)}(2(x^2-10x+24)) \geq \log_{(x-3)}(x^2-9)$

17. $\sqrt{\log_{1/2}^2 x + 4\log_2\sqrt{x}} < \sqrt{2}\,(4-\log_{16}x^4)$.

18. $\log_a(x-1) + \log_a x > 2$.

19. $\log_x 2x \leq \sqrt{\log_x(2x^3)}$

20. $\log_x 2 \,.\, \log_{2x} 2 \,.\, \log_2 4x > 1$.

21. $\log_3 \dfrac{|x^2-4x|+3}{x^2+|x-5|} \geq 0$.

22. $\dfrac{1}{\log_3(x+1)} < \dfrac{1}{2\log_9\sqrt{x^2+6x+9}}$.

23. $\log_a(1-8a^{-x}) \geq 2(1-x)$

24. $\log_{\frac{2}{3}|x-2|} 2^{1-x^2} \geq 0$

25. $\log_{x^2}\dfrac{4x-5}{|x-2|} \geq \dfrac{1}{2}$.

1.17 | TRIGONOMETRIC FUNCTIONS

Trigonometric functions

There are six trigonometric ratios. In this section, we describe each trigonometric function with corresponding domain, range and graph. In particular, we shall come to know that some of these trigonometric functions are not defined for all values of angles. Further, we shall deliberately denote an angle by variable x not by θ as conventionally denoted. This is to emphasise that an angle is a real number.

Besides, domain and range, we shall also discuss periodicity of each trigonometric function. We refer to a function as periodic if its values are repeated after certain intervals. Graphically, periodic function has a fundamental segment, which can be used to draw plot of the function by repeat-

ing that fundamental segment again and again. Mathematically, we say that f(x+T) = f(x), where T is the fundamental period.

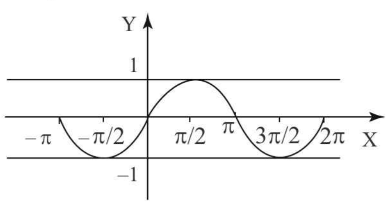
Here, we shall make use of one important rule about periodic function. If T is the period of function f(x), then period of function af(kx ± b) is $\frac{T}{|k|}$, where a,b and k are real numbers. Important points to note that a and b do not affect period, but coefficient of x i.e. k affects period and is given by $\frac{T}{|k|}$.

On the other hand, we find whether the function is even or odd. If f(x) = f(–x), then function is even and its plot is symmetric about y-axis. If f(x) = –f(x), then function is odd and its plot is symmetric about origin.

(a) y = cos x x ∈ R; y ∈ [–1, 1]
(b) y = tan x x ∈ R – (2n + 1) π/2, n ∈ I; y ∈ R
(c) y = cot x x ∈ R – nπ, n ∈ I; y ∈ R
(d) y = cosec x x ∈ R – n ∈ I; y ∈ (–∞, –1] ∈ [1, ∞)
(e) y = sec x x ∈ R – (2n + 1) π/2 n ∈ I ; y ∈ (–∞, –1] ∈ [1, ∞)

Sine function

For each real number x, there is a sine function defined as f(x) = sin x.
The plot of y = sin x is shown here.



The plot, here, is continuous and period is 2π. We think period of the function in terms of the minimum segment which can be used to extend the plot on either side. Further as sin(–x) = –sinx, sine function is an odd function. This fact is also substantiated by the fact that plot is symmetric about origin - not y-axis.

Since function holds for all values of x, its domain is R. On the other hand, the values of sine function is bounded between –1 and 1, inclusive of end points. Hence, domain and range of sine function are:

Domain: x ∈ R
Range: y ∈ [–1, 1]

Let us now consider sine function which is given as f (x) = A sin x.

Multiplying sine function by a constant A does not change the periodicity of the function. However, it changes the maximum and minimum values of the function. The plot extends from –A to A along y-axis as against from –1 to 1 when function is not multiplied bythe constant. This, in turn, changes the range of the function:

Range y ∈ [–A, A]

We now consider yet another form of sine function which is given as:

f(x) = A sin (kx)

Multiplying argument x of sine function by a constant k does not change the nature of plot. However, it changes the periodicity of the function. Recall that if T is the period of function f(x), then period of function af (kx ± b) is $\frac{T}{|k|}$ Clearly, the period of sin(kx) is $\frac{T}{|k|}$. If |k| is less than 1, then period is more than 2π and if |k| is greater than 1, then period is less than 2π.

EXAMPLE 1.1 Find domain and range of function

f (x) = sinx + 2.

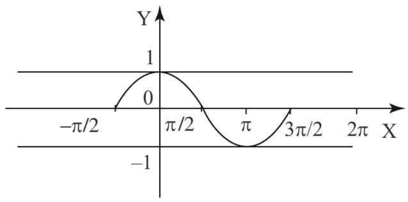
SOLUTION We know that the domain of sinx is the real number set R and range is [–1,1]. The given function is real for all real values of x. Hence, its domain remains R. On the other hand, minimum and maximum values of function changes from that corresponding to sinx function:

$y_{min} = -1 + 2 = 1$
$y_{max} = 1 + 2 = 3$

Hence, range of the given function is [1, 3]. It is evident that graph of the function is that of graph of sinx shifted up by 2 units.

Cosine function

For each real number x, there is a cosine function defined as f(x) = cos x.
The plot of y = cos x is shown here.



The plot, here, is continuous and its period is 2π. Further as cos(–x) = cosx, cosine function is an even function. This is also substantiated by the fact that plot is symmetric about y-axis.

Since the function holds for all values of x, its domain is R. On the other hand, the values of cosine function is bounded between –1 and 1, inclusive of end points. Hence, domain and range of cosine function are:

Domain: $x \in R$

Range: $y \in [-1, 1]$

When cosine function is given as f(x) = Acosx, maximum and minimum values of function becomes –A and A. The range is modified as:

Range: $y \in [-A, A]$

When cosine function is given as f(x) = Acos(kx), the period of cosine function is given by $\frac{T}{|k|}$.

EXAMPLE 1.2 Find domain range of the function

f(x) = 12sinx + 5cosx.

SOLUTION The given function comprises sine and cosine functions. Here, we reduce the function in terms of one trigonometric function and then find range of the function. This reduction is required as otherwise it would be difficult to estimate when two trigonometric functions together evaluate to minimum and maximum values. Let us put,

$a\cos\alpha = 12$

$a\sin\alpha = 5$

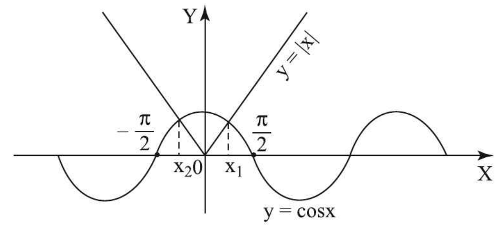
Clearly, $a = \sqrt{12^2 + 5^2} = 13$. Putting these values in the function,

$f(x) = 13(\cos\alpha \sin x + \sin\alpha \cos x) = 13\sin(x + \alpha)$.

We know that range of sine function is [–1, 1]. Hence, range of given function is [–13, 13].

EXAMPLE 1.3 Find the number of solutions of the equation cos x = |x|.

SOLUTION



Clearly the graphs of y = cos x and y = |x| intersect at two points. Hence, the number of solutions is two.

Tangent Function

The tangent function is defined as f (x) = tan x.

The tangent function is not defined for all real values of x. We have

$$\tan x = \frac{\sin x}{\cos x}$$

This is is defined for $\cos x \neq 0$. Now, cos x evaluates to zero for certain values of x, which appears at a certain interval given by the condition,

$$\cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}, \text{ where } n \in I.$$

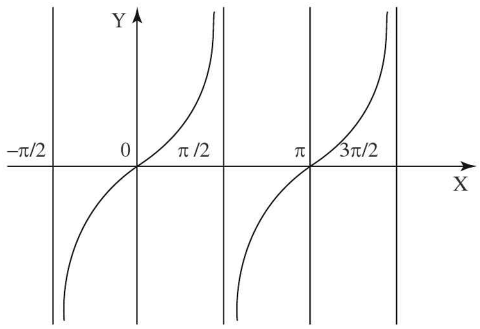
The function cos x is zero for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ ... etc. It means that tangent function is not defined for odd multiples of π/2. On the other hand, the values of tangent function are extended along the real number line on either side of zero.

The range of the function, therefore, is R. Hence, domain and range of tangent function are:

Domain : $x \in R - \{x : x = (2n + 1)\frac{\pi}{2}, n \in I\}$

Range : $y \in R$

The plot of y = tan x is shown here.



The period of tanx is π. Multiplication of a tangent function by a constant A does not change the range as in the case of sine and cosine function. The plot is always extended on either side of x-axis so that its range is R. Multiplying a factor with the argument x like tan(kx), however, changes the points where function is not defined. It is now given by:

$$x = (2n + 1)\frac{\pi}{2k}, n \in I$$

Therefore, domain is now modified as :

Domain : $x \in R - \{x : x = (2n + 1)\frac{\pi}{2k}, n \in I\}$

Cosecant function

The cosecant function is defined as f (x) = cosec x. We have

$$\text{cosec } x = \frac{1}{\sin x}$$

This is defined for $\sin x \neq 0$. Now sin x evaluate to zero for values of x, which appears at a certain interval given by the condition.

$\sin x = 0 \quad \Rightarrow x = n\pi$, where $n \in I$

This means that sin x is zero for x = 0, ± π, ± 2π, ± 3π, ...etc. It means that cosecant function is not defined for integral multiples of π.

On the other hand, values of cosecant function fall at certain intervals. We have seen that values of sine function is between –1 and 1, including end points. Reciprocal of these values of are either lesser than –1 or greater than 1. Symbolically,

$\text{cosec } x \leq -1$ or $\text{cosec } x \geq 1$

Combining the two intervals, using modulus function

$|\text{cosec } x| \geq 1$

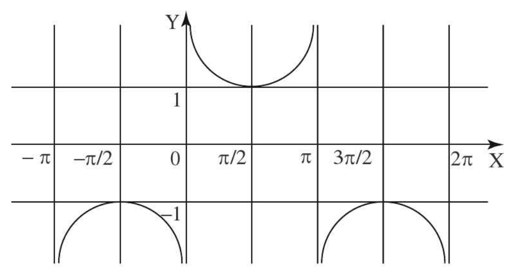
The combined interval of the cosecant function, therefore, is $(-\infty, -1] \cup (1, \infty)$.

Hence, domain and range of cosecant function are :

Domain : $x \in R - \{x : x = n\pi, n \in I\}$

Range : $y \in (-\infty, -1] \in (1, \infty)$

The plot of y = cosec x is shown here.



The period of cosecx is 2π. Important to note here is that function is not defined even within a periodic segment. Since cosec(–x) = – cosecx, we conclude that cosecant function is an odd function. Multiplication of cosecant function by a constant A changes the range as the plot lies on or beyond –A or A. The range is modified as:

Range : $y \in (-\infty, -A] \cup (A, \infty)$.

Multiplying argument x like coses(kx), however, changes the points where the function is not defined. It is now given by:

$$x = \frac{n\pi}{k}, n \in I$$

Therefore, domain is now modified as :

Domain : $x \in R - \{x : x = \frac{n\pi}{k}, n \in I\}$

Secant Function

The secant function is defined as f(x) = sec x. We have

$$\sec x = \frac{1}{\cos x}$$

This is defined for $\cos x \neq 0$.

The function cos x is zero for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ ... etc. It means that secant function is not defined for odd multiples of $\pi/2$.

The values of secant function are bounded by certain intervals. We have seen that values of cosine function is between –1 and 1, including end points. Just like the case of cosecant function, the range of secant function is $|\sec x| \geq 1$

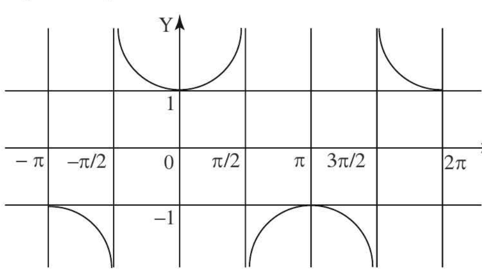
or $(-\infty, -1] \cup (1, \infty)$.

Hence, domain and range of secant function are:

Domain: $x \in R - \{x: x = (2n + 1)\pi/2, n \in I\}$

Range: $y \in (-\infty, -1] \cup (1, \infty)$.

The plot of y = sec x is shown here.



The period of secx is 2π. Important to note here is that function is not defined even within a periodic segment. Since sec(–x) = secx, we conclude that secant function is even function. Multiplication of secant function by a constant A changes the range plot lies on or beyond –A or A. The range is modified as:

Range: $y \in (-\infty, -A] \cup (A, \infty)$

Multiplying argument x like sec(kx), however, changes the points where function is not defined. It is now given by:

$$x = (2n + 1)\pi/2k, n \in I$$

Therefore, domain is now modified as:

Domain: $x \in R - \{x: x = (2n + 1)\pi/2k, n \in I\}$

Cotangent function

The cotangent function is defined as f(x) = cot x. We have

$$\cot x = \frac{\cos x}{\sin x}$$

This is defined for $\sin x \neq 0$.

sin(x) is zero for

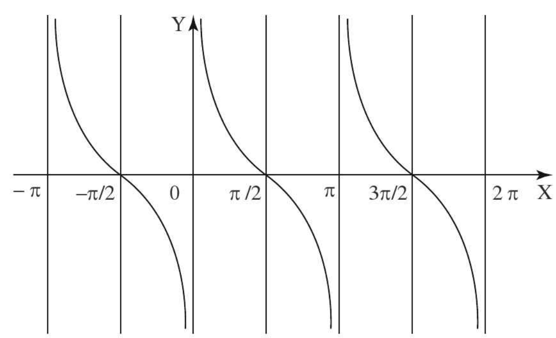
$x = 0, \pm\pi, \pm2\pi, \pm3\pi,$... etc.

It means that cotangent function is not defined for integral multiples of π. On the other hand, the values of cotangent function are extended along the real number line on either side of zero. The range of the function, therefore, is R. Hence, domain and range of cotangent function are:

Domain: $x \in R - \{x: x = n\pi, n \in I\}$

Range: $y \in R$

The plot of y = cot x is shown here



The period of cotx is π. Since cot(–x) = –cotx, we conclude that cotangent function is odd function in each of periodic segment. Multiplication of cotangent function by a constant A does not change the range plot extends either side of x-axis. Multiplying argument x like cot(kx), however, changes the points where function is not defined. It is now given by:

$$x = \frac{n\pi}{k}, n \in I$$

Therefore, domain is now modified as:

Domain: $x \in R - \{x: x = \frac{n\pi}{k}, n \in I\}$

EXAMPLE 1.4 Find range of the function:

$$f(x) = \frac{1}{2 - \sin 2x}.$$

SOLUTION The denominator of given function is positive as value of sin2x cannot exceed 1. We know the range of sin2x. We shall build up expression from this basic trigonometric function to determine range of the given function. Here, $-1 \le \sin 2x \le 1$

Multiplying with –1 to change sign of sine function, we have (note the change in inequality sign):

$$1 \ge -\sin 2x \ge -1$$

$$\Rightarrow 2 + 1 \ge 2 - \sin 2x \ge 2 - 1$$

$$\Rightarrow 3 \ge 2 - \sin 2x \ge 1$$

We need to take reciprocal of each term in the equality to obtain required function form (note the change in inequality sign),

$$\frac{1}{3} \le \frac{1}{2 - \sin 2x} \le 1$$

$$\Rightarrow \frac{1}{3} \le f(x) \le 1$$

$\therefore$ Range $y \in \left(\frac{1}{3}, 1\right)$.

EXAMPLE 1.5 Find range of the function

$$f(x) = 2\sin\sqrt{\left(\frac{\pi^2}{4} - x^2\right)}.$$

SOLUTION Before we attempt to find range, we need to find domain of the function so that we can determine interval of function values. We know that expression within square root is non-negative. Also, expression is a quadratic function. Analysing this quadratic function, domain of quadratic function is found as [–π/2, π/2]. The coefficient of x^2 is negative. Hence, the maximum value of the quadratic function is:

$$y_{max} = -\frac{D}{4a} = -\frac{0 - 4(-1)\frac{\pi}{2}}{4(-1)} = \frac{\pi}{2}.$$

Since the expression is non-negative within square root, the minimum value of the function is 0. Now, sine function is an increasing function in [0, π/2] as is evident from its graph. Thus sine function assumes values in the interval [sin0, sin π/2] i.e. [0,1]. Sine function, however, has a coefficient of 2. Hence the range of the function is [0, 2].

TRIGONOMETRIC IDENTITIES

1. Pythagorean identities

$\cos^2x + \sin^2x = 1$; $1 + \tan^2x = \sec^2x$;
$1 + \cot^2x = \mathrm{cosec}^2x$

2. Trigonometric Functions of Allied Angles

(a) $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$,
$\tan(-x) = -\tan x$

(b) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, $\cos\left(\frac{\pi}{2} - x\right) = \sin x$,

$\tan\left(\frac{\pi}{2} - x\right) = \cot x$.

(c) $\sin\left(\frac{\pi}{2} + x\right) = \cos x$, $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$,

$\tan\left(\frac{\pi}{2} + x\right) = -\cot x$

(d) $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$,
$\tan(\pi - x) = -\tan x$

(e) $\sin(\pi + x) = -\sin x$, $\cos(\pi + x) = -\cos x$,
$\tan(\pi + x) = \tan x$

(f) $\sin(2\pi + x) = \sin x$, $\cos(2\pi + x) = \cos x$,
$\tan(2\pi + x) = \tan x$

3. Sum/difference identities

$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = (\tan x + \tan y)/(1 \mp \tan x \tan y)$;

x, y and (x + y) are not odd multiple of $\pi/2$

$\cot(x \pm y) = (\cot x \cot y \mp 1)/(\cot y \pm \cot x)$;

x, y and (x + y) are not odd multiple of $\pi/2$.

$\sin^2 x - \sin^2 y = \sin(x + y) . \sin(x - y)$

$\cos^2 x - \sin^2 y = \cos(x + y) . \cos(x - y)$

4. Double angle identities

$$\sin 2x = 2\sin x \cos x = \frac{2\tan x}{1+\tan^2 x}$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

$$= 1 - 2\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$$

5. Triple angle identities

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

$$\cot 3x = \frac{3\cot x - \cot^3 x}{1 - 3\cot^2 x}$$

6. Power reduction identities

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\cos^3 x = \frac{\cos 3x + 3\cos x}{4}$$

7. Product to sum identities

$2\sin x \cos y = \sin(x + y) + \sin(x - y)$

$2\cos x \sin y = \sin(x + y) - \sin(x - y)$

$2\cos x \cos y = \cos(x + y) + \cos(x - y)$

$2\sin x \sin y = -\cos(x + y) + \cos(x - y)$

$= \cos(x - y) - \cos(x + y)$

8. Sum to product identities

$$\sin x + \sin y = 2\sin\frac{(x+y)}{2}\cos\frac{(x-y)}{2}$$

$$\sin x - \sin y = 2\cos\frac{(x+y)}{2}\sin\frac{(x-y)}{2}$$

$$\cos x + \cos y = 2\cos\frac{(x+y)}{2}\cos\frac{(x-y)}{2}$$

$$\cos x - \cos y = -2\sin\frac{(x+y)}{2}\sin\frac{(x-y)}{2}$$

$$= 2\sin\frac{(x+y)}{2}\sin\frac{(x-y)}{2}$$

9. Half angle identities

$$\left|\sin\frac{x}{2}\right| = \sqrt{\frac{1-\cos x}{2}}$$

$$\left|\cos\frac{x}{2}\right| = \sqrt{\frac{1+\cos x}{2}}$$

$$\left|\tan\frac{x}{2}\right| = \sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

$$\cot\frac{x}{2} = \frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$$

10. Important Trigonometric Ratios

(a) $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos\frac{5\pi}{12}$;

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin\frac{5\pi}{12};$$

$$\tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot\frac{5\pi}{12}$$

$$\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot\frac{\pi}{12}$$

(b) $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$; $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$;

$\tan\frac{\pi}{8} = \sqrt{2}-1$; $\tan\frac{3\pi}{8} = \sqrt{2}+1$

(c) $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$; $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

Maximum and Minimum values of Trigonometric Functions

(a) Min. value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$

(b) Max. and Min. values of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$.

(c) If $f(\theta) = a\cos(\alpha+\theta) + b\cos(\beta+\theta)$ where a, b, α and β are known quantities then

$$-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le f(\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$$

(d) If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then the maximum values of the expressions
$\cos\alpha\cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha + \sin\beta$ and $\sin\alpha\sin\beta$ occurs when $\alpha = \beta = \sigma/2$.

(e) If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then the minimum values of the expressions
$\sec\alpha + \sec\beta$, $\tan\alpha + \tan\beta$, $\mathrm{cosec}\,\alpha + \mathrm{cosec}\,\beta$ occurs when $\alpha = \beta = \sigma/2$.

(f) If A, B, C are the anlges of a triangle then maximum value of sin A + sin B + sin C and sinA sinB sinC occurs when $A = B = C = \pi/3$

(g) When a quadratic expression in $\sin\theta$ or $\cos\theta$ is given then the maximum or minimum values can be interpreted by making a perfect square.

Summation of Trigonometric Series

Sum of sines or cosines of n angles in A.P.

(i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots$

$$+ \sin(\alpha+\overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots$

$$+ \cos(\alpha+\overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

EXAMPLE 1.6 Graph the function

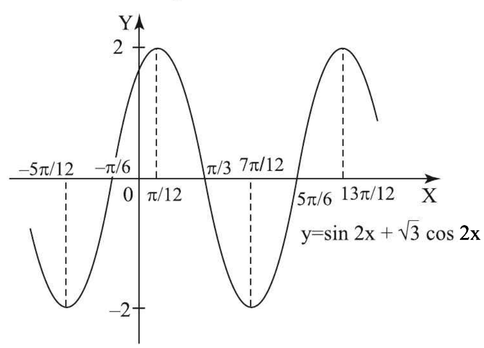
$\sin 2x + \sqrt{3}\cos 2x$.

SOLUTION The most convenient technique here is to reduce this function to the function A sin $(\omega x + \alpha)$ using the method of introducing an auxiliary angle.

$$\sin 2x + \sqrt{3}\cos 2x = 2\left(\frac{1}{2}\sin 2x + \frac{\sqrt{3}}{2}\cos 2x\right)$$

$$= 2\sin\left(2x + \frac{\pi}{3}\right).$$

The fundamental period of this function is π.



(i) The points of intersection of the graph with the x-axis are given by

$$\sin\left(2x+\frac{\pi}{3}\right) = 0, \text{ that is, } x = \frac{\pi}{6}+\frac{\pi k}{2}, k \in I$$

(ii) The points of maxima of the function are:

$\sin\left(2x+\frac{\pi}{3}\right) = 1$. that is, $x = \frac{\pi}{12} + \pi k, k \in I$,

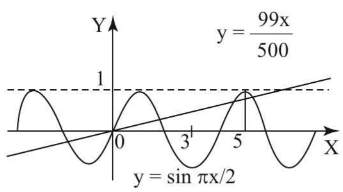
therefore the points of the form $\left(\frac{\pi}{12}+\pi k, 2\right)$, $k \in I$ lie on the graph of the function.

(iii) The points of minima of the function are:

$\sin\left(2x+\frac{\pi}{3}\right) = -1$, that is, $x = -\frac{5\pi}{12} + \pi k$, $k \in I$, therefore the points $\left(-\frac{5\pi}{12}+\pi k, -2\right)$, $k \in I$, lie on the graph.

EXAMPLE 1.7 Find the number of solutions of the equation (i) $\sin\left(\frac{\pi x}{2}\right) = \frac{99x}{500}$, (ii) $\cos \pi x = \frac{3x^2}{4\pi}$.

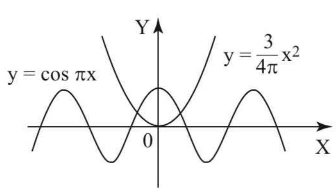
SOLUTION (i) We plot two curves: $y = \sin\left(\frac{\pi x}{2}\right)$ and $y = \frac{99x}{500}$. Note that the second curve is below the line $y = 1$ at $x = 5$.



Plotting the curves on the same axes, we can see that they intersect at 5 points. Hence, the given equation has five solutions.

(ii) We plot two curves: $y = \cos \pi x$, and

$y = \frac{3x^2}{4\pi}$.



Plotting the curves on the same axes, we can see that they intersect at 2 points.

Hence, the given equation has two solutions.

EXAMPLE 1.8 Prove the identity,

$$\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} = \frac{1+\sin x}{\cos x} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right),$$

wherever it is defined.

SOLUTION $\text{LHS} = \frac{(\sin^2 x) - (\cos x - 1)^2}{(\sin x + \cos x - 1)^2}$

$$= \frac{2\cos x - (\cos 2x + 1)}{2 + 2\sin x \cos x - 2\cos x - 2\sin x}$$

$$= \frac{2\cos x(1-\cos x)}{2(1-\sin x)(1-\cos x)}$$

$$= \frac{\cos x}{1-\sin x} = \frac{(1+\sin x)}{\cos x}$$

$$= \frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)} = \frac{1+\tan(x/2)}{1-\tan(x/2)}$$

$$= \tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

EXAMPLE 1.9 The function f (x) is defined by

$$f(x) = \cos^4 x + K\cos^2 2x + \sin^4 x,$$

where K is a constant. If the function f (x) is a constant function, then find the value of K.

SOLUTION Your first impulse might be to complete the square to get the expression $(\cos^2 x + \sin^2 x)^2$ somewhere in the problem. While this works, it is easier to substitute in two values for x and force them to be equal, since the function is to be constant. Any two values will work, but we want to use ones for which we know the sine and cosine. All multiples of $\pi/2$ result in $1 + K$, so we need to put in one value that is not a multiple of $\pi/2$. Try $\pi/4$. This gives us

$$\cos^4(\pi/4) + K\cos^2(\pi/2) + \sin^4(\pi/4)$$
$$= 1/4 + K \cdot 0 + 1/4 = 1/2.$$

So now we know that

$$1 + K = 1/2, \text{ so } K = -1/2.$$

This does in fact give us the function
$$f(x) = (\cos^2 x + \sin^2 x)^2 - 1/2 = 1/2.$$

EXAMPLE 1.10 Given that

$$5\cos^2\alpha - 2\sin\alpha - 2 = 0 \left(\frac{5\pi}{4} < \alpha < \frac{7\pi}{4}\right),$$

then find the value of $\cot\dfrac{\alpha}{2}$.

SOLUTION Making a quadratic equation in sin α,

$5(1 - \sin^2\alpha) - 2\sin\alpha - 2 = 0$

$\sin^2\alpha\ (\sin\alpha + 1)(5\sin\alpha - 3) = 0$ $\sin\alpha = -1$

$\sin\alpha = \dfrac{3}{5}$ not possible as $\left(\dfrac{5\pi}{4} < \alpha < \dfrac{7\pi}{4}\right)$.

$$\alpha = \frac{3\pi}{2}, \frac{\alpha}{2} = \frac{3\pi}{4} \Rightarrow \cot\frac{3\pi}{4} = 1.$$

EXAMPLE 1.11 Calculate $\tan\dfrac{\alpha}{2}$

if $\cos 2\alpha = \dfrac{7}{32}$ and $\alpha \in \left(-\pi, -\dfrac{3\pi}{4}\right)$.

SOLUTION Since cos α is negative for an angle α belonging to the indicated interval, we have

$$\cos\alpha = -\sqrt{\frac{1+\cos 2\alpha}{2}} = -\frac{\sqrt{39}}{8}.$$

Since $\alpha \in \left(-\pi, -\dfrac{3\pi}{4}\right)$, it follows that

$\dfrac{\alpha}{2} \in \left(-\dfrac{\pi}{2}, -\dfrac{3\pi}{8}\right)$. For any angle $\dfrac{\alpha}{2}$ belonging to this interval $\tan\dfrac{\alpha}{2}$ is negative, and therefore

$$\tan\frac{\alpha}{2} = -\frac{8+\sqrt{39}}{5}.$$

EXAMPLE 1.12 Calculate

$$4\sin\left(1+\frac{\pi}{6}\right)\cos\left(1+\frac{\pi}{3}\right).$$

SOLUTION $4\sin\left(1+\dfrac{\pi}{6}\right)\cos\left(1+\dfrac{\pi}{3}\right)$

$$= 2\left[\sin\left(1+\frac{\pi}{6}+1+\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{6}-1-\frac{\pi}{3}\right)\right]$$

$$= 2\left[\sin\left(1+\frac{\pi}{2}\right)+\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= 2\left[\sin\left(\frac{\pi}{2}-(-2)-\frac{1}{2}\right)\right]$$

$$= 2\cos(-2) - 1 = 2\cos 2 - 1.$$

Thus, $4\sin\left(1+\dfrac{\pi}{6}\right)\cos\left(1+\dfrac{\pi}{3}\right) = 2\cos 2 = 1$.

EXAMPLE 1.13 If $0 \le \theta \le \pi$ and

$\sin\dfrac{\theta}{2} = \sqrt{1+\sin\theta} - \sqrt{1-\sin\theta}$, then find the possible values of tan θ.

SOLUTION We have $\sin\dfrac{\theta}{2}$

$$= \left|\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right| - \left|\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right|$$

If $0 \le \theta \le \dfrac{\pi}{2}$, $\sin\dfrac{\theta}{2} = 2\sin\dfrac{\theta}{2} \Rightarrow \theta = 0$.

If $\dfrac{\pi}{2} < \theta \le \pi$, $\sin\dfrac{\theta}{2} = 2\cos\dfrac{\theta}{2}$

$$\tan\frac{\theta}{2} = 2 \Rightarrow \tan\theta = \frac{(2)(2)}{1-4} = \frac{-4}{3}.$$

Hence, the possible values of tan θ are 0 and –4/3.

EXAMPLE 1.14 Prove using induction or otherwise that,

$$2\cos\frac{\theta}{2^n} = \sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots\sqrt{2+2\cos\theta}}}}$$

where R. H. S. contains n radical signs and $\theta \in (0, \pi)$.

SOLUTION We have $2\cos\dfrac{\theta}{2} = \sqrt{2(1+\cos\theta)}$

$$2\cos\frac{\theta}{2^2} = \sqrt{2\left(1+\cos\frac{\theta}{2}\right)}\ \frac{\theta}{2^n} = \sqrt{2+\sqrt{2(1+\cos\theta)}}$$

$$2\cos\frac{\theta}{2^3} = \sqrt{2\left(1+\cos\frac{\theta}{2^2}\right)} = \sqrt{2+2\cos\frac{\theta}{2^2}}$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2\,(1 + \cos\theta)}}} \text{ and so on .}$$

In the same way $2\cos\dfrac{\theta}{2^n}$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots\ldots + \sqrt{2 + 2\cos\theta}}}}\ .$$

Similarly $2\sin\dfrac{\theta}{2^n} = 2\sqrt{\dfrac{1 - \cos\dfrac{\theta}{2^{n-1}}}{2}}$

$$= \sqrt{2 - 2\cos\frac{\theta}{2^{n-1}}}$$

$$= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots\ldots \sqrt{2 + 2\cos\theta}}}}}$$

where R. H. S. contains n radical signs.

EXAMPLE 1.15 Show that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \left(\frac{1}{2}\right)^7.$$

SOLUTION We have $\sin\dfrac{2\pi}{15} = 2\sin\dfrac{\pi}{15}\cos\dfrac{\pi}{15}$,

$$\sin\frac{4\pi}{15} = 2\sin\frac{2\pi}{15}\cos\frac{2\pi}{15},$$

$$\sin\frac{8\pi}{15} = 2\sin\frac{4\pi}{15}\cos\frac{4\pi}{15},\ \sin\frac{16\pi}{15}$$

$$= 2\sin\frac{8\pi}{15}\cos\frac{8\pi}{15}.$$

Multiplying the equalities and noting that

$$\sin\frac{16\pi}{15} = -\sin\frac{\pi}{15},\ \cos\frac{8\pi}{15} = -\cos\frac{7\pi}{15} = \frac{1}{2^4}.$$

Further $\cos\dfrac{5\pi}{15} = \dfrac{1}{2}$.

and $\sin\dfrac{6\pi}{15} = 2\sin\dfrac{3\pi}{15}\cos\dfrac{3\pi}{15}$,

$$\sin\frac{12\pi}{15} = 2\sin\frac{6\pi}{15}\cos\frac{6\pi}{15}.$$

Hence, $\cos\dfrac{3\pi}{15}\,.\,\cos\dfrac{6\pi}{15} = \dfrac{1}{2^2}$.

The rest is obvious.

EXAMPLE 1.16 Simplify the expression

$$\frac{1}{\sqrt{b-a}}\cdot\frac{\sqrt{\dfrac{b-a}{a}}\sin x}{\sqrt{1+\left(\sqrt{\dfrac{b-a}{a}}\sin x\right)^2}}\sqrt{a + b\tan^2 x}$$

where $b > a > 0$.

SOLUTION After a few simple manipulations, this expression (for brevity denote it by P) can be rewritten

$$P = \frac{\sin x\sqrt{a + b\tan^2 x}}{\sqrt{a + (b-a)\sin^2 x}} = \frac{\sin x\sqrt{a + b\tan^2 x}}{\sqrt{a\cos^2 x + b\sin^2 x}}$$

Some students handle this as follows:

$$\sqrt{a + b\tan^2 x} = \sqrt{a + b\frac{\sin^2 x}{\cos^2 x}} = \frac{\sqrt{a\cos^2 x + b\sin^2 x}}{\cos x}$$

and get a wrong answer: P = tan x. In this transformation what we actually have to simplify is the expression $\sqrt{\cos^2 x}$ which is equal to $|\cos x|$. And so the final result is $P = \sin x / |\cos x|$.

EXAMPLE 1.17 Prove that

$$\frac{\tan 8\theta}{\tan\theta} = (1 + \sec 2\theta)(1 + \sec 4\theta)(1 + \sec 8\theta).$$

SOLUTION

$$\text{RHS} = \frac{1+\cos 2\theta}{\cos 2\theta}\times\frac{1+\cos 4\theta}{\cos 4\theta}\times\frac{1+\cos 8\theta}{\cos 8\theta}$$

$$= \frac{2\cos^2\theta \times 2\cos^2 2\theta \times 2\cos^2 4\theta}{\cos 2\theta\cos 4\theta\cos 8\theta}$$

$$= \frac{[8\cos\theta\cos 2\theta\cos 4\theta]\cos\theta}{\cos 8\theta}$$

$$= \frac{\left(\dfrac{\sin 8\theta}{\sin\theta}\right)\cos\theta}{\cos 8\theta}.$$

EXAMPLE 1.18 Eliminate θ between the equation $a \sec\theta + b \tan\theta + c = 0$ and $p \sec\theta + q \tan\theta + r = 0$.

SOLUTION Given $a \sec\theta + b \tan\theta + c = 0$ ...(1)
and $p \sec\theta + q \tan\theta + r = 0$. ..(2)

Solving (1) and (2) by cross multiplication method, we have

$$\frac{\sec\theta}{br - qc} = \frac{\tan\theta}{pc - ar} = \frac{1}{aq - pb}$$

$\because \quad \sec^2\theta - \tan^2\theta = 1$

$$\therefore \quad \left(\frac{br - qc}{aq - pb}\right)^2 - \left(\frac{pc - ar}{aq - pb}\right)^2 = 1$$

or, $\quad (br - qc)^2 - (pc - ar)^2 = (aq - pb)^2$

EXAMPLE 1.19 Find the minimum value of the function

$$f(x) = (3\sin x - 4\cos x - 10)(3\sin x + 4\cos x - 10).$$

SOLUTION $f(x) = 9\sin^2 x - 16\cos^2 x - 10(3\sin x - 4\cos x)$

$- 10(3\sin x + 4\cos x) + 100$

$= 25\sin^2 x - 60\sin x + 84$

$= (5\sin x - 6)^2 + 48$

$\Rightarrow \quad f(x)_{min}$ occurs when $\sin x = 1$ and the minimum value is 49.

EXAMPLE 1.20 Find the minimum and maximum value of $(\sin x - \cos x - 1)(\sin x + \cos x - 1)\ \forall x \in R$.

SOLUTION $y = (\sin x - 1)^2 - \cos^2 x$

$= (\sin x - 1)^2 - (1 - \sin^2 x) = 2\sin^2 x - 2\sin x$

$$y = 2(\sin^2 x - \sin x) = 2\left[\left(\sin x - \frac{1}{2}\right)^2 - \frac{1}{4}\right]$$

Hence, $y_{max} = 2\left[\frac{9}{4} - \frac{1}{4}\right] = 4$ when $\sin x = -1$

$$y_{min} = 2\left[0 - \frac{1}{4}\right] = -\frac{1}{2} \text{ when } \sin x = \frac{1}{2}.$$

EXAMPLE 1.21 Find the sum of the maximum and minimum values of the function

$$f(x) = \frac{1}{1 + (2\cos x - 4\sin x)^2}.$$

SOLUTION $-\sqrt{20} \le 2\cos x - 4\sin x \le \sqrt{20}$

$\Rightarrow \quad 0 \le (2\cos x - 4\sin x)^2 \le 20$

Hence, the minimum value m of $f(x) = \frac{1}{1+20} = \frac{1}{21}$, and the maximum value $M = 1$.

$\Rightarrow M + m = \frac{22}{21}$.

EXAMPLE 1.22 Find the greatest value of c such that system of equations $x^2 + y^2 = 25$ and $x + y = c$ has a real solution.

SOLUTION Put $x = 5\cos\theta$, $y = 5\sin\theta$

$\therefore \quad 5(\cos\theta + \sin\theta) = c$,

but $(\cos\theta + \sin\theta)_{max} = \sqrt{2}$ and

$(\cos\theta + \sin\theta)_{min} = -\sqrt{2}$

Hence, $c_{max} = 5\sqrt{2}$.

EXAMPLE 1.23 Find the minimum and maximum value of $f(x, y) = 7x^2 + 4xy + 3y^2$ subjected to $x^2 + y^2 = 1$.

SOLUTION Let $x = \cos\theta$ and $y = \sin\theta$

$z = f(\theta) = 7\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta$

$= 3 + 2\sin 2\theta + 2(1 + \cos 2\theta)$

$= 5 + 2(\sin 2\theta + \cos 2\theta)$ but

$-\sqrt{2} \le (\sin 2\theta + \cos 2\theta) \le \sqrt{2}$

$\therefore \quad z_{max} = 5 + 2\sqrt{2}$ and $z_{min} = 5 - 2\sqrt{2}$

EXAMPLE 1.24 Let

$$f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$$

for some real number k. Determine

(a) all real numbers k for which f (x) is constant for all values of x.

(b) all real numbers k for which there exists a real number 'c' such that f (c) = 0.

SOLUTION

(a) $f(x) = (\sin^2x + \cos^2x)^3 - 3\sin^2x\cos^2x$
$(\sin^2x + \cos^2x) + k[(\sin^2 + \cos^2x)^2 - 2\sin^2x\cos^2x]$
$= 1 - 3\sin^2x\cos^2x + k(1 - 2\sin^2x\cos^2x)$

$f(x) = (k + 1) - \sin^2x\cos^2x\,(2k + 3)$ ...(1)

For f (x) to be independent of x,

$$k = -\frac{3}{2}.$$

(b) $f(c) = (k + 1) - \sin^2c\cos^2c\,(2k + 3) = 0$

$\therefore\quad \sin^2c\cos^2c = \dfrac{k+1}{2k+3}$

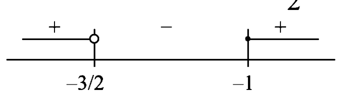
$\Rightarrow\quad \dfrac{1}{4}(\sin^2 2c) = \dfrac{k+1}{2k+3}$

$\Rightarrow\quad \sin^2 2c = \dfrac{4(k+1)}{2k+3}$ but $0 \le \sin^2 2c \le 1$

$\therefore\quad 0 \le \dfrac{4(k+1)}{2k+3} \le 1 \Rightarrow \dfrac{4(k+1)}{2k+3} \ge 0$

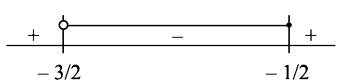
$\Rightarrow\quad \dfrac{k+1}{2k+3} \ge 0$

Hence $\quad k \ge -1$ or $k < -\dfrac{3}{2}$



Again solving $\dfrac{4(k+1)}{2k+3} \le 1 \Rightarrow \dfrac{4(k+1)}{2k+3} - 1 \le 0$

$\Rightarrow\quad \dfrac{4k+4-2k-3}{2k+3} \le 0 \Rightarrow \dfrac{2k+1}{2k+3} \le 0$



Hence, $k \in \left[-1, \dfrac{1}{2}\right]$.

EXAMPLE 1.25 Show that the equation

$2^{\cos x} = x^2 - 2x + 3$ has no real roots.

SOLUTION We have $2^{\cos x} = x^2 - 2x + 3$

$$\text{RHS} = x^2 - 2x + 3 = (x - 1)^2 + 2 \ge 2$$

It has the minimum value 2 at x = 1.

$$\text{LHS} = 2\cos x$$

has a maximum value 2 at $x = 2n\pi$, $n \in I$.

Thus, the given equation can have a solution only if both the sides are equal to 2 for the same x.

This requires $\quad x = 1 = 2n\pi$, $n \in I$

which is not possible for any n.

Hence, the given equation has no solution.

EXAMPLE 1.26 Prove the inequality

$\sin\alpha\sin 2\alpha\sin 3\alpha < \dfrac{3}{4}$.

SOLUTION Let us carry out some transformations of the left-hand side of the inequality. We have

$$(\sin\alpha\sin 2\alpha)\sin 3\alpha = \frac{\cos\alpha - \cos 3\alpha}{2}\sin 3\alpha$$
$$= \frac{2\sin 3\alpha\cos\alpha - 2\sin 3\alpha\cos 3\alpha}{4}$$
$$= \frac{\sin 4\alpha + \sin 2\alpha - \sin 6\alpha}{4}.$$

Since $\sin 4\alpha \le 1$, $\sin 2\alpha \le 1$, $-\sin 6\alpha \le 1$, we have: $\sin 4\alpha + \sin 2\alpha - \sin 6\alpha \le 3$, the equality sign taking place only for those values of α which satisfy the system of equations:

$$\begin{cases} \sin 4\alpha = 1 \\ \sin 2\alpha = 1 \\ \sin 6\alpha = -1 \end{cases}.$$

But this system has no solution.

Indeed, if $\sin 2\alpha = 1$, then $\cos 2\alpha = 0$, and therefore $\sin 4\alpha = 2\sin 2\alpha\cos 2\alpha = 0$.

Thus, $\sin 4\alpha + \sin 2\alpha - \sin 6\alpha < 3$, and, hence,

$$\frac{\sin 4\alpha + \sin 2\alpha - \sin 6\alpha}{4} < \frac{3}{4},$$

from which the given inequality follows.

EXAMPLE 1.27 Prove that for arbitrary α the inequality $4\sin 2\alpha + 5 \ge 4\cos 2\alpha + 5\sin\alpha$ is valid.

SOLUTION $4(3\sin\alpha - 4\sin^3\alpha) + 4$
$\ge 4(1 - 2\sin^2\alpha) + 5\sin\alpha$

or $\quad 16\sin^3\alpha - 8\sin^2\alpha - 7\sin\alpha - 1 \le 0$

The latter inequality should be valid for all values of α. Denoting $\sin\alpha$ by x, we rewrite it as

$$16x^3 - 8x^2 - 7x - 1 \le 0$$

We now have to prove that this inequality is valid for arbitrary values of x in the interval $-1 \le x \le 1$.

The latter inequality can then be rewritten as

$$8x^2(x-1) + 7x(x^2-1) + (x^3 - 1 \le 0$$

or $(x-1)(4x+1)^2 \le 0$. This inequality is clearly valid, and so the original inequality is proved.

EXAMPLE 1.28 Find the sum of the series,

$$\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + \ldots$$

upto n terms.

SOLUTION Let $\theta = \dfrac{\pi}{2n+1}$.

$S = \cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots\ldots \cos(2n-1)\theta$

$(2\sin\theta)S = 2\sin\theta\,[\cos\theta + \cos 3\theta + \cos 5\theta$

$+ \ldots\ldots\ldots \cos(2n-1)\theta]$

$T_1 = \sin 2\theta - 0;\ T_2 = \sin 4\theta - \sin 2\theta;$

$T_3 = \sin 6\theta - \sin 4\theta;\ \ldots\ldots\ldots\ldots$

$T_n = \sin 2n\theta - \sin 2(n-1)\theta$

$$\therefore\ (2\sin\theta)S = \sin 2n\theta \Rightarrow S = \frac{\sin\frac{2n\pi}{2n+1}}{2\sin\frac{\pi}{2n+1}} = \frac{1}{2}.$$

EXAMPLE 1.29 Find the sum of the series,

$\cot 2x \,.\, \cot 3x + \cot 3x \,.\, \cot 4x + \ldots\ldots$
$+ \cot(n+1)x \,.\, \cot(n+2)x$.

SOLUTION $\cot x = \cot[(n+2)x - (n+1)x]$

$$= \frac{\cot(n+2)x \,.\, \cot(n+1)x + 1}{\cot(n+1)x - \cot(n+2)x}$$

or, $\cot x\,[\cot(n+1)x - \cot(n+1)x]$

$= \cot(n+2)x \,.\, \cot(n+1)x + 1$

Hence $\cot(n+1)x - \cot(n+2)x$

$= \cot x\,[\cot(n+1)x - \cot(n+2)x] - 1$

Put $n = 1, 2, 3, \ldots\ldots, n$ and adding we get

$$\text{L.H.S.} = \cot x\,[\cot 2x - \cot(n+2)x] - \frac{n}{2}.$$

EXAMPLE 1.30

Let $f_n(\theta) = \displaystyle\sum_{n=1}^{n} \frac{2\sin 2\theta}{\cos 2\theta - \cos 4n\theta}$ then calculate:

(i) $f_5\left(\dfrac{\pi}{8}\right)$ (ii) $f_3\left(\dfrac{\pi}{12}\right)$

(iii) $f_{2009}\left(\dfrac{\pi}{4}\right)$ (iv) $f_{51}\left(\dfrac{\pi}{6}\right)$.

SOLUTION Consider $\dfrac{2\sin 2\theta}{\cos 2\theta - \cos 4n\theta}$

$$= \frac{2\sin 2\theta}{2\sin(2n+1)\theta\,\sin(2n-1)\theta}$$

$$= \frac{\sin((2n+1)\theta - (2n-1)\theta)}{\sin(2n+1)\theta\,\sin(2n-1)\theta}$$

$= \cot(2n-1)\theta - \cot(2n+1)\theta$

$$f_n(\theta) = \sum_{n=1}^{n} \cot(2n-1)\theta - \cot(2n+1)\theta$$

$= (\cot\theta - \cot 3\theta) + (\cot 3\theta - \cot 5\theta) + \ldots\ldots$
$+ (\cot(2n-1)\theta - \cot(2n+1)\theta)$

$f_n(\theta) = \cot\theta - \cot(2n+1)\theta$

$$\text{Now, } f_5\left(\frac{\pi}{8}\right) = \cos\frac{\pi}{8} - \cot\frac{11\pi}{8} = \cot\frac{\pi}{8} - \cot\frac{3\pi}{8}$$

$$= \cos\frac{\pi}{8} - \tan\frac{\pi}{8} = \left(\sqrt{2}+1\right) - \left(\sqrt{2}-1\right) = 2$$

$$f_3\left(\frac{\pi}{12}\right) = \cot\frac{\pi}{12} - \cot\frac{7\pi}{12} = \cot\frac{\pi}{12} + \tan\frac{\pi}{12}$$

$$= \left(2+\sqrt{3}\right) + \left(2-\sqrt{3}\right) = 4.$$

$$f_{2009}\left(\frac{\pi}{4}\right) = \cot\frac{\pi}{4} - \cot\frac{(4019)\pi}{4}$$

$$= \cot\frac{\pi}{4} - \cot\frac{3\pi}{4} = 1 + 1 = 2$$

$$f_{51}\left(\frac{\pi}{6}\right) = \cot\frac{\pi}{6} - \cot\frac{(103)\pi}{6}$$

$$= \cot\frac{\pi}{6} - \cot(102+1)\frac{\pi}{6}$$

$$= \cot\frac{\pi}{6} - \cot\frac{\pi}{6} = 0.$$

PRACTICE PROBLEMS [M]

1. Find range of $y = \sin x,\ x \in \left[\frac{2\pi}{3}, 2\pi\right]$.
2. Find range of the following functions:
   (i) $y = 2\sin^2 x + 5\sin x + 1,\ \forall\ x \in R$
   (ii) $y = \cos^2 x - \cos x + 1,\ \forall\ x \in R$
3. Prove the identity, $\cos\left(\frac{3\pi}{2} + 4\alpha\right) + \sin(3\pi - 8\alpha) - \sin(4\pi - 12\alpha) = 4\cos 2\alpha \cos 4\alpha \sin 6\alpha$.
4. If $\cos\alpha = \frac{2\cos\beta - 1}{2 - \cos\beta}$ then find the value of $\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$ $(0 < \alpha < \pi$ and $0 < \beta < \pi)$
5. Calculate $\sin\frac{\alpha}{2}$ if $\sin\alpha = \frac{4\sqrt{2}}{9}$ and $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$.
6. If $\frac{\tan\theta}{\tan\theta - \tan 3\theta} = \frac{1}{3}$, then find the value of $\frac{\cot\theta}{\cot\theta - \cot 3\theta}$.
7. Which is greater: sin (tan 1) or tan (sin 1) ?
8. If $0 < \pi < \frac{\pi}{2}$, then prove that
   (i) $\sin^6 x \cos^6 x \le \frac{1}{64}$
   (ii) $\sec^6 x \operatorname{cosec}^6 x \ge 64$
   (iii) $\sec^6 x + \operatorname{cosec}^6 x \ge 16$
   (iv) $\sec^6 x + \operatorname{cosec}^6 x + \sec^6 x \operatorname{cosec}^6 x \ge 80$.
9. Let [a] denote the greatest integer less than or equal to 'a'. Find the set of all values of 'x' satisfying the inequality $|2x^2 - 4x - 7| < \left[1 + \frac{1}{2}\left(\frac{\cos\theta}{\cos(\theta/2) - \sin(\theta/2)}\right)^2\right]$, where $-\frac{\pi}{2} \le \theta < \frac{\pi}{2}$.
10. Find all values of a for which the equation $\cos x = \frac{a - 1.5}{2 - 0.5a}$ possesses solutions.
11. If $\cos\theta + \cos\phi = a$, $\sin\theta + \sin\phi = b$, prove that $\tan\frac{\theta}{2} + \tan\frac{\phi}{2} = \frac{4b}{a^2 + b^2 + 2a}$.
12. Find maximum and minimum values of following
    (i) $10\cos^2 x - 6\sin x\cos x + 2\sin^2 x$
    (ii) $\cos\theta + 3\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) + 6$
13. If the expression $\cos\left(x - \frac{3\pi}{2}\right) + \sin\left(\frac{3\pi}{2} + x\right) + \sin(32\pi + x) - 18\cos(19\pi - x) + \cos(56\pi + x) - 9\sin(x + 17\pi)$ is expressed in the form of $a\sin x + b\cos x$ find the value of a + b.
14. Find sum of the series: $\sin 2\alpha + \sin 3\alpha + \sin 4\alpha + \ldots + \sin n\alpha$, where $(n + 2)\alpha = 2\pi$

1.18 | TRIGONOMETRIC EQUATIONS

Trigonometric functions are many-one relation. We are required to find angles corresponding to a given trigonometric value.

For example, what are angles corresponding to sine value of $-\sqrt{3}/2$. In other words, we need to find angles whose sine evaluates to this value. Note that these values corresponds to the intersection of parallel line $y = -\sqrt{3}/2$ with the graph of sine curve.

For the time being, let us concentrate the interval [0,2π], which corresponds to one cycle of four quadrants.

For $\sin x = -\sqrt{3}/2$, we first consider only the magnitude:

$$\sin\theta = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}.$$

Now, sine function is negative in third and fourth quadrants. The angle in third quadrant is:

$$x = \pi + \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

Similarly, angle in the fourth quadrant is:

$$x = 2\pi - \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

EXAMPLE 1.1 Find angles in [0, 2π], if

$$\cot x = \frac{1}{\sqrt{3}}.$$

SOLUTION Considering only the magnitude, we have

$$\cot\theta = \frac{1}{\sqrt{3}} = \cot\frac{\pi}{3}.$$

Now, cotangent function is positive in first and third quadrants. The angle in third quadrant is:

$$x = \pi + \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

Hence the angles are π/3 and 4π/3.

Negative Angles

When we consider angle as a real number entity, we need to express angles as negative angles as well. The corresponding negative angle (y) is obtained as:

$$y = x - 2\pi$$

Thus, negative angles corresponding to 4π/3 and 5π/3 are:

$$\Rightarrow \quad y = \frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$$

$$\Rightarrow \quad y = \frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

We deduct 2π from the positive value.

Trigonometric equations are formed by equating trigonometric functions to zero. A solution of a given trigonometric equation is the value of the unknown angle that satisfies the equation.

For example if $\sin\theta = \frac{1}{\sqrt{2}}$

then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots\ldots$ are solutions.

The trigonometric equation may have infinite number of solutions because of their periodic nature.

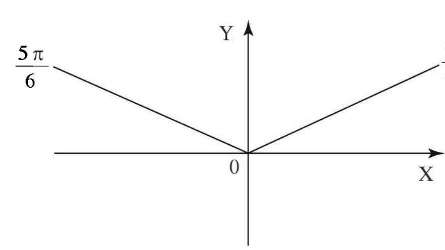
The solutions can be classified as:

(i) Principal solution (ii) General solution.

The solutions of trigonometric equation which lie in the interval [0, 2π) are called **principal solutions**.

EXAMPLE 1.2 Find the principal solutions of the equation $\sin x = \frac{1}{2}$.

SOLUTION $\sin x = \frac{1}{2}$



Since, there exists two values $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\frac{1}{2}$, the principal solutions of the equation $\sin x = \frac{1}{2}$ and $\frac{\pi}{6}, \frac{5\pi}{6}$.

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called **general solution**.

Solutions of some standard equations involving sine, cosine and tangent functions are listed here:

1. If $\sin\theta = \sin\alpha$

$\Rightarrow \theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$

2. If $\cos\theta = \cos\alpha$

$\Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

3. If $\tan\theta = \tan\alpha$

$\Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

4. If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
5. $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
6. $\tan^2\alpha = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

Note that α is called the principal angle.

Note

1. $\sin x = 0 \quad \Rightarrow \quad x = n\pi$; $n \in I$
2. $\cos x = 0 \quad \Rightarrow \quad x = (2n + 1)\frac{\pi}{2}$; $n \in I$

In order to understand the working with trigonometric equation, let us consider an equation:

$$\sin x = \frac{\sqrt{3}}{2}$$

As worked out earlier, $-\sqrt{3}/2$ is sine value of two angles in the interval $[0, 2\pi]$. The important question here is to know which angle should be used in the solution set. Here,

$$\sin\frac{4\pi}{3} = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$$

We can write general solution using either of two values.

$$x = n\pi + (-1)^n\frac{4\pi}{3};\ n \in I$$

or, $$x = n\pi + (-1)^n\frac{5\pi}{3};\ n \in I$$

The solution sets appear to be different, but are same on expansion. Conventionally, however, we use the smaller of two angles which lie in the interval $[0, 2\pi]$.

EXAMPLE 1.3 Solve $4\tan^2\theta = 3\sec^2\theta$.

SOLUTION $4\tan^2\theta = 3\sec^2\theta$ ...(1)

For equation (1) to be defined $\theta \neq (2n + 1)\frac{1}{2}$, $n \in I$.

The equation (1) can be written as

$$\frac{4\sin^2\theta}{\cos^2\theta} = \frac{3}{\cos^2\theta}$$

$$\Rightarrow \quad 4\sin^2\theta = 3$$

$$\Rightarrow \quad \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \quad \sin^2\theta = \sin^2\frac{\pi}{3}$$

$$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3},\ n \in I$$

EXAMPLE 1.4 Solve $\frac{\cos 3\pi x}{1 + \sqrt{3}\tan\pi x} = 0$

SOLUTION $\cos 3\pi x = 0$

$$\Rightarrow \quad 3\pi x = (2n + 1)\frac{\pi}{2} \Rightarrow x = \frac{(2n+1)}{6},\ n \in I$$

$$\Rightarrow \quad x = \frac{1}{6}, \frac{3}{6}, \frac{5}{6}, \frac{7}{6}, \frac{9}{6}, \frac{11}{6}$$

Since, $\tan\pi x$ must be defined, $\pi x \neq (2m + 1)\frac{\pi}{2}$

$$\Rightarrow \quad x \neq \frac{(2m+1)}{2} \Rightarrow x \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\ldots$$

Also, $1 + \sqrt{3}\tan\pi x \neq 0$

$$\Rightarrow \quad \tan\pi x \neq -\frac{1}{\sqrt{3}} \Rightarrow \pi x \neq k\pi - \frac{\pi}{6}$$

$\Rightarrow \quad x \neq k - \frac{1}{6} \Rightarrow x \neq \frac{-1}{6}, \frac{5}{6}, \frac{11}{6}, \ldots$

Hence, $\quad x = \frac{1}{6}, \frac{7}{6}, \ldots..$

i.e. $\quad x = \frac{6\ell+1}{6}, l \in I.$

1. Solution by Factorisation

EXAMPLE 1.5 Solve

$(2 \sin x - \cos x)(1 + \cos x) = \sin^2 x$

SOLUTION $(2 \sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\Rightarrow \quad (2 \sin x - \cos x)(1 + \cos x) - (1 - \cos x)(1 + \cos x) = 0$

$\Rightarrow \quad (1 + \cos x)(2 \sin x - 1) = 0$

$\Rightarrow \quad 1 + \cos x = 0$ or $2 \sin x - 1 = 0$

$\Rightarrow \quad \cos x = -1$ or $\sin x = \frac{1}{2}$

$\Rightarrow \quad x = (2\pi + 1)\pi, n \in I$ or $\sin x = \sin \frac{\pi}{6}, n \in I$

$\Rightarrow \quad x + n\pi (-1)n \frac{\pi}{6}, n \in I$

$\therefore$ The solution of the given equation is $(2n + 1)\pi, n \in I$ or $n\pi + (-1)^n \frac{\pi}{6}, n \in I.$

EXAMPLE 1.6 Find the general solution of the equation

$\sin^3 x(1 + \cot x) + \cos^3 x(1 + \tan x) = \cos 2x.$

SOLUTION

$\sin^2 x(\cos x + \sin x) + \cos^2 x(\cos x + \sin x) = \cos 2x$

$(\cos x + \sin x)(\cos^2 x + \sin x)$

$= (\cos x + \sin x)(\cos x - \sin x)$

$\Rightarrow \quad (\cos x + \sin x)[\cos x - \sin x - 1)] = 0$

$\Rightarrow \quad$ either $\cos x + \sin x = 0 \quad \ldots(1)$

or $\quad \cos x - \sin x = 1 \quad \ldots(2)$

From (1), $\tan x = -1$ or $1 - \sin 2x = 1$

$\Rightarrow \sin 2x = 0$

If $\tan x = -1 = \tan\left(-\frac{\pi}{4}\right)$

$\therefore \quad x = n\pi - \frac{\pi}{4}, n \in I$

If $\sin 2x = 0$, then $2x = n\pi$

But $x = \frac{n\pi}{2}$ is to be rejected because tan x or cot x will be undefined.

$\therefore \quad x = n\pi - \frac{\pi}{2}, n \in I.$

EXAMPLE 1.7 Find the solutions of the equation $\log_{\sqrt{2}\sin x}(1+\cos x) = 2$ in the interval $[0, 2\pi]$.

SOLUTION If the logarithmic expression is to make sense, then $\sqrt{2}\sin x > 0$, $\sqrt{2} \sin \neq 1$ and $1 + \cos x > 0$. For this we must have

$$x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right).$$

Now, if x belong to this set

$\log_{\sqrt{2}\sin x}(1+\cos x) = 2 \Rightarrow 2 \sin^2 x = 1 + \cos x.$

Using $\sin^2 x = 1 - \cos^2 x$, the last equality occurs if and only if

$(2\cos x - 1)(\cos x + 1) = 0.$

If $\cos x + 1 = 0$, then $x = \pi$, a value that must be discarded. If $\cos x = \frac{1}{2}$, then $x = \frac{\pi}{3}$, which is the only solution in $[0, 2\pi]$.

2. Equations reducible to quadratic equations

EXAMPLE 1.8 Solve the equation

$\sin^2 x(\tan x + 1) = 3 \sin x (\cos x - \sin x) + 3.$

SOLUTION The given equation makes no sense when $\cos x = 0$; therefore we can suppose that $\cos x \neq 0$. Noting that the right-hand member of the equation is equal to $3 \sin x \cos x + 3 \cos^2 x$, and dividing both members by $\cos^2 x$, we obtain

$\tan^2 x (\tan x + 1) = 3 (\tan x + 1)$

$\Rightarrow (\tan^2 x - 3)(\tan x + 1) = 0.$

$\Rightarrow x_1 = -\dfrac{\pi}{4} + k\pi,\ x_2 = \dfrac{\pi}{3} + k\pi,\ x_3 = -\dfrac{\pi}{3} + k\pi,\ k \in I$

EXAMPLE 1.9 Find the general solution set of the equation $\log_{\tan x}(2 + 4\cos^2 x) = 2$.

SOLUTION $2 + 4\cos^2 x = \tan^2 x$

$\Rightarrow 3 + 4\cos^2 x = \sec^2 x$

$\Rightarrow 4\cos^4 x + 3\cos^2 x - 1 = 0$

Let $\cos^2 x = t$. Then,

$4t^2 + 3t - 1 = 0 \qquad \Rightarrow (4t - 1)(t + 1) = 0$

$\Rightarrow t = 1/4$ or $t = -1$

$\Rightarrow \cos^2 x = \dfrac{1}{4}$ or $\cos^2 x = -1$ (not possible)

$\Rightarrow \cos^2 x = \cos^2 \dfrac{\pi}{3}$

Hence, $x = n\pi + \dfrac{\pi}{3},\ n \in I.$

EXAMPLE 1.10 Solve the equation

$2 - (7 + \sin 2x)\sin^2 x + (7 + \sin 2x)\sin^4 x = 0.$

SOLUTION The left member of the equation being equal to

$2 - (7 + \sin 2x)(\sin^2 x - \sin^4 x)$

$= 2 - (7 + \sin 2x)\sin^2 x \cdot \cos^2 x$

$= 2 - (7 + \sin 2x)\dfrac{1}{4}\sin^2 2x,$

we can put $t = \sin 2x$ and rewrite the equation in the form $\qquad t^3 + 7t^2 - 8 = 0. \qquad (1)$

It is readily seen that equation (1) has the roots $t_1 = 1$. The other two roots are found from the equation $t^2 + 8x + 8 = 0 \qquad (2)$

Solving this equation we find $t = -4 + 2\sqrt{2}$ and $t = -4 - 2\sqrt{2}$.

These roots should be discarded because they are greater than unity in their absolute values. Consequently, the roots of the original equation coincide with the roots of the equation $\sin 2x = 1$.

Hence, $x = \pi/4 + k\pi,\ k \in I.$

3. Solving equations by introducing an auxilliary argument

EXAMPLE 1.11 Solve $\sin x + \cos x = \sqrt{2}$

SOLUTION $\sin x \cdot \dfrac{1}{\sqrt{2}} + \cos x \cdot \dfrac{1}{\sqrt{2}} = 1$

$\Rightarrow \sin x \cdot \sin\dfrac{\pi}{4} + \cos x \cdot \cos\dfrac{\pi}{4} = 1$

$\Rightarrow \cos\left(x - \dfrac{\pi}{4}\right) = 1$

$\Rightarrow x - \dfrac{\pi}{4} = 2n\pi,\ n \in I \Rightarrow x = 2n\pi + \dfrac{\pi}{4},\ n \in I.$

EXAMPLE 1.12 Solve the equation

$\cos 7x - \sin 5x = \sqrt{3}(\cos 5x - \sin 7x).$

SOLUTION Rewrite the equation in the form

$\dfrac{1}{2}\cos 7x + \dfrac{\sqrt{3}}{2}\sin 7x = \dfrac{\sqrt{3}}{2}\cos 5x + \dfrac{1}{2}\sin 5x$

$\Rightarrow \sin\dfrac{\pi}{6}\cos 7x + \cos\dfrac{\pi}{6}\sin 7x$

$= \sin\dfrac{\pi}{3}\cos 5x + \cos\dfrac{\pi}{3}\sin 5x,$

$\Rightarrow \sin\left(\dfrac{\pi}{6} + 7x\right) = \sin\left(\dfrac{\pi}{3} + 5x\right).$

But $\sin\alpha = \sin\beta$ if and only if either $\alpha - \beta = 2k\pi$ or $\alpha + \beta = (2m + 1)\pi$ ($k, m = 0, \pm 1, \pm 2, \ldots$). Hence,

$\dfrac{\pi}{6} + 7x - \dfrac{\pi}{3} - 5x = 2k\pi$

or $\dfrac{\pi}{6} + 7x - \dfrac{\pi}{3} - 5x = (2m + 1)\pi.$

Thus, the roots for the equation are

$$\left.\begin{aligned} x &= \frac{\pi}{12}(12k + 1), \\ x &= \frac{\pi}{24}(4m + 1) \end{aligned}\right\},\ k, m \in I.$$

EXAMPLE 1.13 Solve the equation

$$\sin^3 x + \cos^3 x = 1 - \frac{1}{2} \sin 2x.$$

SOLUTION Using the formula for the sum of cubes of two members we transform the left-hand side of the equation in the following way:

$$(\sin x + \cos x)\ (1 - \sin x \cos x)$$

$$= \left(1 - \frac{1}{2}\sin 2x\right)\ (\sin x + \cos x).$$

Hence, the original equation takes the from

$$\left(1 - \frac{1}{2}\sin 2x\right)\ (\sin x + \cos x - 1) = 0.$$

The expression in the first bracket is different from zero for all x. Therefore, it is sufficient t consider the equation $\sin x + \cos x - 1 = 0$. The latter is reduced to the form

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

$$x_1 = 2\pi k,\ x^2 = \frac{\pi}{2} + 2\pi k,\ k \in I.$$

4. Solving equations by transforming a sum of trigonometric functions into a product

EXAMPLE 1.14 Solve

$$\cos 3x + \sin 2x - \sin 4x = 0.$$

SOLUTION $\cos 3x + \sin 2x - \sin 4x = 0$

$\Rightarrow \cos 3x + 2 \cos 3x . \sin (-x) = 0$

$\Rightarrow \cos 3x - 2 \cos x . \sin x = 0$

$\Rightarrow \cos 3x (1 - 2 \sin x) = 0$

$\Rightarrow \cos 3x = 0$ or $1 - 2 \sin x = 0$

$\Rightarrow 3x = (2n + 1) \frac{\pi}{2}, n \in I$ or $\sin x = \frac{1}{2}$

$\Rightarrow x = (2n + 1) \frac{\pi}{6}, n \in I$

or $x = n\pi + (-1)^n \frac{\pi}{6}, n \in I.$

5. Solving equations by transforming a product of trigonometric functions into a sum

EXAMPLE 1.15 Solve

$$\sin 5x . \cos 3x = \sin 6x . \cos 2x.$$

SOLUTION $\sin 5x . \cos 3x = \sin 6x . \cos 2x$

$\Rightarrow 2\sin 5x . \cos 3x = 2\sin 6x . \cos 2x$

$\Rightarrow \sin 8x + \sin 2x = \sin 8x + \sin 4x$

$\Rightarrow \sin 4x - \sin 2x = 0$

$\Rightarrow 2 \sin 2x . \cos 2x - \sin 2x = 0$

$\Rightarrow \sin 2x (2 \cos 2x - 1) = 0$

$\Rightarrow \sin 2x = 0$ or $2 \cos 2x - 1 = 0$

$\Rightarrow 2x = n\pi, n \in I$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow x = \frac{n\pi}{2}, n \in I$ or $2x = 2n\pi \pm \frac{\pi}{3}, n \in I$

$\Rightarrow x = n\pi \pm \frac{\pi}{6}, n \in I.$

$\therefore$ Solution of the given equation is $\frac{n\pi}{2}$, $n\pi \pm \frac{\pi}{6}, n \in I.$

6. Solving equations by a change of variable

(i) Equations of the form
$P(\sin x \pm \cos x , \sin x . \cos x) = 0$, where $P(y, z)$ is a polynomial, can be solved by the change.
$\cos x \pm \sin x = t \Rightarrow 1 \pm 2 \sin x . \cos x = t^2.$

(ii) Equations of the form of

$$a . \sin x + b . \cos x + d = 0,$$

where a , b and d are real numbers and $a , b \neq 0$ can be solved by changing sin x and cos x into their corresponding tangent of half the angle.

(iii) Many equations can be solved by introducing a new variable . For example, the equation $\sin^4 2x + \cos^4 2x = \sin 2x . \cos 2x$ changes to $2(y + 1)\left(y - \frac{1}{2}\right) = 0$ by substituting, $\sin 2x . \cos 2x = y$.

EXAMPLE 1.16 Solve

$$\sin x + \cos x = 1 \sin x . \cos x$$

SOLUTION $\sin x + \cos x = 1 \sin x . \cos x$ (1)

Let $\sin x + \cos x = t$

$\Rightarrow \sin^2 x + \cos^2 x + 2 \sin x . \cos x = t^2$

$\Rightarrow \sin x . \cos x = \frac{t^2 - 1}{2}$

Now putting in (1), we get $t = 1 + \frac{t^2 - 1}{2}$

$\Rightarrow t^2 - 2t + 1 = 0$

$\Rightarrow t = 1 \quad \Rightarrow \quad \sin x + \cos x = 1$ (2)

Dividing both sides of equation (2) by $\sqrt{2}$, we get

$\Rightarrow \sin x . \frac{1}{\sqrt{2}} + \cos x . \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$

$\Rightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

(i) if we take positive sign, we get
$x = 2n\pi +, n \in I$

(ii) if we take negative sign, we get
$x = 2n\pi, n \in I$

EXAMPLE 1.17 Solve the equation

$$1 + 2 \operatorname{cosec} x = -\frac{\sec^2 \frac{x}{2}}{2}.$$

SOLUTION Transform the given equation to the form $2\cos^2\frac{x}{2}(2 + \sin x) + \sin x = 0$.

Using the formula $2\cos^2\frac{x}{2} = 1 + \cos x$ and ope-ing the brackets, we obtain

$2 + 2(\sin x + \cos x) + \sin x . \cos x = 0.$ (1)

By the substitution $\sin x + \cos x = t$ equation (1) is reduced to the quadratic equation $t^2 + 4t + 3 = 0$ whose roots are $t_1 = -1$ and $t_2 = -3$.

Since $|\sin x + \cos x| \le \sqrt{2}$, the original equation can only be satisfied by the roots of the equation.

$\sin x + \cos x = -1.$ (2)

Solving equation (2), we obtain

$x_1 = -\frac{\pi}{2} + 2k\pi$ and $x2 = (2k + 1)\pi$.

Here x_2 should be discarded because $\sin x^2 = 0$, and therefore the original equation makes no sense for $x = x_2$.

Hence, $x = -\frac{\pi}{2} + 2k\pi, k \in I$.

6. Solving equations using boundedness of the functions

EXAMPLE 1.18 Solve the equation

$$\frac{1 - \tan x}{1 + \tan x} = 1 + \sin 2x.$$

SOLUTION The equation makes no sense for $x = \frac{\pi}{2} + k\pi$ and for $x = -\frac{\pi}{4} + k\pi$. For all the other valus of x it is equivalent to the equation

$$\frac{\cos x - \sin x}{\cos x + \sin x} = 1 + \sin 2x.$$

After simple transformations we obtain

$\sin x(3 + \sin 2x + \cos 2x) = 0.$

It is obvious that the equation

$\sin 2x + \cos 2x + 3 = 0$

has no solution, and therefore, the original equation is reduced to the equation $\sin x = 0$.

Hence, $x = k\pi$.

EXAMPLE 1.19 Solve

$\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right).$

$\cos x = 0.$

SOLUTION

$\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right).$

$\cos x = 0$ (1)

$\Rightarrow \sin x \cos\frac{x}{4} - 2\sin^2 x + \cos x$

$$+ \sin\frac{x}{4} . \cos x - 2\cos^2 x = 0$$

$$\Rightarrow \left(\sin x.\cos\frac{x}{4} + \sin\frac{x}{4}.\cos x\right) - 2(\sin^2 x + \cos^2 x) + \cos x = 0$$

$$\Rightarrow \sin\frac{5x}{4} + \cos x = 2 \quad (2)$$

Now equation (2) will be true if

$$\sin\frac{5x}{4} = 1 \text{ and } \cos x = 1$$

$$\Rightarrow \frac{5x}{4} = 2n\pi + \frac{\pi}{2}, n \in I \text{ and } x = 2m\pi, m \in I$$

$$\Rightarrow x = \frac{(8n+2)\pi}{5}, n \in I \quad (3)$$

and $\quad x = 2m\pi, m \in I \quad (4)$

Now to find general solution of equation (1)

$$\frac{(8n+2)\pi}{5} = 2m\pi$$

$$\Rightarrow 8n + 2 = 10m$$

$$\Rightarrow n = \frac{5m-1}{4}$$

If $m = 1$ then $n = 1$

if $m = 5$ then $n = 6$

....

If $m = 4p - 3, p \in I$ then $n = 5p - 4, p \in I$

$\therefore$ The general solution of given equation can be obtained by substituting either $m = 4p - 3$ in equation (4) or $n = 5p - 4$ in equation (3)

$\therefore$The general solution of equation (1) is $(8p - 6)\pi, p \in I$.

EXAMPLE 1.20 Solve the equation

$$(\sin x + \cos x)\sqrt{2} = \tan x + \cot x.$$

SOLUTION Let us transform the equation to the form

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{2\sin x\cos x}$$

$$\Rightarrow \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sin 2x},$$

$$\Rightarrow \sin\left(x + \frac{\pi}{4}\right)\sin 2x = 1. \quad (1)$$

We have $|\sin\alpha| \le 1$, and therefore (1) holds if either

$$\sin\left(x + \frac{\pi}{4}\right) = 1 \text{ and } \sin 2x = -1.$$

$$\text{or } \sin\left(x + \frac{\pi}{4}\right) = 1 \text{ and } \sin 2x = 1.$$

But the first two equations have no roots in common while the second two equations have the common roots $\quad x = \frac{\pi}{4} + 2k\pi$.

Consequently the roots of the given equation are

$$x = \frac{\pi}{4} + 2k\pi, k \in I.$$

EXAMPLE 1.21 Find the solutions of the equation

$$\cos^{2001}x - \sin^{2000}x = 1.$$

SOLUTION Since $|\cos x| \le 1$ and $|\sin x| \le 1$, we have

$$1 = \cos^{2001}x - \sin^{2001}x$$
$$= \cos^{2001}(-x) + \sin^{2001}(-x)$$
$$\le |\cos^{2001}(-x)| + |\sin^{2001}(-x)|$$
$$= |\cos^{1999}(-x)|\cos^2(-x) + |\sin^{1999}(-x)|\sin^2(-x)$$
$$\le \cos^2(-x) + \sin^2(-x) = 1.$$

The inequalities are tight, and so equality holds throughout. The first inequality above is true if and only if $\cos(-x) \ge 0$ and $\sin(-x) \ge 0$.

The second inequality is true if and only if $|\cos(-x)| = 1$ or $|\sin(-x)| = 1$. Hence we must have either $\cos(-x) = 1$ or $\sin(-x) = 1$. This means $x = 2n\pi$ or $x = -\pi/2 + 2n\pi$ where $n \in I$.

EXAMPLE 1.22 Solve the equation

$$(\tan x)^{\sin x} = (\cot x)^{\cos x}.$$

SOLUTION For the tangent and cotangent to be defined, we must have $x \ne \frac{n\pi}{2}$ $n \in I$. Then

$(\tan x)^{\sin x} = (\cot x)^{\cos x} = \dfrac{1}{(\tan x)^{\cos x}}$

$\Rightarrow (\tan x)^{\sin x + \cos x} = 1.$

Thus either tanx = 1, in which case $x = \dfrac{\pi}{4} + n\pi$, n $\in$ I or sin x + cos x = 0, which implies tan x = – 1, but this does not give real values for the expressions in the original equation. The solution is thus $x = \dfrac{\pi}{4} + n\pi,\ n \in I$.

PRACTICE PROBLEMS [N]

1. Solve the equation for x, $5^{\frac{1}{2}} + 5^{\frac{1}{2} + \log_5(\sin x)} = 15^{\frac{1}{2} + \log_{15}(\cos x)}$
2. If $(1 + \sec^2 \pi x) \,.\, (1 + \sec^2 y) = -x^2 + 2x + 3$ then find x and y.
3. Solve the following equations:
   (i) $\cos^3 x + \cos^2 x - 4\cos^2 \dfrac{x}{2} = 0$ (ii) $\cot^2\theta + 3\operatorname{cosec}\theta + 3 = 0$
4. Solve the following equations:
   (i) $\sin 7\theta = \sin 3\theta + \sin\theta$
   (ii) $5\sin x + 6\sin 2x + 5\sin 3x + \sin 4x = 0$
   (iii) $\cos\theta - \sin 3\theta = \cos 2\theta$
5. Solve the following equations:
   (i) $\sin 2x + 5\sin x + 1 + 5\cos x = 0$
   (ii) $3\cos x + 3\sin x + \sin 3x - \cos 3x = 0$
   (iii) $(1 - \sin 2x)(\cos x - \sin x) = 1 - 2\sin^2 x$.
6. Solve the following equations:
   (i) $\sin 3x + \cos 2x = -2$
   (ii) $\sqrt{3\sin 5x - \cos^2 x - 3} = 1 - \sin x$
7. Solve the equation $\sin^3 x \cos x - \sin x \cos^3 x = \dfrac{1}{4}$.

Solve the following equations:

8. $2\sin 17x + \sqrt{3}\cos 5x + \sin 5x = 0$
9. $\cot x - 2\sin 2x = 1$
10. $3\cos x + 4\sin x = 5$
11. $2 + \cos x = 2\tan \dfrac{x}{2}$
12. $2\cot 2x - 3\cot 3x = \tan 2x$.
13. $\tan\left(x - \dfrac{\pi}{4}\right)\tan x \tan\left(x + \dfrac{\pi}{4}\right) = \dfrac{4\cos^2 x}{\tan\dfrac{x}{2} - \cot\dfrac{x}{2}}$
14. $(\cos 4x - \cos 2x)^2 = \sin 3x + 5$.
15. Find all solutions of the equation $1 + (\sin x - \cos x)\sin\dfrac{\pi}{4} = 2\cos^2 \dfrac{5}{2}x$, which satisfy the condition $\sin 6x < 0$.

1.19 | TRIGONOMETRIC INEQUALITIES

Evaluating trigonometric ratios is a direct process in which we make use of known values, trigonometric identities and transformations or even predefined trigonometric tables. The evaluation of trigonometric inequalities is somewhat inverse of this process. Consider an inequality:

$$\tan x \geq -\sqrt{3}$$

Clearly, we need to know x for which this inequality holds. Trigonometric functions are many-one relations. The value of x satisfying a given inequality is not an unique interval, but a series of intervals. Incidentally, however, trigonometric function values repeat after certain period. So this enables us to define periodic intervals in generic manner for which trigonometric inequality holds.

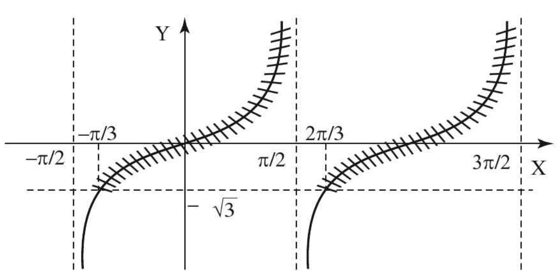
Solution of Trigonometric Inequality

Determination of base or fundamental interval is central to solving trigonometric inequality. The function values in this interval is repeated with a periodicity of trigonometric function. The base interval depends on the nature of trigonometric function and inequality in question.

In order to understand the process, let us solve the inequality given by:

$$\tan x \geq -\sqrt{3}$$

We observe that line $y = -\sqrt{3}$ intersects tangent graph at multiple points. The sections of plots satisfying the inequality are easily identified on the graph.



Since the function tan x is periodic with period π, it suffices to find all the solutions of the inequality in question belonging to a definite interval of length π, since all the remaining solutions will differ from the found ones by a shift to the right or left by integral multiple of π.

To get the shortest possible answer, it is desired that the initial interval of length π be chosen so that the solutions belonging to that interval, in turn, constitute a continuous interval. Here we may take, for instance, $(-\pi/2, \pi/2)$ as such an interval.

The corresponding trigonometric equation, is:

$$\tan x = -\sqrt{3}$$

The angle lying in $(-\pi/2, \pi/2)$ is $-\frac{\pi}{3}$.

Tangent values are greater than $-\sqrt{3}$ for angle greater than $-\pi/3$, but its value asymptotes to infinity at $\pi/2$. Tangent function is not a continuous function.

Thus, the basic interval satisfying the inequality is:

$$-\frac{\pi}{3} \leq x < \frac{\pi}{2}.$$

It is also clear that the solution in this interval is repeated with a period of π, which is the period of tangent function. Hence, solution of the given inequality is:

$$n\pi - \frac{\pi}{3} \leq x < n\pi + \frac{\pi}{2};\ n \in I.$$

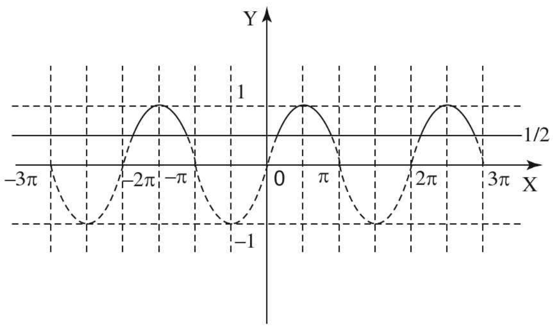
EXAMPLE 1.1 Solve trigonometric inequality:

$$\sin x \geq \frac{1}{2}.$$

SOLUTION The solutions of the corresponding equation are:

$$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots..$$

The sine curve is defined for all values of x. The valid intervals on sine plot are shown in the figure.



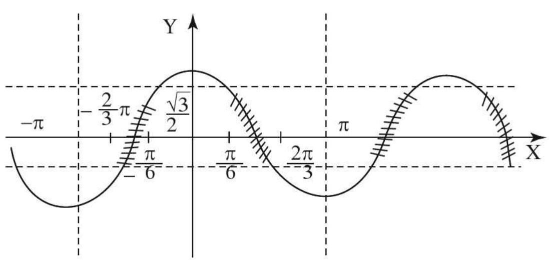
The base interval, therefore, is $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$.

The periodicity of sine function is 2π. Hence, we add $2n\pi$ on either side of the base interval:

$$2n\pi + \frac{\pi}{6} \le x \le 2n\pi + \frac{5\pi}{6},\ n \in I.$$

EXAMPLE 1.2 Solve $-\frac{1}{2} \le \cos x < \frac{\sqrt{3}}{2}$.

SOLUTION



From the figure, the solution set is:

$$\begin{cases} 2n\pi - \frac{2\pi}{3} \le x \le 2n\pi - \frac{\pi}{6} \\ 2n\pi + \frac{\pi}{6} < x \le 2n\pi + \frac{2\pi}{3},\ n \in I \end{cases}$$

EXAMPLE 1.3 Solve

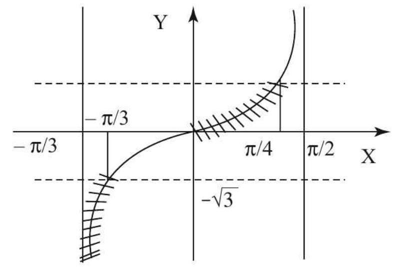
$$\sqrt{1 - \tan x}\,(\sqrt{3}\cot x + 1) > 0$$

SOLUTION $\sqrt{1 - \tan x}\ \ (\sqrt{3}\cot x + 1) > 0$

$\tan x \le 1$, and $\sqrt{3}\cot x + 1 > 0$

The second inequality implies $\frac{\sqrt{3} + \tan x}{\tan x} > 0$

Hence, $\tan x < -\sqrt{3}$, $0 < \tan x < 1$



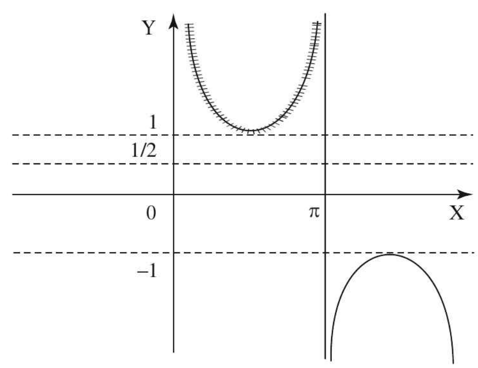
From the figure, the solution set is:

$$\begin{cases} n\pi - \frac{\pi}{2} < x < n\pi - \frac{\pi}{3} \\ n\pi < x < n\pi + \frac{\pi}{4} \end{cases},\ n \in I.$$

EXAMPLE 1.4 Solve $\log_2\left(\operatorname{cosec}\frac{x}{2}\right) > -1$

SOLUTION Let $\operatorname{cosec}\frac{x}{2} = t$ We have $t > 0$ and $\log_2 t > -1$

$\Rightarrow t > \frac{1}{2}$, Hence, $\operatorname{cosec}\frac{x}{2} > \frac{1}{2}$



From the figure, the solution set is:

$$2n\pi < \frac{x}{2} < 2n\pi + \pi$$

$$4n\pi < x < (4n + 2)\,\pi\,.$$

EXAMPLE 1.5 Solve $\sqrt{3-4\cos^2 x} > 2\sin x + 1$

SOLUTION Let $\sin x = t$, $\sqrt{4t^2-1} > 2t+1$ (1)

$4t^2 - 1 \ge 0 \Rightarrow t^2 \ge \frac{1}{4} \Rightarrow t \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$.

Case 1: $2t + 1 \ge 0 \Rightarrow t \ge -\frac{1}{2}$

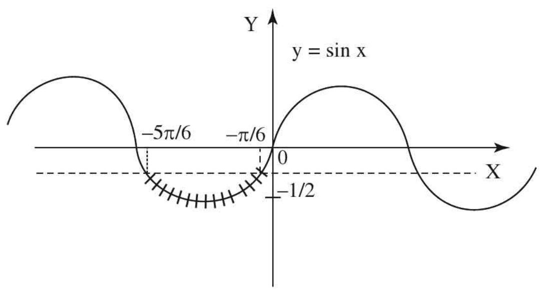
Squaring (1), $4t^2 - 1 > 4t^2 + 1 + 4t$

$\Rightarrow 4t < -2 \Rightarrow t < -\frac{1}{2}$

We have no solution.

Case 2: $2t + 1 < 0$

$\Rightarrow t < -\frac{1}{2}$



From the figure, the solution set is:

$$2n\pi - \frac{5\pi}{6} < x < 2n\pi - \frac{\pi}{6}, n \in I$$

EXAMPLE 1.6 Solve the inequality $\cos^2 x < 3/4$.

SOLUTION The given inequality is equivalent to the inequalities $-\sqrt{3}/2 < \cos x < \sqrt{3}/2$

The set representing the solution of this system is the intersection of the sets which are the solutions of two simple trigonometric inequalities. In order to find this intersection, it is convenient to consider the closed interval $[-\pi, \pi]$ and mark on it separately the solutions of the first and second inequalities. Then we get two subintervals.

$$-\frac{5\pi}{6} < x < -\frac{\pi}{6} \text{ and } \frac{\pi}{6} < x < \frac{5\pi}{6}.$$

Noting that one interval can be obtained from the other by shifting the latter by π and taking into account that $\cos 2x$ is a periodic function with period π, we may write the answer in the following form:

$$\frac{\pi}{6} + \pi n < x < \frac{5\pi}{6} + \pi n, \qquad n \in I.$$

EXAMPLE 1.7 Solve the inequality $\sin x > \cos x$.

SOLUTION In order to solve this inequality, it is required to convert it in terms of inequality of a single trigonometric function.

$\sin x > \cos x$

$\Rightarrow \sin x - \cos x > 0$

$\Rightarrow \sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4} > 0$

$\Rightarrow \sin\left(x - \frac{\pi}{4}\right) > 0$

Let $y = x - \pi/4$. Then, $\sin y > 0$

Thus, we see that problem finally reduces to solving the sine inequality. The base interval is $0 < y < \pi$.

The periodicity of sine function is 2π. Hence, we add $2n\pi$ on either side of the base interval

$2n\pi < y < 2n\pi + \pi, n \in I$

Now substituting for $y = x - \pi/4$, we have:

$2n\pi < x - \frac{\pi}{4} < 2n\pi + \pi, n \in I$

$\Rightarrow 2n\pi + \frac{\pi}{4} < x < 2n\pi + \frac{5\pi}{4}, n \in I.$

EXAMPLE 1.8 For which real numbers x is

(i) $\log_{\cos x} x$

(ii) $\log_x \cos x$, a real number ?

SOLUTION

(i) If $\log_a t$ is defined and real, then $a > 0$, $a \ne 1$ and $t > 0$. Hence, one must have $\cos x > 0$,

$\cos x \neq 1$ and $x > 0$. All this happens when $x \in (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2} + 2\pi n,\ 2\pi\ (n+1))$ $\cup(\ 2\pi\ (n+1),\ \frac{5\pi}{2} + 2\pi n)$, for $n \geq 0$, $n \in I$.

(ii) In this case one must have $x > 0$, $x \neq 1$ and $\cos x > 0$, Hence,

$$x \text{ Œ } (0, 1) \text{ » } (1, \frac{\pi}{2}) \text{ » } (\frac{3\pi}{2} + 2\pi n,\ \frac{5\pi}{2} + 2\pi n), \text{ for } n \geq 0,\ n \in I.$$

EXAMPLE 1.9 If domain of a function $f(x)$ is $[0, 1]$, then find the domain of the function $f(2\sin x - 1)$.

SOLUTION The domain of the function is given here. we need to find the domain when argument (input) to the function is a trigonometric expression. The given domain is:

$$0 \leq x \leq 1.$$

Changing argument of the function the domain condition becomes:

$$0 \leq 2\sin x - 1 \leq 1 \Rightarrow\ 1 \leq 2\sin x \leq 2$$

$$\Rightarrow\ 1/2 \leq \sin x \leq 1$$

However, the range of sinx is $[-1,1]$. It means that the above interval is equivalent to a trigonometric inequality given by:

$$\sin x \geq \frac{1}{2}.$$

Two values of x between 0 and 2π are $\frac{\pi}{6}$, $\frac{5\pi}{6}$.

The value of x satisfying the above condition is

$$2n\pi + \pi/6 \leq x \leq 2n\pi + 5\pi/6,\ n \in I$$

Hence, the required domain is:

$$\left[2n\pi + \frac{\pi}{6},\ 2n\pi + \frac{5\pi}{6}\right],\ n \in I.$$

EXAMPLE 1.10 Solve the inequality

$$\tan\frac{x}{2} > \frac{\tan x - 2}{\tan x + 2}$$

SOLUTION Put $\tan\frac{x}{2} = t$. Then the inequality takes the form

$$t > \frac{2t - 2 + 2t^2}{2t + 2 - 2t^2}$$

$$\Rightarrow \frac{(t-1)(t^2 + t + 1)}{t^2 - t - 1} > 0. \qquad \text{...(1)}$$

Since $t^2 + t + 1 > 0$ for all real values of t, inequality (1) is equivalent to the inequality

$$\frac{t-1}{t^2 - t - 1} > 0 \qquad \text{...(2)}$$

The trinomial $t^2 - t - 1$ has the roots

$$\frac{1-\sqrt{5}}{2} \text{ and } \frac{1+\sqrt{5}}{2}$$

Solving (2), we find that either

$$\tan\frac{x}{2} > \frac{1+\sqrt{5}}{2}.$$

or $\frac{1-\sqrt{5}}{2} < \tan\frac{x}{2} < 1$

Hence, $2k\pi + 2\tan^{-1}\frac{1+\sqrt{5}}{2}$

$< x < \pi + 2k\pi,$

$2k\pi - 2\tan^{-1}\frac{\sqrt{5}-1}{2}$

$< x < \frac{\pi}{2} + 2k\pi.$

EXAMPLE 1.11 Solve the inequality

$$\cos^3 x \cos 3x - \sin^3 x \sin 3x > \frac{5}{8}.$$

SOLUTION From the formulas for sin 3x and cos 3x we find

$$\cos^3 x = \frac{\cos 3x + 3\cos x}{4},$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Using these formulas, we rewrite the given inequality in the form

$$(\cos 3x + 3 \quad \cos x)\cos 3x -$$
$$(3\sin x - \sin 3x)\sin 3x > \frac{5}{2},$$

or $\sin^2 3x + \cos^2 3x + 3$

$$(\cos 3x \cos x - \sin 3x \sin x) > \frac{5}{2},$$

i.e. $\cos 4x > \frac{1}{2}$

$$-\frac{\pi}{3} + 2\pi n < 4x < \frac{\pi}{3} + 2\pi n$$

or $-\frac{\pi}{12} + \frac{1}{2}\pi n < x < \frac{\pi}{12} + \frac{1}{2}\pi n$, $n \in I$.

EXAMPLE 1.12 Solve the inequality

$$\sin x \sin 2x < \sin 3x \sin 4x \text{ if } 0 < x < \frac{\pi}{2}.$$

SOLUTION Transforming the product of sines into the sum, we replace the given inequality by the equivalent inequality

$\cos 3x > \cos 7x$ or $\sin 5x \sin 2x > 0$

But for $0 < x < \frac{\pi}{2}$ we have $\sin 2x > 0$ and, consequently, the original inequality is equivalent to $\sin 5x > 0$. Hence, the solution is

$$0 < x < \frac{\pi}{2} \text{ and } \frac{\pi}{2}\pi < x < \frac{\pi}{2}.$$

EXAMPLE 1.13 If $1 + \log_2(1 + \cos 2x) > \log_4 2$ then find x.

SOLUTION Here, $1 + \log_2(2\cos^2 x) > \frac{1}{2}$

$\log_2 2 = \frac{1}{2}$

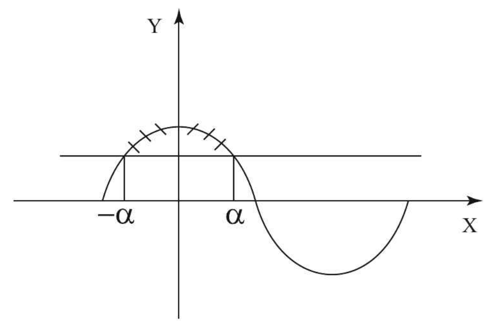
or $2 + \log_2 \cos^2 x > \frac{1}{2}$

or $2\log_2 \cos x > \frac{-3}{2}$

$\therefore \log_2 \cos x > \frac{-3}{4} = \log_2 2^{\frac{-3}{4}}$

$\Rightarrow \cos x > 2^{\frac{-3}{4}} = \frac{1}{2^{3/4}}$

Let $\alpha = \cos^{-1}\frac{1}{2^{3/4}}$



$$2n\pi - \alpha < x < 2n\pi + \alpha$$

$\therefore 2n\pi - \cos^{-1}\frac{1}{2^{3/4}} < x < 2n\pi + \cos^{-1}\frac{1}{2^{3/4}}$

where $n \in I$.

Note that $\log_2(1 + \cos 2x)$ is defined for all obtained values of x.

EXAMPLE 1.14 Find all positive values of x not exceeding 2π for which the inequality

$\cos x - \sin x - \cos 2x > 0$ is satisfied.

SOLUTION Let us write the inequality in the form

$$(\cos x - \sin x)[1 - (\cos x + \sin x)]$$
$$= 2\sin\frac{x}{2}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)$$
$$(\cos x - \sin x) > 0. \qquad ...(1)$$

But $\sin\frac{x}{2} > 0$, since $0 < x < 2\pi$. Let us consider the following two possible cases when inequality (1) is fulfilled.

Case 1:

$$\left.\begin{aligned} &\cos x - \sin x > 0, \\ &\sin\frac{x}{2} - \cos\frac{x}{2} > 0 \end{aligned}\right\} \quad ...(2)$$

By the hypothesis, we have $0 < x < 2\pi$. Taking this into account, we find from (2) that the first inequality is fulfilled if $0 < x < \frac{\pi}{4}$ or $\frac{5}{4}\pi < x < 2\pi$ and the second if $\frac{\pi}{2} < x < 2\pi$. Hence, in this case $\frac{5}{4}\pi < x < 2\pi$.

Case 2:

$$\left.\begin{aligned} &\cos x - \sin x < 0, \\ &\sin\frac{x}{2} - \cos\frac{x}{2} < 0 \end{aligned}\right\} \quad ...(3)$$

Taking into consideration that $0 < x < 2\pi$, we see that system (3) is satisfied if $\frac{\pi}{4} < x < \frac{\pi}{2}$.

$$\frac{\pi}{4} < x < \frac{\pi}{2} \text{ and } \frac{5}{4}\pi < x < 2\pi.$$

EXAMPLE 1.15 Solve the inequality

$$\frac{5}{4}\sin^2 x + \frac{1}{4}\sin^2 2x > \cos 2x.$$

SOLUTION Using the half-angle formula for sine, we rewrite our inequality in the form

$$5(1 - \cos 2x) + 2(1 - \cos^2 2x) > 8\cos 2x,$$

or $2\cos^2 2x + 13\cos 2x - 7 < 0$.

Setting $y = \cos 2x$, we get the quadratic inequality $2y^2 + 13y - 7 < 0$ whose solution is the interval $-7 < y < 1/2$.

Thus, the problem has been reduced to solving the inequality $-7 < \cos 2x < 1/2$. The inequality

$-7 < \cos 2x$ is satisfied for any x.

Solving the inequality $\cos 2x < 1/2$, we get

$$\frac{\pi}{3} + 2\pi n < 2x < \frac{5\pi}{3} + 2\pi n,\ n \in I, \text{ that is ,}$$

$$\frac{\pi}{6} + \pi n < x < \frac{5\pi}{6} + \pi n,\ n \in I.$$

EXAMPLE 1.16 Solve the inequality

$$5 + 2\cos 2x \le 3\,|\,2\sin x - 1\,|.$$

SOLUTION Using the formula for cosine, we reduce the given inequality to the form

$$7 - 4\sin^2 x \le 3\,|\,2\sin x - 1|.$$

Setting $y = \sin x$, we get

$$7 - 4y^2 \le 3\,|\,2y - 1|.$$

(a) Let $y \ge 1/2$, then $7 - 4y^2 \le 3(2y - 1)$ or $2y^2 + 3y - 5 \ge 0$. Solving this inequality, we get $y \ge 1$ and $y \le -5/2$, but from the condition $y \ge 1/2$ we have $y \ge 1$.

(b) Let $y < 1/2$. Then the inequality is rewritten as follows : $7 - 4y^2 \le -3(2y - 1)$ or $2y^2 - 3y - 2 \ge 0$. Solving the last inequality, we get $y \ge 2$ and

$y \le -1/2$.

Thus, all x's satisfying the inequalities $\sin x \ge 1$ and $\sin x \le -1/2$ are solutions of the original inequality. The first inequality holds true only for x's satisfying the equation $\sin x = 1$, that is,

$$x = \frac{\pi}{2} + 2\pi n,\ n \in I$$

Solving the second inequality, we get

$$-\frac{5\pi}{6} + 2\pi n \le x \le -\frac{\pi}{6} + 2\pi n,\ n \in I.$$

Thus, $\quad x = \frac{\pi}{2} + 2\pi n,\ n \in I$, and

$$-\frac{5\pi}{6} + 2\pi n \le x \le -\frac{\pi}{6} + 2\pi n,\ n \in I.$$

EXAMPLE 1.17 Solve the inequality

$$\sin^6 x + \cos^6 x > 13/16.$$

SOLUTION Transforming the left hand side, we have $\sin^6 x + \cos^6 x$

$= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x)$

$= (\sin^2 x + \cos^2 x)^2 - 3\sin^2 x\cos^2 x$

$= 1 - \frac{3}{4}\sin^2 2x = 1 - \frac{3}{4}\cdot\frac{1-\cos 4x}{2}$

$= \frac{5}{8} + \frac{3\cos 4x}{8}$.

Hence, the problem reduces to solving the inequality $\frac{5}{8} + \frac{3\cos 4x}{8} > \frac{13}{16}$ or $\cos 4x > 1/2$.

Hence, $-\frac{\pi}{3} + 2\pi n < 4x < \frac{\pi}{3} + 2\pi n,\ n \in I$,

$\Rightarrow -\frac{\pi}{12} + \frac{\pi n}{2} < x < \frac{\pi}{12} + \frac{\pi n}{2},\ n \in I.$

EXAMPLE 1.18 Solve the inequality

$\sin 2x + \tan x \geq 2.$

SOLUTION The left hand side is defined for

$x \neq \frac{\pi}{2} + \pi n,\ n \in I.$

The substitution

$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$ leads to the equivalent inequality

$\frac{2\tan x}{1+\tan^2 x} + \tan x - 2 \geq 0.$

Setting $y = \tan x$, we have $\frac{2y}{1+y^2} + y - 2 \geq 0$,

$\Rightarrow 2y + (y-2)(y^2+1) \geq 0$

$\Rightarrow y^3 - 2y^2 + 3y - 2 \geq 0$

$\Rightarrow y^2(y-1) - y(y-1) + 2(y-1) \geq 0.$

$\Rightarrow (y^2 - y + 2)(y-1) \geq 0.$

The quadratic function $y^2 - y + 2$ is positive for any y (since the discriminant is negative), therefore the last inequality is equivalent to the inequality

$y - 1 \geq 0$, or $\tan x \geq 1$. Hence

$\frac{\pi}{4} + \pi n \leq x < \frac{\pi}{2} + \pi n,\ n \in I.$

EXAMPLE 1.19 Find the solution set of the equation

$\log_{|\sin x|}(x^2 - 8x + 23) > \log_{|\sin x|}(8),\ x \in [0, 2\pi].$

SOLUTION Note that $x^2 - 8x + 23 > 0\ \forall\ x \in R$

$x \neq n\pi$ and $x \neq (2n+1)\frac{\pi}{2}$.

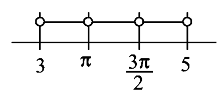
$|\sin x| < 1$ hence above inequality holds if

$\therefore\ x \neq \left\{0, \pi, 2\pi, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$

$\therefore\ x^2 - 8x + 23 < 8$

$x^2 - 8x + 15 < 0$

$(x-3)(x-5) < 0$



$\therefore\ x \in (3, \pi) \cup \left(\pi, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 5\right)$.

EXAMPLE 1.20 Find the solutions of the equation $(\tan^2 x - 1)^{-1} = 1 + \cos 2x$, which satisfy the inequality $2^{x+1} - 8 > 0$.

SOLUTION Let us reduce the trigonometric equation to the form $(1+\cos 2x)\left(1 + \frac{1}{2\cos 2x}\right) = 0$.

The following values of x are solutions of this equation

$x = -\frac{\pi}{2} + n\pi,\ x = \pm\frac{\pi}{3} + \pi k,\ n,\ k \in I.$

We must choose those values of x which satisfy the inequalities

$2^{x+1} - 8 > 0, \cos x \neq 0.$

The values we need are $x = \pm\frac{\pi}{3} + \pi n,\ n \in N.$

EXAMPLE 1.21 Solve the inequality

$$2\sin^2(x+\pi/4)+\sqrt{3}\cos 2x>0. \qquad ...(1)$$

SOLUTION Applying the formula $1-\cos 2\alpha = 2\sin^2\alpha$, we transform the inequality (1) to

$$1-\cos\,(2x+\pi/2)+\sqrt{3}\cos 2x>0,$$

and further $-\cos(2x+\pi/2)+\sqrt{3}\ \cos 2x>-1$,

$$\sin 2x+\sqrt{3}\ \cos 2x>-1,$$

$$\cos(2x-\pi/6)>-1/2 \qquad ...(2)$$

We then solve inequality (2).

Setting $t=(2x-\pi/6)$, we get the inequality: $\cos t>-1/2$ whose solution is

$$-\frac{2\pi}{3}+2\pi k<t<\frac{2\pi}{3}+2\pi k.$$

Returning to the variable x, we get :

$$-\frac{2\pi}{3}+2\pi k<2x-\frac{\pi}{6}<\frac{2\pi}{3}+2\pi k,$$

i.e. $-\frac{\pi}{4}+\pi k<x<\frac{5\pi}{12}+\pi k$ which is the solution.

EXAMPLE 1.22 Solve the inequality

$$\sqrt{3+2\tan x-\tan^2 x}\geq\frac{1+3\tan x}{2}.$$

SOLUTION Denoting tan x by y, rewrite the inequality as $\sqrt{3+2y-y^2}\ \geq 1+3y \qquad ...(1)$

The domain is the interval $-1\leq y\leq 3$. Our inequality is obvious for those values of y in the domain for which

$1+3y<0$;

that is all values of y in the interval $-1\leq y<-1/3$ are solutions of inequality (1).

It remains to consider $-1/3\leq y\leq 3$. ...(2)

Here both members of (1) are non-negative and so squaring in the case at hand yields the equivalent inequality $13y^2-2y-11\leq 0$.

The solution of the last inequality consists of all values of y in the interval $-11/13\leq y\leq 1$. Taking into consideration condition (2), we find that in this case the solution of inequality (1) consists of all values of y in the interval $-1/3\leq y\leq 1$.

Combining both cases we find the solution to be all x satisfying the inequality $-1\leq \tan x\leq 1$. We solve this elementary trigonometric inequality to get the answer:

$$-\frac{\pi}{4}+k\pi\leq x\leq\frac{\pi}{4}+k\pi,\ \text{where } k\in I.$$

EXAMPLE 1.23 Solve the inequality

$$6\sin^2 x-\sin x\cos x-\cos^2 x>2. \qquad ...(1)$$

SOLUTION Since $2=2(\sin^2 x+\cos^2 x)$, we transform the inequality (1) to

$$4\sin^2 x-\sin x\cos x=3\cos^2 x>0. \qquad ...(2)$$

Since $\cos^2 x\geq 0$, the inequality (2) is equivalent to the following collection of systems :

$$\begin{cases}\cos^2 x=0\\ 4\sin^2 x>0\end{cases};$$

$$\begin{cases}\cos^2 x>0\\ 4\tan^2 x-\tan x-3>0.\end{cases} \qquad ...(3)$$

The first system of collection (3) has the following solution: $x=\pi/2+\pi k$. the second system of this collection is equivalent to the following system :

$$\begin{cases}x\neq\frac{\pi}{2}+\pi k\\ (\tan x-1)\left(\tan x+\frac{3}{4}\right)>0,\end{cases}$$

which, in turn, is equivalent to the collection of inequalities $\tan x<-3/4$; $\tan x>1$.

On solving this we get

$$\pi/4+\pi k<x<\pi-\tan{-1}\ 3/4+\pi k.$$

PRACTICE PROBLEMS [O]

Solve the following inequalities :

1. $\sin 2x < 1/\sqrt{2}$.
2. $\sin \frac{x}{5} \geq -1/2$
3. $\tan(3x - 1) < 1/\sqrt{3}$
4. $|\tan x| \geq \sqrt{3}$
5. $\tan^2 x \geq 1/3$
6. $2\cos^2 x + \cos x < 1$
7. $|\sin x| > |\cos x|$
8. $|\sin x| \cos x > 1$
9. $|\sin x| + |\cos x| > 1$
10. $\dfrac{\sin x + \cos x}{\sin x - \cos x} > \sqrt{3}$
11. $\cot x + \dfrac{\sin x}{\cos x - 2} \geq 0$
12. $\sin x + \cos x > \sqrt{2} \cos 2x$.
13. $4 \sin x \sin 2x \sin 3x > \sin 4x$.
14. $\dfrac{\cos^2 2x}{\cos^2 x} \geq 3\tan x$
15. $\dfrac{\cos x + 2\cos^2 x + \cos 3x}{\cos x + 2\cos^2 x - 1} > 1$
16. $\begin{cases} \sin \dfrac{x}{2} < \dfrac{1}{2} \\ \cos 2x > -\dfrac{1}{2} \end{cases}$.
17. $2\left(\sqrt{2} - 1\right)\sin x - 2\cos 2x + 2 - \sqrt{2} < 0$
18. $\sin 2x > \sqrt{2}\sin^2 x + (2 - \sqrt{2})\cos^2 x,\ 0 < x < 2\pi$
19. $1 - \cos x < \tan x - \sin x$
20. Solve : $\log_{10} \sin x + \log_{100} (8 \cos x) = \dfrac{1}{4} \log_{\sqrt{10}} 3$.
21. Solve : $1 + \log_4 \sin x + 2 \log_{16} \cos x > 0$.
22. If $\log_2 \left(\sin \dfrac{x}{2}\right) < -1$ then find the set of general values of x.
23. Solve : $1 + \log_2 \sin x + \log_2 \sin 3x > 0,\ 0 \leq x \leq 2\pi$.

2.20 | GREATEST INTEGER FUNCTION

In this section, we shall study a family of functions which return integers based on certain rule, corresponding to a real number. Greatest integer function (floor), least integer function (ceiling) and nearest integer function form part of this family.

Greatest Integer Function

The function y = [x] is called the greatest integer function where [x] denotes the greatest integer just less than or equal to x. It is also known as integer floor function.

In other words, we can say that greatest integer function rounds "down" any number to the nearest integer. This function is also known by the names of "floor" or "step" function. The greatest integer function (GIF) is denoted by the symbol [x] or $\lfloor x \rfloor$.

Interpretation of greatest integer function is straight forward for positive numbers. Consider the values 0.23 and 1.7. The greatest integers for the two numbers are 0 and 1. Now, consider a negative number –0.54 and –2.34. The greatest integers just less than these negative numbers are –1 and –3 respectively.

Interpretation for negative numbers need some explanation. We interpret these values in the context of the fact that every real number can be decomposed to have two parts (i) integral and (ii) fractional part. From this point of view, the negative number can be thought as:

– 0.54 (real number)

= –1 (integral part) + 0.36 (fraction part)

–2.34 (real number)

= –3 (integral part) + 0.66 (fraction part)

We may be tempted to disagree (why not

–2 + –0.34 = –2.34?).

But, we should know that this is how greatest integer function (GIF) treats a negative number. It returns –3 for –2.34 and not –2. Subsequently, we shall define a function called fraction part function (FPF) that returns fraction part of real number. We shall find that the function exactly returns the same fraction for negative number as has been worked out. The fraction part function (FPF) returns a fraction, which is always positive. It is denoted as {x}. Because of these aspects of GIF and FPF, we can understand the reason why negative number is treated the way it has been presented above. In terms of integral and fraction parts, we write a real number x as :

$x = [x] + \{x\}$

In the nutshell, we can use any of the following interpretations of greatest integer function :

[x] = Greatest integer less than equal to x

= Greatest integer not greater than x

= Integral part of x

In general, if n is an integer and x is any real number between n and (n + 1)

i.e. $n \le x < n + 1, n \in I$.

then $[x] = n$

Note

1. If x is an integer, then [x] = x.
2. If x is not an integer, then [x] evaluates to the greatest integer less than x.

Thus, $[3.245] = 3, [0.75] = 0, [3] = 3$

$[-5.86] = -6, [-0.57] = -1, [\pi] = 3$

$[e] = 2, [-\pi] = -4$

Graph of f(x) = [x]

The function values for some intervals are as follows:

For $-1 \le x < 0$, $f(x) = [x] = -1$

For $0 \le x < 1$, $f(x) = [x] = 0$

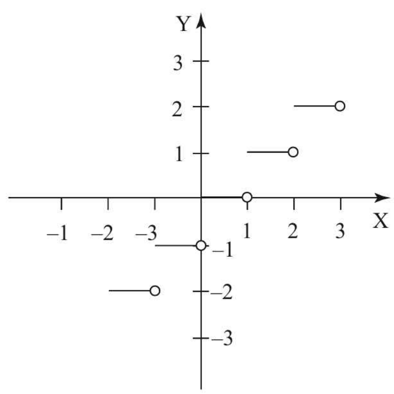
For $1 \le x < 2$, $f(x) = [x] = 1$

For $2 \le x < 3$. $f(x) = [x] = 2$

Thus,

$$y = [x] = \begin{cases} -1 & ; \quad -1 \le x < 0 \\ 0 & ; \quad 0 \le x < 1 \\ 1 & ; \quad 1 \le x < 2 \\ n & ; \quad n \le x < n+1 \end{cases}$$

Graph of $y = [x]$



This function is known as step function as values of function steps by 1 as we switch values of x from one interval to another. We see that there is no restriction on values of x and as such its domain is the set of real numbers. On the other hand, the step function or greatest integer function evaluates only to integer values. It means that the range of the function is set of integers, I. Hence,

Domain : $x \in R$

Range : $y \in I$

GIF is not a periodic function. Even if the function is defined for all real x, the graph is not continuous. It breaks at integral values of x.

Note

(i) $[x] = x$ if and only if x is an integer.

(ii) $-[-x]$ is the least integer greater than or equal to x.

(iii) $[x + 0.5]$ is the nearest integer to x. If two integers are equally near to x, $[x + 0.5]$ denotes the larger of the two.

Definition of a General Step Function

A function f, whose domain is a closed interval [a, b], is called a step function if there is a partition $P = \{x_0, x_1, ..., x_n\}$ of [a, b] such that f is a constant on each open subinterval of P. That is to say, for each k = 1, 2, ..., n, there is a real number f_k such that

$$f(x) = f_k \text{ if } \quad x_{k-1} < x < x_k.$$

Step functions are sometimes called piecewise constant functions.

Note

At each of the endpoints x_{k-1} and x_k the function must have some well-defined value, but this need not be the same as f_k.

EXAMPLE 1.1 Let $x = \log_4 9 + \log_9 28$ show that $[x] = 3$, where [x] denotes the greatest integer less than or equal to x.

SOLUTION Now $x = \log_4 9 + \log_9 28$

$$= \log_2 3 + \frac{\log_3 28}{\log_3 9} > \log_2 3 + \frac{\log_3 27}{2}$$

$$= \log_2 3 + \frac{3}{2}$$

$$> \log_2 2\sqrt{2} + \frac{3}{2} = \frac{3}{2} + \frac{3}{2} = 3$$

Hence $x > 3$ ....(1)

Again $x = \log_2 3 + \log_9 28 < \log_2 4 + \log_9 81$

$= 2 + 2 = 4$

Hence $x < 4$ ...(2)

From (1) and (2), $3 < x < 4 \Rightarrow [x] = 3$.

EXAMPLE 1.2 If [x] denotes the integral part of x for real x, then find the value of

$$\left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{200}\right] + \left[\frac{1}{4} + \frac{1}{100}\right] + \left[\frac{1}{4} + \frac{3}{200}\right] + \ldots\ldots + \left[\frac{1}{4} + \frac{199}{200}\right]$$

SOLUTION The given expression can be written as

$$0 + ...+ 0 + \left[\frac{1}{4} + \frac{150}{200}\right] + \left[\frac{1}{4} + \frac{151}{200}\right] + ...$$

$$+ \left[\frac{1}{4} + \frac{199}{200}\right] = 1 \times 50 = 50.$$

EXAMPLE 1.3 If $f(x) = \cos[\pi^2]\, x + \cos\,[-\pi^2]\, x$, where [x] stands for the greatest integer function then show that $f\left(\frac{\pi}{2}\right) = -1$.

SOLUTION $[\pi^2] = 9$, and $[-\pi^2] = -10$

$f(x) = \cos[\pi^2]\, x + \cos\,[-\pi^2]\, x$
$= \cos 9x + \cos(-10x)$
$= \cos 9x + \cos 10x$

$\therefore\ f\left(\frac{\pi}{2}\right) = \cos\frac{9\pi}{2} + \cos 5\pi = -1.$

EXAMPLE 1.4 Find the sum of all positive integral values of a, $a \in [1, 500]$ for which the equation $[x]^3 + x - a = 0$ has solution, (where [.] denotes the greatest integer function).

SOLUTION $x = a - [x]^3 \Rightarrow x \in I$

$\therefore\ a = x^3 + x$

$$\sum a = \sum_{r=1}^{7} r^3 + \sum_{r=1}^{7} r = \left(\frac{7\times 8}{2}\right)^2 + \frac{7\times 8}{2}$$

$= 784 + 28 = 812.$

EXAMPLE 1.5 Find the domain of the function given by $f(x) = \frac{1}{\sqrt{\pi - [x]}}$.

SOLUTION For the function to be defined

$\pi - [x] > 0$

$\Rightarrow [x] < \pi$

π is approximately 3.14. Here, [x] returns integral value. Clearly, it can assume a maximum value of 3. But, GIF returns integer value n for $x < n + 1$.

The inequality, therefore, has solution given by :

$x < 4$

Hence the domain of the function is $(-\infty, 4)$.

Properties of Greatest Integer Function

Let x and y be real numbers. Then we have

(i) $[x] \le x < [x] + 1$

(ii) $x - 1 < [x] \le x,\ 0 \le x - [x] < 1$

(iii) $[x + m] = [x] + m$ if m is an integer.

(iv) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

If and only if the sum of fractional parts of x and y is less than 1, we have

$[x + y] = [x] + [y]$

For example, let $x = -2.27$ and $y = 0.63$. Then,

$\Rightarrow\ [x + y] = [-2.27 + 0.63] = [-1.64] = -2$

$\Rightarrow\ [x] + [y] = [-2.27] + [0.63] = -3 + 0 = -3$

However, if one $x = 2.2$ and $y = 0.63$, then the proposed identity is true.

(v) $[-x] = -[x]$, if $x \in I$

$= -[x] - 1$, if $x \notin I$.

In another form,

$$[x] + [-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$$

For example, let x = 2. Then

$[2] + [-2] = 2 - 2 = 0$

Let x = 2.7. Then $[2.7] + [-2.7] = 2 - 3 = -1$

(vi) $[x] \ge n \Rightarrow x \ge n,\ n \in I$

(vii) $[x] > n \Rightarrow x \ge n + 1,\ n \in I$

(viii) $[x] \le n \Rightarrow x < n + 1,\ n \in I$

(ix) $[x] < n \Rightarrow x < n,\ n \in I$

(x) $\left[\frac{[x]}{m}\right] = \left[\frac{x}{m}\right]$ if m is a positive integer

(xi) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + ..$

$\ldots + \left[x + \frac{n-1}{n}\right] = [nx]$

(xii) If n and a are positive integers, [n/a] is the number of integers among 1,2....., n that are divisible by a.

(xiii) **de Polignac's Formula** : To find the highest power of a prime p which is contained in n!.

Let p denote a prime. Let E be the largest exponent of p such that p^E divides n!.

Then $$E = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right].$$

Out of the first n integers, as many are divisible by p as the number of times p is contained in n, that is, p, 2p, 3p,..... out of these, some may contain the factor p again, namely, p^2, $2p^2$, $3p^2$.... The number of such integers is the number of times p^2 is contained in n. Some of these may contain p a third time. The number of such integers is the number of times p^3 is contained in n. We proceed in this manner examining higher and higher powers of p, stopping when $p^r > n$.

Hence the highest power required

$$= \left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\ldots\ldots$$

where [ ] is the greatest integer function.

EXAMPLE 1.6 Find the highest power of 5 that divides 518!

SOLUTION The highest power of 5 less than 518 is $125 = 5^3$. Therefore the highest power of 5 in 518! is

$$\left[\frac{518}{5}\right]+\left[\frac{518}{25}\right]+\left[\frac{518}{125}\right]$$

$$= 103 + 20 + 4 = 127.$$

EXAMPLE 1.7 How many zeros are there at the end of 60! ?

SOLUTION Let [x] be the greatest integer function. With this notation, the highest power of 2 in 60!

$$= \left[\frac{60}{2}\right]+\left[\frac{60}{2^2}\right]+\left[\frac{60}{2^3}\right]+\left[\frac{60}{2^4}\right]+\left[\frac{60}{2^5}\right],$$

$$= 30 + 15 + 7 + 3 + 1 = 56$$

Also, the highest power of 5 in 60!

$$= \left[\frac{60}{5}\right]+\left[\frac{60}{5^2}\right],$$

$$= 12 + 2 = 14.$$

The highest power of

10 (i.e. 2×5) in 60! = min. (56, 14) = 14.

EXAMPLE 1.8 Prove that for $n \in N$.

$$\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\left[\frac{n+8}{16}\right]+\ldots = n.$$

SOLUTION Note that $[x] = \left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]$.

Hence,

$$n = \left[\frac{n}{2}\right]+\left[\frac{n+1}{2}\right];\ \left[\frac{n}{2}\right] = \left[\frac{n}{4}\right]+\left[\frac{n+2}{4}\right];$$

$$\left[\frac{n}{4}\right] = \left[\frac{n}{8}\right]+\left[\frac{n+4}{8}\right];\ldots\ldots\ldots$$

Adding these equations we get the required result.

EXAMPLE 1.9 Find domain of the function

$$f(x) = \frac{1}{[x-2]}.$$

SOLUTION The given function has having GIF in its denominator. The denominator should not evaluate to zero for real values of x. The domain of GIF is real number set R. But, we know that GIF evaluates to zero in an interval which is spread over unit value. In order to know this interval, we determine interval of x for which

$[x - 2] = 0$

We can write this function as :

$\Rightarrow [x + (-2)] = 0$

Using property $[x + m] = [x] + m$,

$[x] - 2 = 0$

$[x] = 2 \Rightarrow 2 \le x < 3 \Rightarrow x \in [2, 3)$

Hence, domain of given function is :

$x \in R - [2, 3)$.

EXAMPLE 1.10 If $y = 2[x] + 3 = 3[x - 2] + 5$, then find the value of $[x + y]$.

SOLUTION $y = 2[x] + 3 = 3[x - 2] + 5$ ...(1)

or $2[x] + 3 = 3[x - 2] + 5$

$\Rightarrow 2[x] + 3 = 3\{[x] - 2\} + 5$

$\Rightarrow 2[x] + 3 = 3[x] - 6 + 5$

$\Rightarrow [x] = 4$

From (1), $y = 2 . 4 + 3 = 11$

Now, $[x + y] = [x + 11] = [x] + 11 = 4 + 11 = 15$.

EXAMPLE 1.11 Let $S_r = \sum_{r=1}^{n} r!$, show that

$S_n - 7\left[\frac{S_n}{7}\right]$ is a constant for $n > 6$.

SOLUTION We have

$S_6 = 1! + 2! + 3! + 4! + 5! + 6! = 873$.

Forn > 6, we have $S_n = 7m + 873$, $m \in I$.

$= 7m + 868 + 5 = 7(m + 124) + 5$

$= 7k + 5$, $k \in I$.

Thus, we have

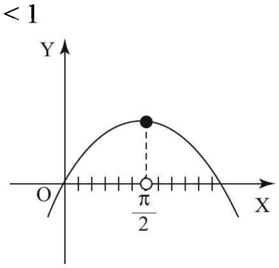
$$S_n - 7\left[\frac{S_n}{7}\right] = (7k + 5) - 7\left[k + \frac{5}{7}\right]$$

$= 7k + 5 - 7k = 5$ which is a constant.

EXAMPLE 1.12 Solve the equation $[\sin x] = 0$.

SOLUTION $[\sin x] = 0$

$\Rightarrow 0 \le \sin x < 1$



From the figure the solution set is :

$$\begin{cases} 2n\pi \le x < 2n\pi + \dfrac{\pi}{2} \\ 2n\pi + \dfrac{\pi}{2} < x \le (2n+1)\pi \end{cases}, n \in I$$

EXAMPLE 1.13 Let $\left[\sqrt{n^2+1}\right] = \left[\sqrt{n^2+\lambda}\right]$ where $n, \lambda \in N$. Show that λ can have $2n$ different values.

SOLUTION We have $n^2 + 1$

$= (n + 1)^2 - 2n < (n + 1)^2 \;\; n \in N$

i.e. $\sqrt{n^2+1} < n + 1$.

Thus, we have $n < \sqrt{n^2+1} < n + 1$

i.e. $\left[\sqrt{n^2+1}\right] = n$.

Now, the given equation reduces to $\left[\sqrt{n^2+\lambda}\right] = n$,

i.e. $n < \sqrt{n^2+\lambda} < (n + 1)$

i.e. $n^2 < n^2 + \lambda < (n + 1)^2$

i.e. $0 < \lambda < 2n + 1$.

Hence, λ can have $2n$ different values.

Fractional Part Function

We define a fraction part function (FPF) denoted by $\{x\}$ as :

$\{x\} = x - [x]$

This function returns fraction part of the number, when x is not an integer. This exception of non-integral x is important. Zero is not a fraction. For integer x, the function evaluates to zero :

$\Rightarrow \{5\} = 5 - [5] = 5 - 5 = 0$

$\Rightarrow \{-5\} = 5 - [-5] = -5 + 5 = 0$

Though zero is not a fraction, but FPF evaluates to zero for integral values. We should keep this exception in mind, while working with FPF. Let us, now, work out with numbers that we earlier used for evaluating greatest integer function.:

$\Rightarrow \quad \{0.23\} = 0.23 - [0.23] = 0.23 - 0 = 0.23$

$\Rightarrow \quad \{1.7\} = 1.7 - [1.7] = 1.7 - 1 = 0.7$

$\Rightarrow \quad \{-0.54\} = -0.54 - [-0.54] = -.54 - (-1)$
$= -0.54 + 1.0 = 0.36$

$\Rightarrow \quad \{-2.34] = -2.34 - [-2.34] = -2.34 - (-3)$
$= -2.34 + 3.0 = 0.66$

We can see that interpretation of fraction for the negative number is consistent with what has been explained earlier.

Graph of f(x) = {x}

The simplification of the function in some intervals are :

For $-2 \le x < -1$, $f(x) = \{x\} = x - [x] = x - (-2)$
$= x + 2$

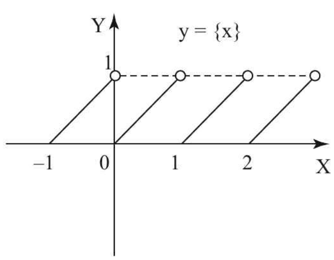
For $-1 \le x < 0$, $f(x) = \{x\} = x - [x] = x - (-1)$
$= x + 1$

For $0 \le x < 1$, $f(x) = \{x\} = x - [x] = x - 0 = x$

For $1 \le x < 2$, $f(x) = \{x\} = x - [x] = x - 1$

For $2 \le x < 3$, $f(x) = \{x\} = x - [x] = x - 2$

$$y = \{x\} = \begin{cases} x+1 & ; \quad -1 \le x < 0 \\ x-0 & ; \quad 0 \le x < 1 \\ x-1 & ; \quad 1 \le x < 2 \\ x-n & ; \quad n \le x < n+1 \end{cases}, \ n \in I$$



We see that there is no restriction on values of x and as such its domain the set of real numbers. The fractional part function can only evaluate to non-negative values between $0 \le y < 1$. Hence,

Domain : $x \in R$

Range : $0 \le y < 1$

FPF is a periodic function. The values repeat with a period of 1. Further, even if the function is defined for all real x, the graph is not continuous. It breaks at integral values of x.

Properties of Fractional Part Function

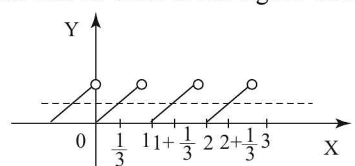
Let x be a real number. Then we have

(i) $0 \le \{x\} < 1$

(ii) $\{x + m\} = \{x\}$ if m is an integer.

(iii) $\{-x\} = 1 - \{x\}, x \in I$
$= 0, \quad x \in I$

(iv) Consider the equation $\{x\} = f, 0 < f < 1$.
The solution is $x = n + f, n \in I$.
For example, the solution of $\{x\} = 1/3$ is
$x = n + \frac{1}{3}, n \in I$.
This can be seen in the figure below.



(v) Consider the inequation $\{x\} < f, 0 < f < 1$
The solution is $n \le x < n + f, n \in I$.
For example, the solution of $\{x\} < \frac{1}{3}$ is
$n \le x < n + \frac{1}{3}, n \in I$.

(vi) If the inequation is $\{x\} > f$ then the solution is
$n + f < x < n + 1$
The solution of $\{x\} > \frac{1}{3}$ is
$n + \frac{1}{3} < x < n + 1, n \in I$.

EXAMPLE 1.14 Solve $(2\{x\}-1)(3\{x\}-2) \le 0$

SOLUTION Solving the quadratic inequality in {x} we get

$$\frac{1}{2} \le \{x\} \le \frac{2}{3}$$

The values of x satisfying this inequality are

$$n + \frac{1}{2} \le x \le n + \frac{2}{3}, \quad n \in I.$$

EXAMPLE 1.15 If {x} and [x] represent fractional and integral part of x, then find the value of

$$[x] + \sum_{r=1}^{2000} \frac{\{x+r\}}{2000}.$$

SOLUTION $[x] + \sum_{r=1}^{2000} \frac{\{x+r\}}{2000}$ we know that

$\{x + r\} = \{x\}$ as $r \in I$

$$= [x] + \sum_{r=1}^{2000} \frac{\{x\}}{2000}$$

$$= [x] + \left[\frac{\{x\}}{2000} + \frac{\{x\}}{2000} + \ldots\ldots + \text{upto 2000 times}\right]$$

$$= [x] + \frac{2000\{x\}}{2000} = [x] + \{x\} = x$$

Thus, $[x] + \sum_{r=1}^{2000} \frac{\{x+r\}}{2000} = x.$

EXAMPLE 1.16 Solve the equations:

(i) $\{x\} + \{\sin x\} = 2$

(ii) $\{x\} + \{\sin x\} = 0.$

SOLUTION

(i) We have $\{x\} < 1$ and $\{\sin x\} < 1$

Hence, $\{x\} + \{\sin x\} < 2.$

Thus, the equation has no solution.

(ii) $\{x\} + \{\sin x\} = 0$

We have $\{x\} \geq 0$ and $\{\sin x\} \geq 0$

Hence, $\{x\} + \{\sin x\} \geq 0.$

Thus, the equation holds only when

$\{x\} = 0 \Rightarrow x \in I$ and

$\{\sin x\} = 0 \Rightarrow \sin x \in I$

$\Rightarrow \sin x = 0, \pm 1$

$\Rightarrow x = \frac{n\pi}{2}, n \in I.$

The only integer in this list is x = 0.
Hence, the solution is x = 0.

Least Integer Function

We have seen that greatest integer function represents the integer, which can be considered to be the floor integral value of a real number. Correspondingly, we define a ceiling function called least integer function (LIF), which returns the least integer greater than or equal to the number x.

We denote least integer function as (x) or $\lceil x \rceil$.

If x is an integer, then (x) = x. If x is not an integer, then (x) evaluates to least integer greater than x.

Thus (3.55) = 4, (0.62) = 1,

(0.265) = 0, (–6.28) = –6

Graph of least integer function

Few initial values of the function are :

For $-3 < x \leq -2$, f(x) = (x) = –2

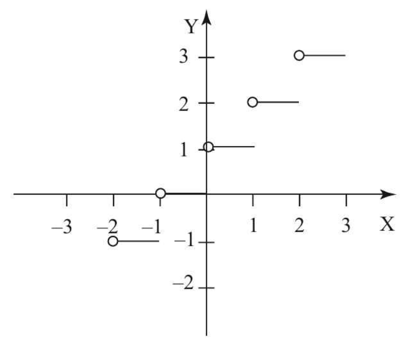
For $-2 < x \leq -1$, f(x) = (x) = –1

For $-1 < x \leq 0$, f(x) = (x) = 0

For $0 < x \leq 1$, f(x) = (x) = 1

For $1 < x \leq 2$, f(x) = (x) = 2

Graph of y = (x)



We see that there is no restriction on values of x and as such its is the set of real numbers. On the other hand, the least integer function evaluates only to integer values. It means that the range of the function is set of integers, denoted by I. Hence,

Domain : $x \in R$

Range : $y \in I$

LIF is not a periodic function. Though function is defined for all real x, the graph is not continuous. It breaks at integral values of x.

Properties of Least Integer Function

(i) If and only if x is an integer, then :
$(x) = x$

(ii) $(x) = [x]$, if $x \in I$ and
$(x) = [x] + 1$, if $x \notin I$.
In fact $(x) = -[-x]$

(iii) $(x + n) = (x) + n, n \in I$
If and only if at least either x or y is an integer, then
$(x + y) = (x) + (y)$
For example, let x = 2.27 and y = 0.63. Then,
$(x + y) = (2.27 + 0.63) = (2.9) = 3$
$(x) + (y) = (2.27) + (0.63) = 3 + 1 = 4$
However, if one of two numbers is integer like x = 2 and y = 0.63, then the proposed identity as above is true.

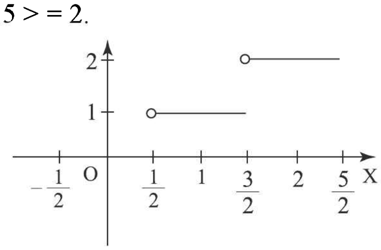
(iv) $(-x) = -(x), x \in I$
$(-x) = -(x) + 1, x \notin I$

Nearest Integer Function

The nearest integer function, as the name suggests, returns the nearest integer. It is denoted by the symbol, $< x >$.
The value of $< x >$ is an integer n such that :

$f(x) = < x >$
$= n$; if $n \le x \le n + 1/2, n \in I$
$= n + 1$; if $n + 1/2 < x \le n + 1, n \in I$

For example $< 2.3 > = 2$, $< 2.6 > = 3$
$< -2.3 > = -2$, $< -2.6 > = -3$
$< 2.5 > = 2$.



Note

The following definitions help in simplification of functions involving GIF.

$$f([x]) = \begin{cases} ---- & ----- \\ f(-2) & -2 \le x < -1 \\ f(-1) & -1 \le x < 0 \\ f(0) & 0 \le x < 1 \\ f(1) & 1 \le x < 2 \\ f(2) & 2 \le x < 3 \\ ---- & ----- \end{cases}$$

$$[f(x)] = \begin{cases} -- & ------ \\ -2 & -2 \le f(x) < -1 \\ -1 & -1 \le f(x) < 0 \\ 0 & 0 \le f(x) < 1 \\ 1 & 1 \le f(x) < 2 \\ 2 & 2 \le f(x) < 3 \\ ---- & ----- \end{cases}$$

$$[nx] = \begin{cases} n[x] \quad \text{if} \quad 0 \le \{x\} < 1/n \\ n[x]+1 \text{ if } 1/x \le \{x\} < 2/n \\ n[x]+2 \text{ if } 2/n \le \{x\} < 3/n \\ -------------- \\ n[x]+n-1 \text{ if } \frac{n-1}{n} \le \{x\} < n \end{cases}$$

Equations in [x] and {x}

To solve equations is one variable (say x) involving greatest integer and fractional part of x, we follow the following steps :

1. Simplify the given equation / inequality in the form of only greatest integer part and fractional part using the fact $x = [x] + \{x\}$.
2. Now find the fractional part in terms of greatest integer part i.e. $\{x\} = f([x])$.
3. Now use the fact $0 \le \{x\} < 1$ to write the given inequality in the form $0 \le f([x]) < 1$.

4. Simplify the above inequality so that [x] is obtained in some interval $a \le [x] < b$.

   Now as [x] is an integer, find the integers satisfying the above inequality.

5. Now from $\{x\} = f([x])$, find the corresponding values of $\{x\}$ for each value of [x] thus obtained.

6. Now using $x = [x] + \{x\}$, we get the values of x, the solution(s) of the given equation.

EXAMPLE 1.17 Solve $4\{x\} = x + [x]$, where $\{x\}$ and $[x]$ denote the fractional and integral part of a real number x respectively.

SOLUTION As $x = [x] + \{x\}$

$$4\{x\} = x + [x]$$

$$\Rightarrow 4\{x\} = [x] + \{x\} + [x]$$

$$\Rightarrow \{x\} = \frac{2[x]}{3}.$$

As [x] is always an integer and $\{x\} \in [0, 1)$, the possible values are

[x]	{x}	x = [x] + {x}
0	0	0
1	$\frac{2}{3}$	$\frac{5}{3}$

$\therefore$ There are two solutions of the given equation

$$x = 0, \frac{5}{3}.$$

n - f method

For simplicity, we can use the following notations :

$[x] = n, n \in I$

$\{x\} = f, 0 \le f < 1$

$\therefore\ x = n + f$

Let us solve the equation $2x - [x] = 3$.

Using the above notations, we get

$$2(n + f) - n = 3$$

$$n + 2f = 3$$

$$2f = 3 - n \qquad ...(1)$$

Here, R.H.S. is an integer, hence L.H.S. should also be an integer.

So, 2f is an integer.

Since, $0 \le f < 1$ we have $0 \le 2f < 2$.

The only integers in this interval are 0 and 1.

Let $2f = 0$ Let $2f = 1$

$\Rightarrow f = 0$ $\Rightarrow f = 1/2$

From (1)

$n = 3$ $\qquad n + 1 = 3 \Rightarrow n = 2$

Since $x = n + f$

$x = 3 + 0$ $\qquad x = 2 + 1/2$

$x = 3$ $\qquad x = 5/2$.

Hence, the solutions are 3 and 5/2.

EXAMPLE 1.18 Solve for x: $2x+3\{x\}=4[x]-2$

SOLUTION $2x + 3\{x\} = 4[x] - 2$

$$\Rightarrow 2(n + f) + 3f = 4n - 2$$

$$\Rightarrow f = \frac{2(n-2)}{5}$$

Now, $0 \le f < 1 \Rightarrow 0 \le \frac{2(n-2)}{5} < 1$

$\Rightarrow 2 \le n < 9/2$

$\Rightarrow n = 2, 3, 4$

For $n = 2, f = 0 \Rightarrow x = 2$

$n = 3, f = 2/5 \Rightarrow x = 3 + 2/5 = 17/5$

$n = 4, f = 4/5 \Rightarrow x = 4 + 4/5 = 24/5$

Therefore, solution to given equation are

2, 17/5, 24/5.

EXAMPLE 1.19 Solve for x: $3x+5\{x\}=4[2x]+3$

SOLUTION $3x + 5\{x\} = 4[2x] + 3$

As $x = [x] + \{x\}$ where $0 \le \{x\} < 1$

Now $2x = 2[x] + 2\{x\}$ for $0 \le 2\{x\} < 2$

$\Rightarrow [2x] = 2[x]$ for $0 \le 2\{x\} < 1$

or $0 \le \{x\} < 1/2$

and $[2x] = 2[x] + 1$ for $1 \le 2\{x\} < 2$

or $1/2 \le \{x\} < 1$

Case 1: $0 \le \{x\} < 1/2$

$3x + 5\{x\} = 4[2x] + 3$

$\Rightarrow 3[x] + 3\{x\} + 5\{x\} = 8[x] + 3$

$\Rightarrow 8\{x\} = 5[x] + 3$

$\Rightarrow \{x\} = (5[x] + 3)/8$ ...(1)

Now as $0 \le \{x\} < 1/2$

$\Rightarrow 0 \le (5[x] + 3)/8 < 1/2$

$\Rightarrow 0 \le 5[x] + 3 < 4$

$\Rightarrow -3 \le 5[x] < 1$

$\Rightarrow -3/5 \le [x] < 1/5$

$\Rightarrow [x] = 0$

From (1), for $[x] = 0$, $\{x\} = (5.0 + 3)/8 = 3/8$

$\Rightarrow x = 0 + 3/8 = 3/8$

Case 2: $1/2 \le \{x\} < 1$

$3x + 5\{x\} = 4[2x] + 3$

$\Rightarrow 3[x] + 3\{x\} + 5\{x\} = 4(2[x] + 1) + 3$

$\Rightarrow 8\{x\} = 5[x] + 7$

$\Rightarrow \{x\} = (5[x] + 7)/8$ ...(2)

Now as $1/2 \le \{x\} < 1$

$\Rightarrow 1/2 \le (5[x] + 7)/8 < 1$

$\Rightarrow 4 \le 5[x] + 7 < 8$

$\Rightarrow -3 \le 5[x] < 1$

$\Rightarrow -3/5 \le [x] < 1/5$

$\Rightarrow [x] = 0$

From (2), for $[x] = 0$, $\{x\} = (5.0 + 7)/8 = 7/8$

$\Rightarrow x = 0 + 7/8 = 7/8$

Therefore, the solutions to given equation are 3/8 and 7/8.

EXAMPLE 1.20 Solve $|2x - 1| = 3[x] + 2\{x\}$.

SOLUTION $|2x - 1| = 3[x] + 2\{x\}$. ...(1)

Case 1: Let $x \ge 1/2$

Equation (1) reduces to

$2x - 1 = 3[x] + 2\{x\}$.

$\Rightarrow 2([x] + \{x\}) - 1 = 3[x] + 2\{x\}$ ...(2)

$\Rightarrow [x] = -1$

which is not acceptable since $x \ge 1/2$.

Case 2: $x < 1/2$

Equation (1) reduces to

$1 - 2x = 3[x] + 2\{x\}$

$\Rightarrow 1 - 2([x] + \{x\}) = 3[x] + 2\{x\}$

$\Rightarrow 1 - 5[x] = 4\{x\}$ ...(3)

Since, $0 \le \{x\} < 1$, therefore we have

$0 \le 1 - 5[x] < 4$

$\Rightarrow 0 \le 5[x] - 1 > -4$

$\Rightarrow 1 \le 5[x] > -3$

$\Rightarrow \dfrac{-3}{5} < [x] \le \dfrac{1}{5}$

Hence $[x] = 0$, since 0 is the only integer lying in $\left(\dfrac{-3}{5}, \dfrac{1}{5}\right]$.

Putting the above value in equation (3), we have

$\{x\} = \dfrac{1}{4}$

Hence, $x = \dfrac{1}{4}$ is the only solution.

EXAMPLE 1.21 Let F (x) be a function defined by

$F(x) = x - [x], x \in R - \{0\}$

where [x] is the greatest integer less than or equal to x . Then find the number of solutions of $F(x) + F(1/x) = 1$.

SOLUTION $\because F(x) = x - [x]$, $x \ne 0$

$$F(x) + F\left(\frac{1}{x}\right) = 1$$

$$\Rightarrow x - [x] + \frac{1}{x} - \left[\frac{1}{x}\right] = 1$$

$$\Rightarrow x + \frac{1}{x} = [x] + \left[\frac{1}{x}\right] + 1$$

$\Rightarrow \left(x+\frac{1}{x}\right)-\left([x]+\frac{1}{x}\right)=1$ ...(1)

$\therefore$ R. H.S. is an integer, hence L. H.S. is also integer

Let $[x]+\left[\frac{1}{x}\right]+1=A$ (integer)

then equation (1) becomes $x+\frac{1}{x}=A$

$\Rightarrow x^2-Ax+1=0$

$\Rightarrow x=\frac{A\pm\sqrt{A^2-4}}{2}$

For real x, $A^2-4\geq 0$

$\Rightarrow A\geq 2$ and $A\leq -2$

$A=2$ and $A=-2$ does not satisfy (1)

The equation (1) has infinitely many solutions.

EXAMPLE 1.22 Solve the equation $(x)^2=[x]^2+2x$, where [x] and (x) are the integer just less than or equal to x and just greater than or equal to x repectively.

SOLUTION **Case 1:** If $x\notin I$

then $(x)=[x]$

$\therefore (x)^2=[x]^2+2x$

$\Rightarrow [x]^2=[x]^2+2x$

$\therefore x=0$

Case 2: If $x\notin I$

then $(x)=[x]+1$

$\Rightarrow (x)^2=[x]^2+2x$

$\Rightarrow [x]^2+1+2[x]=[x]^2+2x$

$\Rightarrow x=[x]+\frac{1}{2}=n+\frac{1}{2}, n\in I$

Hence the solution of the original equation is

$x=0, n+\frac{1}{2}, n\in I$

EXAMPLE 1.23 Solve the equation,

$x^2-4x+[x]+3=0.$

SOLUTION We are given that $x^2-4x+[x]+3=0$

$\Rightarrow x^2-4x+x-\{x\}+3=0$

$\Rightarrow x^2-3x+3=\{x\}$

$\Rightarrow 0\leq x^2-3x+3<1$

Now $x^2-3x+3>0\ \forall\ x\in R$ since $D<0$.

Also $x^2-3x+3<1$

$\Rightarrow x^2-3x+2<0$

$\Rightarrow 1<x<2, [x]=1.$

Now from the original equation we have

$x^2-4x+4=0 \Rightarrow (x-2)^2=0$

$\Rightarrow x=2,$

which does not satisfy $1<x<2$

Thus, the given equation does not have any solution.

EXAMPLE 1.24 Find the solution set of

$(x)^2+(x+1)^2=25,$

where (x) is the least integer greater than or equal to x.

SOLUTION

Case 1: If $x\in I$ then $(x)=[x]$,

and $(x)^2+(x+1)^2=25$ reduces to

$[x]^2+[x+1]^2=25$

$\Rightarrow [x]^2+([x]+1)^2=25$

$\Rightarrow 2[x]^2+2[x]-24=0$

$\Rightarrow [x]^2+[x]-12=0$

$\Rightarrow ([x]+4)([x]-3)=0$

$\therefore [x]=-1$ and $[x]=3$

$\Rightarrow x\in[-4,-3)\cup[3,4)$

$\because x\in I, x=-4, 3$ ...(1)

Case 2: If $x\notin I$ then $(x)=[x]+1$

and $(x)^2+(x+1)^2=25$ reduces to

$([x]+1)^2+([x+1]+1)^2=25$

$\Rightarrow ([x]+1)^2+([x]+2)^2=25$

$\Rightarrow 2[x]^2+6[x]-20=0$

$\Rightarrow$ $[x]^2 + 3[x] - 10 = 0$

$\Rightarrow$ $([x] + 5)([x] - 2) = 0$

$\therefore$ $[x] = -5$ and $[x] = 2$

$\Rightarrow$ $x \in [-5, -4) \cup [2, 3)$

$\because$ $x \notin I$, $x \in (-5, -4] \cup (2, 3)$ ...(2)

Combining (1) and (2), we get $x \in (-5, -4] \cup [2, 3]$.

EXAMPLE 1.25 Solve the equations

$$y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]] \text{ and}$$

$$[y + [y]] = 2\cos x.$$

SOLUTION We have

$$y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$$

$$= \frac{1}{3}[\sin x + [\sin x] + [\sin x]]$$

[$\because$ $[n + x] = n + [x]$ for any integer n]

$$= \frac{1}{3}([\sin x]+[\sin x]+[\sin x])=[\sin x]$$

and $[y + [y]] = 2\cos x$

$\Rightarrow$ $2[y] = 2\cos x$

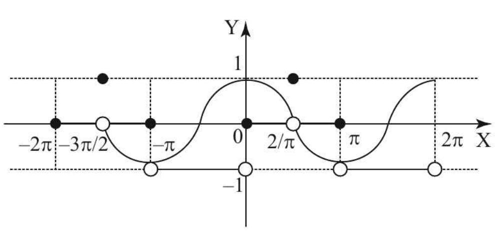
$\Rightarrow$ $[y] = \cos x$

From equations (1) and (2), we have

$[[\sin x]] = \cos x$

$\Rightarrow$ $[\sin x] = \cos x$

Now, we plot the curves $y = [\sin x]$ and $y = \cos x$ on the same axes.



We can see that the two curves have no intersection points. Hence, the given equation has no solution.

Inequations in [x] and {x}

EXAMPLE 1.26 Solve $[x]^2 - [x] - 6 > 0$

SOLUTION $[x]^2 - [x] - 6 > 0$

or $\{[x] - 3\}\{[x] + 2\} > 0$.

Using sign-scheme,

$\therefore$ $[x] < -2$ or $[x] > 3$.

But $[x] < -2$ $\Rightarrow$ $[x] = -3, -4, -5, \ldots$.

$\therefore$ $x < -2$

$[x] >$ $\Rightarrow$ $[x] = 4, 5, 6, \ldots$..

$\therefore$ $x \geq 4$

$\therefore$ $x \in (-\infty, -2) \cup [4, \infty)$.

EXAMPLE 1.27 Solve the inequality

$x[x] - x^2 - 3[x] + 3x > 0$.

SOLUTION We have $x[x] - x^2 - 3[x] + 3x > 0$

$\Rightarrow$ $x(x - \{x\}) - x^2 - 3(x - \{x\}) + 3x > 0$

$\Rightarrow$ $(3 - x)\{x\} > 0$

We have $0 \leq \{x\} < 1$

If $x \in I$ then $\{x\} = 0$ and the inequality cannot hold.

If $x \in I$ then $\{x\}$ is a positive number and then

$3 - x > 0$ $\Rightarrow$ $x < 3$.

Hence, the solution consists of all x less than 3 except integers.

EXAMPLE 1.28 If $a_1, a_2, \ldots a_n$ are integers with

$s = a_1 + a_2 + \ldots + a_n$ then prove that

$$\left[\frac{s}{a}\right] \geq \left[\frac{a_1}{a}\right] + \left[\frac{a_2}{a}\right] + \ldots\ldots + \left[\frac{a_n}{a}\right]$$

for any integer $a > 0$.

SOLUTION We may write $a_j = aq_j + r_j$ with

$0 \leq r_j < a$ for $j = 1, 2, \ldots n$. Then

$s = a_1 + a_2 + \ldots + a_n$ gives

$s = a(q_1 + q_2 + \ldots + q_n) + (r_1 + r_2 + \ldots\ldots + r_n)$.

$\therefore$ $\left[\frac{s}{a}\right]$

$$= \left[q_1 + q_2 + \dots + q_n + \frac{(r_1 + r_2 + \dots + r_n)}{a}\right]$$

$$\geq q_1 + q_2 + \dots\dots q_n$$

$$= \left[\frac{a_1}{a}\right] + \left[\frac{a_2}{a}\right] + \left[\frac{a_n}{a}\right]$$

EXAMPLE 1.29 Prove the following properties of the greatest integer function:

(i) $[x + y] \geq [x] + [y]$

(ii) $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$ where n is a natural number

(iii) $\left[x + \frac{1}{2}\right] = [2x] - [x]$

SOLUTION

(i) Let x be an arbitrary real number. Then we can write $x = [x] + \alpha$ where α is a nonnegative number less than 1. Now let us represent y in the form

$y = [y] + \beta (0 \leq \beta < 1)$.

Then $x + y = [x] + [y] + \alpha + \beta$. Since $\alpha + \beta \geq 0$ the last equality shows that $[x] + [y]$ is an integer not exceeding $x + y$. Further, since $(x + y)$ is the greatest of the integers not exceeding $x + y$ we have

$[x + y] \geq [x] + [y]$.

(ii) Let us represent x in the form $x = [x] + \alpha$ where $0 \leq \alpha < 1$. The division of the integer $[x]$ by n results in a quotient q and a remainder r, that is

$[x] = qn + r$ $(0 \leq r \leq n - 1)$. Thus we have

$\frac{[x]}{n} = q + \frac{r}{n}$, $\left[\frac{[x]}{n}\right] = q$ and x

$= qn + r + \alpha = qn + r_1$

where $r_1 = r + \alpha < n$. Hence $x/n = q + r_1/n$ $(0 \leq r_1/n < 1)$ and $[x/n] = q = [[x]/n]$, which is what we intended to prove.

Alternative

Let us consider all whole numbers which do not exceed x and are divisible by n. The number of these whole numbers is obviously equal to $[x/n]$. Let us also consider all whole numbers which do not exceed $[x]$ and are divisible by n. Their number is equal to $[[x]/n]$. Now, since these groups of whole numbers coincide, the numbers in these groups are equal.

Hence $[[x]/n] = [x/n]$.

(iii) If $x - [x] < 1/2$ then $[x + 1/2] = [x]$, and we have $[2x] = 2[x]$ and

$[2x] - [x] = 2[x] - [x] = [x] = [x + 1/2]$.

If $x - [x] \geq 1/2$ then $[x + 1/2] = [x] + 1$, $[2x] = 2[x] + 1$,
and we again have $[2x] - [x]$
$= 2[x] + 1 - [x] = [x] + 1 = [x + 1/2]$

EXAMPLE 1.30 Prove that for every real number x,

$$[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \dots + \left[x + \frac{n-1}{n}\right] = [nx].$$

SOLUTION Let $x = [x] + y$, where $0 \leq y < 1$.
Let p be an integer such that $p - 1 \leq ny < p$
(This is always possibe because given a real number, we can always find two consecutive integers between which the number lies.)

Now $x + \frac{k}{n} = [x] + y + \frac{k}{n}$.

Also $y + \frac{k}{n}$ lies between $\frac{p-1+k}{n}$

and $\frac{p+k}{n}$.

So long as $\frac{p-1+k}{n} < 1$, i.e., $k < n - (p - 1)$,

$y + \frac{k}{n}$ is less than 1, and consequently,

$$\left[x + \frac{k}{n}\right] = [x],$$

$$\left[x + \frac{k}{n}\right] = [x] \text{ for } k = 0, 1,..., n - p.$$

But $\left[x + \frac{k}{n}\right] = [x] + 1$ for $k = n - p,....., n - 1$

$$\therefore\ [x] + \left[x + \frac{1}{n}\right] +... + \left[x + \frac{n-1}{n}\right]$$

$= [x] +..... +[x]$ $(n - p + 1$ times$)$

$+ ([x] + 1) + ([x] + 1) + (p-1)$ times

$= n[x] + (p-1).$ ...(1)

Also, $[nx] = [n[x] + ny], = n[y] + (p + 1),$

since $p - 1 \le ny < p,$ ...(2)

Form (1) and (2), we find that

$$[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] +.... + \left[x + \frac{n-1}{n}\right] = [nx].$$

EXAMPLE 1.31 If a, b and c are integers with $c > 0$, then prove that

$$\left[\frac{2a}{c}\right] + \left[\frac{2b+1}{c}\right] = \left[\frac{a}{c}\right] + \left[\frac{b}{c}\right] + \left[\frac{a+b+1}{c}\right].$$

SOLUTION Let $a = mc + r$ and $b = nc + s$, where m, n, r and s are integers and $0 \le r < c$ and $0 \le s < c$.

If $s \ge r$, then $2s + 1 \ge r + s + 1$.

so $\left[\frac{2s+1}{c}\right] \ge \left[\frac{r+s+1}{c}\right]$;

if $s < r$, then $2r \ge r + s + 1$, and $\left[\frac{2r}{c}\right] \ge \left[\frac{r+s+1}{c}\right]$;

In either case, $\left[\frac{2r}{c}\right] + \left[\frac{2s+1}{c}\right] \ge \left[\frac{r+s+1}{c}\right]$.

Further more, since $\left[\frac{r}{c}\right] = \left[\frac{s}{c}\right] = 0$, we can also say

that $\left[\frac{2r}{c}\right] = \left[\frac{2s+1}{c}\right] \ge \left[\frac{r}{c}\right] + \left[\frac{s}{c}\right] + \left[\frac{r+s+1}{c}\right]$

Now the proof :

$$\left[\frac{2a}{c}\right] + \left[\frac{2b+1}{c}\right] = \left[2m + \frac{2r}{c}\right] + \left[2n + \frac{2s+1}{c}\right] + \left[\frac{r+s+1}{c}\right]$$

$$= 2m + 2n + \left[\frac{2r}{c}\right] + \left[\frac{2s+1}{c}\right]$$

$$\ge m + \left[\frac{r}{c}\right] + n\left[\frac{s}{c}\right] + m + n + \left[\frac{r+s+1}{c}\right]$$

$$= \left[m + \frac{r}{c}\right] + \left[n + \frac{s}{c}\right] + \left[m + n + \frac{r+s+1}{c}\right]$$

$$= \left[\frac{a}{c}\right] + \left[\frac{b}{c}\right] + \left[\frac{a+b+1}{c}\right].$$

EXAMPLE 1.32 Let

$$f(n) = \frac{1}{n}\left(\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + ... + \left[\frac{n}{n}\right]\right).$$

Prove that $f(2n) > f(n)$.

SOLUTION We have $[x + y] \ge [x] + [y]$

$$f(2n) = \frac{1}{2n}\left(\left[\frac{2n}{1}\right] + \left[\frac{2n}{2}\right] + \left[\frac{2n}{3}\right] + ... \left[\frac{2n}{2n}\right]\right)$$

$$\ge \frac{1}{2n}\left(2\left[\frac{n}{1}\right] + 2\left[\frac{n}{2}\right] + ... 2\left[\frac{n}{n}\right] + \left[\frac{2n}{n+1}\right] + ... \left[\frac{2n}{2n}\right]\right) > = f(n)$$

PRACTICE PROBLEMS [P]

1. Find the value of $\left[\frac{3}{4}\right] + \left[\frac{3}{4}+\frac{1}{100}\right] + \left[\frac{3}{4}+\frac{2}{100}\right] + \ldots\ldots\ldots\ldots \left[\frac{3}{4}+\frac{99}{100}\right]$.
2. Find the exact value of
   $[\log_2 1] + [\log_2 2] + [\log_2 3] + \ldots. + [\log_2 66]$
   where [.] denotes the greastest integer function.
3. Prove that there exists no value of $n \in N$ such that $[e] + [e^2] + \ldots..[e^n] = 36$.
4. If x be a real number for which
   (x) = the integer just greater than or equal to x,
   [x] = the integer just less than or equal to x
   then solve $(x)^2 = [x]^2 + 2x$.
5. Solve $[x] + |x-2| \le 0$ and $-1 \le x \le 3$ where [x] is the greatest integer function.
6. Discuss the graph of $f(x) = \dfrac{1}{(x)-[x]}$
   where (x) is the least integer and [x] is the greatest integer function.
7. Find the number of different values which $\left[\dfrac{2x^2}{x^2+1}\right]$ can assume.
8. Prove that the equation $\{x\} + x^2 = 0$ has only one solution.
9. Find the number of values of x satisfying
   $\{x^2\} + [x^4] = 1$.
10. If $y = 4 - [x]^2$ where x is real then find x for which $[y] + y = 6$ where [x] is the greatest integer $\le$ x.
11. Let [x] = the greatest integer $\le$ x, and {x}, (x) be defined as follows :
    $\{x\} = x - [x]$; $(x) = \begin{cases} 2[x]-\{x\}, & x<0 \\ [x]+3[x], & x \le 0 \end{cases}$
    then solve the equation $(x) = x + \{x\}$.
12. Solve $[x] + |x-2| \le 0$, $-1 \le x \le 3$
    where [x] is the greatest integer function.
13. If $\{x\} = x - [x]$ where [x] = the greatest integer $\le$ x and $f(x) = \{x\}\,[x]$ then find the set
    $A = \{x \mid f(x) = 1, x \in R\}$.

14. Prove that the greatest integer function has the properties indicated.

    (a) $[x + y] = [x] + [y]$ or $[x] + [y] + 1$. (b) $[2x] = [x] + \left[x + \frac{1}{2}\right]$

    (c) $[3x] = [x] + \left[x + \frac{1}{3}\right] + \left[x + \frac{2}{3}\right]$.

15. Find the number of solutions of the equation

    $\{x\} + \{\tan \pi x\} = 0$.

16. Show that $\left[\frac{x-1}{2}\right] + \left[\frac{x}{2}\right]$ is identically equal to $[x - 1]$.

17. If $\frac{\pi}{6} \le x \le \frac{\pi}{3}$ and [.] denotes greatest integer function, then find the number of values taken by $[\sec x] + [\text{cosec } x]$.

18. Find the complete set of x such that

    $f(x) = \log_{[x-1]} \sin x$,

    is defined where [.] denotes greatest integer function.

19. Find the values of x satisfying the equation

    $\left\{\frac{\cos^2 x - 2}{2}\right\} = \frac{1}{4}$.

20. Find the complete set of value of x satisfying

    $\text{sgn}([x]) > \text{sgn}(x)$.

21. Find the range of the function $y = \left[\sin \frac{\pi}{2[x]}\right]$.

22. Find the number of solutions of the equation

    $2[x] - x = \{x\} + 1$.

23. Which of the following functions is not identically zero ?

    (i) $y = [\{x\}]$ (ii) $y = \left[\frac{\sin^2 x}{2}\right]$

    (iii) $\left\{\cos \frac{\pi}{2}[x]\right\}$ (iv) $y = \left[\cos^2 \frac{\pi}{2} x\right]$

24. Find the complete solution set of the inequality $[\sqrt{x}] \ge [x^2]$.

25. Find the number of points at which the function $y = \left\{\frac{x^2+1}{2}\right\}$, $0 \le x \le 2$ assumes integral values.

26. If 'x' satisfies the equation

    $[x + 0.19] + [x + 0.20] \ldots. + [x + 0.91] = 542$ where [.] stands for G.I.F then find the value of $[100x]$.

TARGET PROBLEMS for JEE ADVANCED

PROBLEM 1.1 For $x \in R$, let $||x||$ be defined as follows

$$||x|| = \begin{cases} x+1, & 0 \le x < 2 \\ |x-4|, & 2 \le x \end{cases}$$

then solve the equation $||x||^2 + 2 = ||x|| + x^2$.

SOLUTION If $0 \le x < 2$, the equation is

$(x + 1)^2 + x = x + 1 + x^2$

or $x^2 + 2x + 1 + x = x + 1 + x^2$

or $2x = 0$, $\therefore$ $x = 0$ which satisfies $0 \le x < 2$.

If $x \ge 2$, the equation is

$|x - 4|^2 + 2 = |x - 4| + x^2$

or $(x - 4)^2 + x = |x - 4| + x^2$

or $x^2 - 8x + 16 + x = |x - 4| + x^2$

or $|x - 4| = 16 - 7x$

$\therefore$ $x - 4 = \pm (16 - 7x)$

provided $16 - 7x \ge 0$, i.e., $x \le \frac{16}{7}$.

When $x - 4 = 16 - 7x$ then $8x = 20$

or $x = \frac{5}{2}$ which satisfies $x \ge 2$ and $x \le \frac{16}{7}$.

$\therefore$ The solutions are $x = 0, 2$.

PROBLEM 1.2 Find two values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation

$$(1 - \tan\theta)(1 + \tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0.$$

SOLUTION Here, $(1-\tan^2\theta)(1+\tan^2\theta)+2^{\tan^2\theta}=0$.

Put $\tan^2\theta = x$; then $(1 - x)(1 + x) + 2^x = 0$

or $1 - x^2 + 2x = 0$

or $2x = x^2 - 1$

By observation $x = 3$ is a root because

$2^x = 2^3 = 8$ and $x^2 - 1 = 3^2 - 1 = 8$.

$\therefore$ $x = 3$ is a solution, so $\tan^2\theta = 3$.

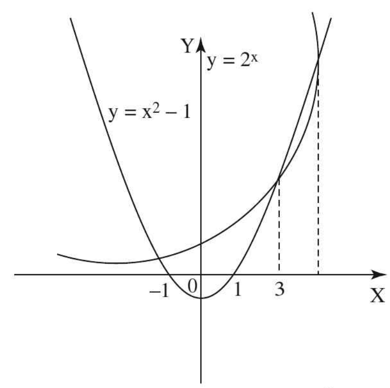
$\therefore$ $\tan\theta = \sqrt{3}$, $\therefore$ $\theta = \frac{\pi}{3}, \frac{-\pi}{3}$.

Thus we find the equation has two solutions

$\theta = \frac{\pi}{3}, \frac{-\pi}{3}$

Now, we want to find values of θ satisfying the given equation other than the values $\theta = \pm\frac{\pi}{3}$.

The solutions of (1) are given by the intersections of the graphs of $y = 2^x$ and $y = x^2 - 1$. The graphs are as follows:



Clearly the curves $y = 2^x$ and $y = x^2 - 1$ intersect at three points only, one of which has negative x-coordinate.

As $x = \tan^2\theta$, x must be positive.

One value of x found on trial is 3.

$\Rightarrow$ $\tan^2\theta = 3$ $\Rightarrow$ $\tan\theta = \pm\sqrt{3}$

$\therefore$ $\theta = \frac{\pi}{3}, \frac{-\pi}{3}$.

Note that there are two more solutions but they cannot be found by inspection.

PROBLEM 1.3 If $x = \frac{4}{9}$ satisfies the inequality

$$\log_a (x^2 - x + 2) > \log_a (-x^2 + 2x + 3)$$

then find all real solutions of the inequation.

SOLUTION Putting $x = \frac{4}{9}$ in the inequation,

$$\log_a\left\{\frac{16}{81}-\frac{4}{9}+2\right\} > \log_a\left\{\frac{16}{81}-\frac{8}{9}+3\right\}$$

or $\log_a \frac{16-36+162}{81} > \log_a \frac{-16+72+243}{81}$

or $\log_a \frac{142}{81} > \log_a \frac{299}{81}$

As $\frac{142}{81} < \frac{299}{81}$ we can conclude $0 < a < 1$.

Thus we have

$$\log_a (x^2 - x + 2) > \log_a (-x^2 + 2x + 3)$$

where $0 < a < 1$.

$\therefore\ x^2 - x + 2 < -x^2 + 2x + 3$

or $2x^2 - 3x - 1 < 0$.

The corresponding equation is $2x^2 - 3x - 1 = 0$

or $x = \frac{3 \pm \sqrt{9+8}}{4} = \frac{3 \pm \sqrt{17}}{4}$

$\therefore$ the sign-scheme of the expression $2x^2 - 3x - 1$, is as follows:

$$+ \quad \frac{3-\sqrt{17}}{4} \quad - \quad \frac{3+\sqrt{17}}{4} \quad +$$

$\therefore\ 2x^2 - 3x - 1 < 0$

$\therefore\ \frac{3-\sqrt{17}}{4} < x < \frac{3+\sqrt{17}}{4}$

But, in order that the given logarithms may be defined we must have

$x^2 - x + 2 > 0$ and $-x^2 + 2x + 3 > 0$.

$D = 1 - 8 < 0$ and coefficient of $x^2 = 1 > 0$

$\therefore\ x^2 - x + 2 > 0$ is true for all $x \in R$.

The sign-scheme of $-x^2 + 2x + 3$, $x \in R$ is as follows:

$$- \quad -1 \quad + \quad 3 \quad -$$

$\therefore\ -x^2 + 2x + 3 > 0 \ \Rightarrow\ -1 < x < 3$ ...(2)

$\therefore$ The required values of x form the set of values of x common to (1) and (2).

$\therefore$ The solution is $\left(\frac{3-\sqrt{17}}{4}, \frac{3+\sqrt{17}}{4}\right)$.

PROBLEM 1.4 Solve for x :

$$\log_{x-2} 6 + \log_{x+2} 6 > \log_{x-2} 6 \,.\, \log_{x+2} 6.$$

SOLUTION The logarithms will be defined if

$x - 2 > 0,\ x - 2 \neq 1,\ x + 2 > 0,\ x + 2 \neq 1$

$\Rightarrow\ x > 2,\ x \neq 3$

Now, $\frac{\log 6}{\log(x-2)} + \frac{\log 6}{\log(x+2)} > \frac{\log 6}{\log(x-2)} \times \frac{\log 6}{\log(x+2)}$

or $\frac{\log(x+2)+\log(x-2)}{\log(x-2).\log(x+2)} > \frac{\log 6}{\log(x-2).\log(x+2)}$

or $\frac{\log(x^2-4)-\log 6}{\log(x-2).\log(x+2)} > 0$

or $\log_{10}\frac{x^2-4}{6} \,.\, \log_{10}(x-2)\,.\, \log_{10}(x+2) > 0$,

taking the base 10.

Now $\log_{10}\frac{x^2-4}{6} = 0$ when $x = \pm\sqrt{10}$

$\log_{10}(x-2) = 0$ when $x = 3$

$\log_{10}(x+2) = 0$ when $x = -1$

Within the domain of inequality, the sign scheme of the product is as follows :

$$2 \quad + \quad 3 \quad - \quad \sqrt{10} \quad +$$

Hence, the solution set is $x \in (2, 3) \cup (\sqrt{10}, \infty)$.

PROBLEM 1.5 Find all values of the parameter $a \in R$ for which the following inequality is valid for all $x \in R$, $1 + \log_5(x^2 + 1) \geq \log_5(ax^2 + 4x + a)$.

SOLUTION Here $1 \geq \log_5(ax^2+4x+a) - \log_5(x^2+1)$

or $1 \geq \log_5 \dfrac{ax^2+4x+a}{x^2+1}$

or $\log_5 5 \geq \log_5 \dfrac{ax^2+4x+a}{x^2+1}$

$\therefore$ we have $5 \geq \dfrac{ax^2+4x+a}{x^2+1}$ ...(1)

Also, for the logarithms to be defined,

$x^2 + 1 > 0$ and $ax^2 + 4x + a > 0$.

$x^2 + 1 > 0$ is true for all $x \in R$.

Now, $ax^2 + 4x + a > 0$ ...(2)

$\therefore$ the values of the parameter $a \in R$ should be such that both (1) and (2) hold for all $x \in R$.

Now, (2) holds for all $x \in R$ if

$D = 4^2 - 4 . a . a < 0$ and coefficient of x^2 is $a > 0$,

i.e. $16 - 4a^2 < 0, a > 0$

$\Rightarrow a^2 > 4, a > 0 \Rightarrow a > 2$.

Now, from (1),

$5(x^2 + 1) \geq ax^2 + 4x + a \quad \{\because x^2 + 1 > 0\}$

or $(5 - a)x^2 - 4x + 5 - a \geq 0$.

This holds for all $x \in R$ if

$D = (-4)^2 - 4(5 - a)(5 - a) \leq 0$

and the coefficient of $x^2 = 5 - a > 0$,

i.e. $16 - 4(5 - a)^2 \leq 0, 5 - a > 0$

$\Rightarrow (5 - a)^2 \geq 4, a < 5$

$\Rightarrow \{5 - a \leq -2 \text{ or } 5 - a \geq 2\}, a < 5$

$\Rightarrow \{a \geq 7 \text{ or } a \leq 3\}, a < 5$

$\Rightarrow a \leq 3$.

Hence, the required values of a satisfy

$a > 2$ and $a \leq 3$.

$\therefore a \in (2, 3]$.

PROBLEM 1.6 Solve the equation

$$\sqrt{x - 4a + 16} = 2\sqrt{x - 2a + 4} - \sqrt{x}.$$

For what real values of a is the equation solvable ?

SOLUTION Transposing $\sqrt{x}$ to the left-hand side and squaring both members of the equation we get

$$\sqrt{x} . \sqrt{x - 4a + 16} = x - 2a.$$

Squaring then both sides of the resulting equation we find that $x = \dfrac{a^2}{4}$ is the only root of the equation. Substituting it into the equation we obtain

$$\sqrt{a^2 - 16a + 64},$$

$$= 2\sqrt{a^2 - 8a + 16} - \sqrt{a^2}$$

which implies, since the radicals are positive, the relation

$$|a - 8| = 2|a - 4| - |a| \quad ...(1)$$

For $a \geq 8$ equality (1) is fulfilled. Consequently, for $a \geq 8$ the original equation has a root $x = \dfrac{a^2}{4}$.

For $4 \leq a < 8$ condition (1) is not fulfilled because

$8 - a \neq 2(a - 4) - a$.

For $0 \leq a < 4$ condition (1) takes the form

$8 - a = 2(4 - a) - a$

and is only fulfilled for $a = 0$. Finally, for $a < 0$ condition (1) turns into the identity

$8 - a = 2(4 - a) + a$.

Hence, for $a \geq 8$ and $a \leq 0$ the equation has the only root $x = \dfrac{a^2}{4}$ for $0 < a < 8$ there are no roots at all.

PROBLEM 1.7 For what real values of x is the inequality

$$\frac{1 - \sqrt{1 - 4x^2}}{x} < 3 \text{ fulfilled ?}$$

SOLUTION The radicand must be ≥ 0, and therefore

$$-\frac{1}{2} \leq x \leq \frac{1}{2} \qquad \ldots(1)$$

For nonzero values of x satisfying condition (1) we have $\sqrt{1-4x^2} < 1$. Therefore, if $-\frac{1}{2} \leq x < 0$, the inequality indicated in the problem is fulfilled, because its left-hand side is negative.

But if $0 < x \leq \frac{1}{2}$, then rationalizing the numerator of the left hand side we obtain

$$\frac{1-\sqrt{1-4x^2}}{x} = \frac{4x^2}{(1+\sqrt{1-4x^2})x}$$

$$= \frac{4x}{1+\sqrt{1-4x^2}}$$

It is readily seen that the numerator of the fraction on the right-hand side does not exceed 2 for $0 < x \leq \frac{1}{2}$, and the denominator is not less than unity. Therefore,

$$\frac{1-\sqrt{1-4x^2}}{x} \leq 2 < 3.$$

Thus, the inequality in question is true for the values $x \neq 0$ satisfying condition (1). For $x = 0$ and $|x| > \frac{1}{2}$ the left member of the inequality makes no sense.

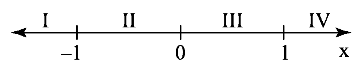
PROBLEM 1.8 Solve for x

$$\frac{(|x-1|-1)(|x|-3)}{4-|1+x|} < 0.$$

SOLUTION Let $S = \frac{(|x-1|-1)(|x|-3)}{4-|1+x|}$

Since
$$|x-1| = x-1, \quad x \geq 1$$
$$= -(x-1), \quad x < 1$$
$$|1+x| = 1+x, \quad x \geq -1$$
$$= -(1+x), \quad x < -1$$
$$|x| = x, \quad x \geq 0$$
$$= -x, \quad x < 0$$

As the expressions $|1+x|$, $|x|$ and $|x-1|$ change their signs at $x = -1$, 0 and 1 respectively, we divide the number line into four parts as follows:



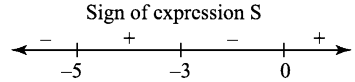
Case I: $x < -1$

$$S = \frac{(-(x-1)-1)(-x-3)}{4-(-(1+x))}$$

$$\Rightarrow S = \frac{(-x+1-1)(-x-3)}{4+1+x}$$

$$\Rightarrow S = \frac{x(x+3)}{5+x}$$

Sign of expression S



For $S < 0$, $x < -5$ and $-3 < x < 0$

But as $x < -1$, therefore solution of given inequality is $x < -5$ and $-3 < x < -1$.

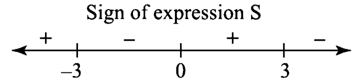
Case II: $-1 \leq x < 0$

$$S = \frac{(-(x-1)-1)(-x-3)}{4-(1+x)}$$

$$\Rightarrow S = \frac{(-x+1-1)(-x-3)}{4-1-x}$$

$$\Rightarrow S = \frac{x(x+3)}{(3-x)}$$

Sign of expression S



For $S < 0$, $-3 < x < 0$ and $x > 3$

But as $-1 \leq x < 0$, therefore solution of given inequality is $-1 \leq x < 0$.

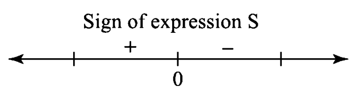
Case III: $0 \le x < 1$

$$S = \frac{(-(x-1)-1)(-x-3)}{4-(1+x)}$$

$$\Rightarrow S = \frac{(-x+1-1)(x-3)}{(4-1-x)}$$

$$\Rightarrow S = \frac{-x(x-3)}{(3-x)}$$

$\Rightarrow S = x$ for $x \ne 3$



For $S < 0$, $x < 0$

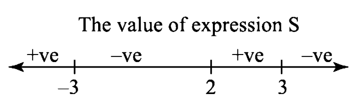
But as $0 \le x < 1$, we have no solution of the given inequality.

Case IV: $x \ge 1$

$$S = \frac{(x-1-1)(x-3)}{4-(1+x)}$$

$$\Rightarrow S = \frac{(x-2)(-x-3)}{(4-1-x)}$$

$$\Rightarrow S = \frac{(x-2)(x+3)}{(3-x)}$$



For $S < 0$, $-3 < x < 2$ and $x > 3$

But as $x \ge 1$, therefore solution of given inequality is $1 \le x < 2$ and $x > 3$.

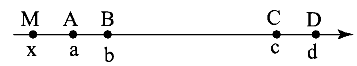
Therefore, the solution of the given inequality is $x \in (-\infty, 5) \cup (-3, -1] \cup (-1, 0) \cup [1, 2) \cup (3, \infty)$ i.e. $x \in (-\infty, -5) \cup (-3, 0) \cup [1, 2) \cup (3, \infty)$

PROBLEM 1.9 Find the least value of the function

$$f(x) = |x-a| + |x-b| + |x-c| + |x-d|,$$

where $a < b < c < d$ are fixed real numbers and x takes arbitrary real values.

SOLUTION Let us take a number scale and mark on it the points A, B, C and D corresponding to the numbers a, b, c and d. Let M denote a point with variable abscissa x. There can be the following five cases here :



(i) If $x \le a$, then we have

$\varphi(x) = MA + MB + MC + MD$

$= AB + 2MB + 2BC + CD$,

which clearly shows that $\varphi(x)$ attains the least value when the point M coincides with the point A and that this value is equal to

$3AB + 2BC + CD$.

(ii) If $a \le x \le b$, then

$\varphi(x) = AM + MB + MC + MD$

$= AB + 2MB + 2BC + CD$.

In this case the least value is attained by the function $\varphi(x)$ when the point M coincides with the point B, this value being equal to

$AB + 2BC + CD$.

(iii) If $b \le x \le c$, then for these values of x the function $\varphi(x)$ is constant and is equal to

$AB + 2BC + CD$

(iv) If $c \le x < d$, then the least value of the function $\varphi(x)$ is attained at the point $x = c$, and it is also equal to $AB + 2BC + CD$

(v) If $x \ge d$, then the least value of the function $\varphi(x)$ is equal to

$AB + 2BC + 3CD$

comparing the results thus obtained we see that the least value of the function $\varphi(x)$ is equal to $AB + 2BC + CD$, that is to

$b - a + 2(c - b) + d = d + c - b - a$.

This value the function $\varphi(x)$ takes on provided

$b \le x \le c$.

PROBLEM 1.10 Find range of the function:

$$f(x) = \frac{e^x - e^{-|x|}}{e^x + e|x|}$$

SOLUTION We observe that for $x \le 0$,

$f(x) = 0$.

For $x > 0$

$$\Rightarrow y = f(x) = \frac{e^x - e^{-|x|}}{e^x + e|x|} = \frac{2^{2x} - 1}{2e^{2x}}$$

$$\Rightarrow y.2e^{2x} = e^{2x} - 1 \quad \Rightarrow \quad (1 - 2y)\, e^{2x} = 1$$

$$\Rightarrow e^{2x} = \frac{1}{1 - 2y}$$

We can see that $e^{2x} \ge 1$ for all x. Hence,

$$\Rightarrow \frac{1}{1-2y} \ge 1 \quad \Rightarrow \quad \frac{1}{1-2y} - 1 \ge 0$$

$$\Rightarrow \frac{1 - 1 + 2y}{1 - 2y} \ge 0$$

$$\Rightarrow \frac{2y}{1-2y} \ge 0$$

$$\Rightarrow \frac{2y}{2y-1} \le 0$$

Here, critical points are 0,1. Thus, range of the given function is :

$$\text{Range} = \left[0, \frac{1}{2}\right)$$

PROBLEM 1.11 If the equation $\left(\frac{1}{4}\right)^x + \left(\frac{1}{2}\right)^{x-1} + b = 0$ has a positive solution, then find the interval in which the real number b lies.

SOLUTION $\left(\frac{1}{2}\right)^{2x} + 2\cdot\left(\frac{1}{2}\right)^x + b = 0$

Let $\left(\frac{1}{2}\right)^x = y,\ y \in (0, 1)$

Hence, $y^2 + 2y + b = 0$

Since the sum of roots $= -2$ is negative, there can be only one root in (0, 1).

Hence, $f(0) \cdot f(1) < 0$

$\Rightarrow b\,(b + 3) < 0$

$\Rightarrow b \in (-3, 0)$.

PROBLEM 1.12 If $\left(\sqrt{5\sqrt{2} - 7}\right)^x + 6\left(\sqrt{5\sqrt{2} + 7}\right)^x = 7$, then find the value of x.

SOLUTION Note that $\left(5\sqrt{2} - 7\right)\left(5\sqrt{2} + 7\right) = 1$

Let $y = \left(5\sqrt{2} - 7\right) \Rightarrow 5\sqrt{2} + 7 = \frac{1}{y}$

Now, $y^{x/2} + \frac{6}{y^{x/2}} = 7$, assume $p = y^{x/2}$

$$p^2 - 7p + 6 = 0 \qquad \Rightarrow p = 6 \text{ or } p = 1$$

$$\left(5\sqrt{2} - 7\right)^{x/2} = 6 \quad \text{or} \quad \left(5\sqrt{2} - 7\right)^x = 1$$

$$\Rightarrow x = 2\log_{(5\sqrt{2} - 7)} 6 \quad \text{or} \quad 0.$$

PROBLEM 1.13 Find the number of solutions of the equation $\sqrt[4]{|x-3|^{x+1}} = \sqrt[3]{|x-3|^{x-2}}$.

SOLUTION Taking log on both the sides

$$\frac{x+1}{4}\log|x-3| = \frac{x-2}{3}\log|x-3|$$

$$\Rightarrow \log|x-3|\left[\left(\frac{x+1}{4}\right) - \left(\frac{x+2}{3}\right)\right] = 0$$

$$\Rightarrow \log|x-3| = 0 \text{ or } \left[\left(\frac{x+1}{4}\right) - \left(\frac{x+2}{3}\right)\right] = 0$$

$$\Rightarrow x = 4, 2 \quad \text{or} \quad x = 11.$$

PROBLEM 1.14 What conditions must be imposed on the numbers a and b for the equation

$$1 + \log_b(2\log a - x)\log_x b = \frac{2}{\log_b x}$$

to have at least one solution ? Find all the solutions of this equation.

SOLUTION Applying the equality

$$\log_x b = \frac{1}{\log_b x}$$

we have

$$\log_b[x(2\log a - x)] = 2.$$

$$\Rightarrow x^2 - 2\log a \,.\, x + b^2 = 0.$$

Solving this equation we find

$$x_{1,2} = \log a \pm \sqrt{\log^2 a - b^2}\,.$$

For $a \geq 10^b$ and $\log a \neq \frac{1}{2}(b^2+1)$ both roots are positive and unequal to unity and, as is readily verified, satisfy the original equation.

For $\log a = \frac{1}{2}(b^2+1)$

we must only take the root $x_1 = b^2$.

For $a < 10^b$ the equation has no roots.

PROBLEM 1.15 Solve the equation

$$\log_{\sqrt{5}} x\sqrt{\log_x 5\sqrt{5} + \log_{\sqrt{5}} 5\sqrt{5}} = -\sqrt{6}$$

SOLUTION After some simple transformations based on the formula for change of base of logarithms we reduce the given equation to the form

$$\log_{\sqrt{5}} x\sqrt{\frac{3}{\log_{\sqrt{5}} x} + 3} = -\sqrt{6}\,.$$

Putting $\log_{\sqrt{5}} x = t$ we obtain, after performing some simplification and squaring both sides of the equation, the new equation $t^2 + t - 2 = 0$

Its roots are $t_1 = -2$ and $t_2 = 1$. The first root yields the value $x = \frac{1}{5}$ which, as is readily seen, satisfies the original equation. The second root gives the value $x = \sqrt{5}$ which does not satisfy the original equation.

PROBLEM 1.16 Solve the equation

$$\log_{\sin x} 2 \,.\, \log_{\sin^2 x} a + 1 = 0$$

SOLUTION Passing to logarithms to the base base 2 we obtain the equation

$$\frac{1}{\log_2 \sin x}\;\frac{\log_2 a}{2\log_2 \sin x} + 1 = 0$$

Hence, $\log_2^2 \sin x = -\dfrac{\log_2 a}{2}$.

The quantity on the left-hand side being strictly positive ($\sin x \neq 1$ because otherwise the symbol $\log_{\sin x} 2$ makes no sense), we have $\log_2 a < 0$ and, consequently, for $a > 1$ the equation has no solutions at all. Supposing that $0 < a < 1$ we obtain

$$\log_2 \sin x = \pm\sqrt{-\frac{\log_2 a}{2}}\,.$$

The plus sign in front of the radical must be discarded because $\log_2 \sin x < 0$. Thus we have

$$\sin x = 2^{-\sqrt{-\frac{\log_2 a}{2}}}$$

and $$x = (-1)^k \sin^{-1} 2^{-\sqrt{-\frac{\log_2 a}{2}}} + \pi k\,,$$

$(k = 0, \pm 1, \ldots)$.

It can easily be seen that all this infinite sequence of values of x satisfies the original equation.

PROBLEM 1.17 Solve the inequality

$$x^{\log_a x + 1} > a^2 x \;(a > 1)\,.$$

SOLUTION Since $x > 0$, the given inequality is equivalent to the inequality $x^{\log_a x} > a^2$.

But $a > 1$, and therefore taking logarithms of both sides of the last inequality to the base a we get

$$\log_a^2 x > 2.$$

From this we deduce the final result :

either $\log a\, x > \sqrt{2}$, and, consequently, $x > a^{\sqrt{2}}$

or $\log_a x < -\sqrt{2}$, and then $0 < x < a^{-\sqrt{2}}$.

PROBLEM 1.18 If $\log_{10}\left(\sin\left(x+\frac{\pi}{4}\right)\right) = \frac{\log_{10} 6 - 1}{2}$ then find the value of $\log_{10}(\sin x) + \log_{10}(\cos x)$.

SOLUTION $2\log_{10}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right) = \log_{10}\left(\frac{6}{10}\right)$

$\Rightarrow \log_{10}\left(\frac{1 + 2\sin x\cos x}{2}\right) = \log_{10}\left(\frac{6}{10}\right)$

$\Rightarrow \frac{1}{2} + \sin x\cos x = \frac{3}{5}$

$\Rightarrow \sin x \cos x = \frac{1}{10}$

$\Rightarrow \log_{10}(\sin x) + \log_{10}(\cos x) = -1.$

PROBLEM 1.19 If the equation $x^2 + 4 + 3\cos(ax + b) = 2x$ has atleast one solution where $a, b \in [0, 5]$ then find the value of $(a + b)$.

SOLUTION $x^2 - 2x + 4 = -3\cos(ax + b)$

$(x - 1)^2 + 3 = -3\cos(ax + b)$

For above equation to have atleast one solution,

let $f(x) = (x - 1)^2 + 3$ and $f(x) = -3\cos(ax + b)$

If $x = 1$ then L.H.S. = 3

and R.H.S. $= -3\cos(a + b)$

Hence, $\cos(a + b) = -1$

$\therefore a + b = \pi, 3\pi, 5\pi$

but $0 \le a + b \le 10 \Rightarrow a + b = \pi$ or 3π.

PROBLEM 1.20 Find the minimum and maximum value of $f(x, y) = 7x^2 + 4xy + 3y^2$ subjected to $x^2 + y^2 = 1$.

SOLUTION Let $x = \cos\theta$ and $y = \sin\theta$

$y = f(\theta) = 7\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta$

$= 3 + 2\sin 2\theta + 2(1 + \cos 2\theta)$

$= 5 + 2(\sin 2\theta + \cos 2\theta)$

but $-\sqrt{2} \le (\sin 2\theta + \cos 2\theta) \le \sqrt{2}$

$\therefore y_{max} = 5 + 2\sqrt{2}$ and $y_{min} = 5 - 2\sqrt{2}$.

PROBLEM 1.21 Solve the equation

$$5\sin x + \frac{5}{2\sin x} - 5 = 2\sin^2 x + \frac{1}{2\sin^2 x}$$

if $x \in (0, \pi)$.

SOLUTION $5\left(\sin x + \frac{1}{2\sin x}\right) - 5$

$= 2\left(\sin^2 x + \frac{1}{4\sin^2 x}\right)$

$= 2\left(\left(\sin x + \frac{1}{2\sin x}\right)^2 - 1\right)$

Let $\sin x + \frac{1}{2\sin x} = t$

$5t - 5 = 2(t^2 - 1) \Rightarrow 2t^2 - 5t + 3 = 0$

$\Rightarrow (2t - 3)(t - 1) = 0$

$t = 1$, or $t = 3/2$

$t = 1$, $2\sin^2 x - 2\sin x + 1 = 0$

$D < 0$ no solution

If $t = 3/2$, $2\sin^2 x - 3\sin x + 1 = 0$

$\Rightarrow \sin x = 1$ or $\sin = 1/2$

$\therefore x = \frac{\pi}{2}$ or $\frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow x \in \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$.

PROBLEM 1.22 Find the value of x, y, z satisfying the equations

$\log_2 x + \log_4 y + \log_4 z = 2$

$\log_9 x + \log_3 y + \log_9 z = 2$

and $\log_{16} x + \log_{16} y + \log_4 z = 2.$

SOLUTION From the first equation, we get

$\log_2(x^2yz) = 4$

$\Rightarrow \quad x^2yz = 2^4 \quad$ ...(1)

Similarly, $\quad y^2zx = 3^4 \quad$ ....(2)

$$z^2xy = 4^4 \qquad ....(3)$$

(1) × (2) × (3)

$\Rightarrow x^4y^4z^4 = (2 \cdot 3 \cdot 4)^4$

$\Rightarrow \quad xyz = 24$

From (1), $\quad x = \dfrac{16}{24} = \dfrac{2}{3}$

From (2), $\quad y = \dfrac{81}{24} = \dfrac{27}{8}$

From (3), $\quad z = \dfrac{256}{24} = \dfrac{32}{3}$.

PROBLEM 1.23 If m and n are positive integers satisfying

$$1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + \cos 8\theta + \cos 10\theta = \frac{\cos m\theta \cdot \sin n\theta}{\sin\theta},$$ then find the value of m + n.

SOLUTION Let $S = \cos 0° + \cos 2\theta + \cos 4\theta + ... + \cos 10\theta$

$2 \sin\theta \cdot S = 2 \sin\theta\, [\cos 0 + \cos 2\theta + ... + \cos 10\theta]$

$= \sin\theta + \sin\theta$

$= \sin 3\theta - \sin\theta$

$= \sin 5\theta - \sin 3\theta$

$= \sin 7\theta - \sin 5\theta$

$= \sin 9\theta - \sin 7\theta$

$= \sin 11\theta - \sin 9\theta$

$2 \sin\theta \cdot S = \sin 11\theta + \sin\theta$

$2 \sin\theta \cdot S = 2 \sin 6\theta \cdot \sin 5\theta$

$$S = \frac{2\sin 6\theta \cos 5\theta}{2\sin\theta} = \frac{\sin n\theta \cos m\theta}{\sin\theta}$$

$\Rightarrow$ n = 6 and m = 5.

PROBLEM 1.24 Solve

$$\sqrt{2+\sqrt{2+\sqrt{2+x}}} + \sqrt{3}\sqrt{2-\sqrt{2+\sqrt{2+x}}} = 2x \text{ for } x \ge 0.$$

SOLUTION Since $2 - \sqrt{2 + \sqrt{2+x}} \ge 0$, we must have

$0 \le x \le 2$.

Let $x = 2\cos\theta,\ \theta \in \left[0, \dfrac{\pi}{2}\right]$

The equation becomes

$$2\cos\frac{\theta}{8} + 2\sqrt{3}\sin\frac{\theta}{8} = 2 \times 2\cos\theta$$

$$\cos\left(\frac{\pi}{3} - \frac{\theta}{8}\right) = \cos\theta$$

Since $\quad 0 < \dfrac{\pi}{3} - \dfrac{\theta}{8} < \dfrac{\pi}{3}$, we have

$$\frac{\pi}{3} - \frac{\theta}{8} = \theta \Rightarrow \quad \theta = \frac{8\pi}{27}$$

$\therefore \quad x = 2\cos\dfrac{8\pi}{27}$.

PROBLEM 1.25 Show that the expression $\cos\theta\ (\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha})$ always lies between $-\sqrt{1+\sin^2\alpha}$ and $\sqrt{1+\sin^2\alpha}$.

SOLUTION Let

$y = \cos\theta\,(\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha})$

$\Rightarrow y - \cos\theta \sin\theta = \cos\theta\,(\sqrt{\sin^2\theta + \sin^2\alpha})$

$\Rightarrow (y - \cos\theta \sin\theta)^2 = \cos^2\theta\,(\sin^2\theta + \sin^2\alpha)$

$\Rightarrow y^2 - 2y\sin\theta \cos\theta + \cos^2\theta$

$\quad = \cos^2\theta \sin^2\theta + \cos^2\theta \sin^2\alpha$

$\Rightarrow y^2 - 2y\sin\theta \cos\theta + \cos^2\theta$

$\quad = \cos^2\theta + \cos^2\theta\,.\,\sin^2\alpha$

[Here we have added $\cos^2\theta$ on both sides to get $1 + \sin^2\alpha$]

$\Rightarrow y^2 - 2y \sin\theta \cos\theta + \cos^2\theta = \cos^2\theta\,(1 + \sin^2\alpha)$

$\Rightarrow y^2 \sec^2\theta - 2y \tan\theta + \theta = 1 + \sin^2\alpha$

$\Rightarrow y^2\tan^2\theta - 2y\tan\theta + 1 = (1 + \sin^2\alpha) - y^2$

$\Rightarrow$ $(y\tan\theta - 1)^2 = 1 \sin^2\alpha - y^2$

$\because$ square of a real number ≥ 0,

$1 + \sin^2\alpha - y^2 \geq 0$

$\Rightarrow y^2 - (\sqrt{1+\sin^2\alpha})^2 \leq 0$

$\Rightarrow$ y lies between $-\sqrt{1+\sin^2\alpha}$ and $\sqrt{1+\sin^2\alpha}$.

PROBLEM 1.26 If sin (sin x + cos x) = cos(cos x – sin x), then find the largest possible value of sin x.

SOLUTION cos(cos x – sin x) = cos $\left(\frac{\pi}{2}-(\sin x+\cos x)\right)$

$\therefore$ $\cos x - \sin x = 2n\pi \pm \left(\frac{\pi}{2}-(\sin x+\cos x)\right)$

Taking + ve sign

$\cos x - \sin x = 2n\pi + \frac{\pi}{2} - \sin x - \cos x$

$\Rightarrow 2\cos x = 2n\pi + \frac{\pi}{2} \Rightarrow \cos x = n\pi + \frac{\pi}{4}$

$n = 0, \cos x = \frac{\pi}{4}$ (only possible values)

$$\sin x = \sqrt{1-\frac{\pi^2}{16}} = \frac{\sqrt{16-\pi^2}}{4}$$

Taking – ve sign

$\cos x - \sin x = 2n\pi - \frac{\pi}{2} + \sin x + \cos x$

$\Rightarrow 2\sin x = \frac{\pi}{2} - 2n\pi \Rightarrow \sin x = \frac{\pi}{4} - n\pi$

only possible $\sin x = \frac{\pi}{4}$

as $\frac{\pi}{4} > \frac{\sqrt{16-\pi^2}}{4}$

$\Rightarrow (\sin x)_{max.} = \frac{\pi}{4}$.

PROBLEM 1.27 Prove that

$$\sum_{r=1}^{r=4}\left(\sin\frac{(2r-1)\pi}{8}\right)^4 = \sum_{r=1}^{r=4}\left(\cos\frac{(2r-1)\pi}{8}\right)^4.$$

Also find their exact numerical value.

SOLUTION LHS

$= \sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$

$2\left(\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}\right)$

$= 2\left(\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}\right)$

$= 2\left(1-2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)$

$= 2\left(1-\frac{1}{2}\sin^2\frac{\pi}{4}\right) = 2\left(1-\frac{1}{4}\right)$

$= 2 \times \frac{3}{4} = \frac{3}{2}$

RHS $= \cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$

$= 2\left(\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}\right)$

$= 2\left(\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}\right)$

$= 2\left(1-2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)$

$= 2\left(1-\frac{1}{4}\right) = \frac{3}{2}$.

PROBLEM 1.28 Let f (x) denote the sum of the infinite trigonometric series,

$$f(x) = \sum_{n=1}^{\infty}\sin\frac{2x}{3^n}\sin\frac{x}{3^n}.$$

Find f(x) (independent of n) also evaluate the sum of the solutions of the equation f (x) = 0 lying in the interval (0, 629).

SOLUTION

$$f(x) = \sum_{n=1}^{\infty}\sin\frac{2x}{3^n}\sin\frac{x}{3^n}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}2\sin\frac{2x}{3^n}\sin\frac{x}{3^n}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left[\cos\frac{x}{3^n} - \cos\frac{x}{3^{n-1}}\right]$$

Now substituting n = 1, 2, 3, 4........

$$f(x) = \frac{1}{2}\left[\cos\frac{x}{3} - \cos x\right] + \frac{1}{2}\left[\cos\frac{x}{3^2} - \cos\frac{x}{3}\right]$$

$$+ \frac{1}{2}\left[\cos\frac{x}{3^3} - \cos\frac{x}{3^2}\right]$$

$$...+ \frac{1}{2}\left[\cos\frac{x}{3^n} - \cos\frac{x}{3^{n-1}}\right]$$

$$f(x) = \lim_{n\to\infty} \frac{1}{2}\left[\cos\frac{x}{3^n} - \cos x\right] = \frac{1}{2}\ [1 - \cos x]$$

Now $f(x) = 0 \quad \Rightarrow \cos x = 1 \quad \Rightarrow x = 2n\pi,\ n \in I$

The sum of the solutions in (0, 629),

$$S = 2[\pi + 2\pi + 3\pi + ... + 100\pi]$$
$$= 2 \cdot 5050\pi = 10100\pi\ .$$

PROBLEM 1.29 Evaluate $\sum_{n=1}^{89}\frac{1}{1+(\tan n°)^2}$.

SOLUTION

$$S = \frac{1}{1+(\tan 1°)^2} + \frac{1}{1+(\tan 2°)^2} + \frac{1}{1+(\tan 3°)^2}$$
$$+ .... + \frac{1}{1+(\tan 88°)^2} + \frac{1}{1+(\tan 89°)^2}$$

Reversing the sum

$$S = \frac{1}{1+(\cot 1°)^2} + \frac{1}{1+(\cot 2°)^2} + ...$$
$$...... + \frac{1}{1+(\cot 88°)^2} + \frac{1}{1+(\cot 89°)^2}$$

$$2S = \sum_{n=1}^{89}\frac{1}{1+(\tan n°)^2} + \frac{1}{1+(\cot n°)^2}$$

$$= \sum_{n=1}^{89}\frac{1}{1+(\tan n°)^2} + \frac{(\tan n°)^2}{1+(\tan n°)^2}$$

$$= \sum_{n=1}^{89} 1 = 1 + 1 + ... + 1 = 89$$

$\therefore\ S = 44.5.$

PROBLEM 1.30 Solve the equation

$$\cos^2\left[\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right] - \tan^2 .$$
$$\left(x + \frac{\pi}{4}\tan^2 x\right) = 1$$

SOLUTION $\cos^2\left[\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right] - \tan^2 .$

$$\left(x + \frac{\pi}{4}\tan^2 x\right) = 1$$

Since square of the cosine of any argument doesnot exceed 1, the given equation holds true if and only if we have, simultaneously

$$\cos^2\left[\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right] = 1 \qquad ...(1)$$

$$\text{and} \quad \tan\left(x + \frac{\pi}{4}\tan^2 x\right) = 0 \qquad ...(2)$$

From (1), $\sin x + \sqrt{2}\cos^2 x = 4k,\ k \in I$ ...(3)

But $|\sin x + \sqrt{2}\ \cos^2 x\,| \le |\sin x| + \sqrt{2}\ |\cos^2 x|$

$\le 1 + \sqrt{2} < 4$

So, equation (3) has no solution for $k \neq 0$.

For k = 0, $\sin x + \sqrt{2}\ \cos^2 x = 0$

or, $\sqrt{2}\ \sin^2 x - \sin x - \sqrt{2} = 0$

or, $\sin x = \frac{-1}{\sqrt{2}}, \sqrt{2}$ but $\sin x = \sqrt{2}$ is not possible

So the solution to the equation (1) is

$$x_1 = \frac{-\pi}{4} + 2\,n\pi,\ x_2 = \frac{5\pi}{4} + 2n\pi,\ \ n \in I$$

For $x_1 = \frac{-\pi}{4} + 2n\pi$, equation (2) becomes an identity

but $x_2 = \frac{5\pi}{4} + 2\,n\pi$ does not satisfy equation (2)

So, the solution to the original equation is

$$x = \frac{-\pi}{4} + 2n\pi\ \forall\ n \in I.$$

PROBLEM 1.31 Solve

$|\sin 3x + \sin x| + |\sin 3x - \sin x|$

$= \sqrt{3}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$

SOLUTION $|\sin 3x + \sin x| = |2 \sin 2x \cos x|$

$= \sin 3x + \sin x,\ 0 \le x < \frac{\pi}{2}$

$= -\sin 3x - \sin x,\ -\frac{\pi}{2} < x < 0$

$|\sin 3x - \sin x| = |2 \cos 2x \sin x|$

$= \sin 3x - \sin x,$

$0 \le x \le \frac{\pi}{4}, -\frac{\pi}{2} < x < -\frac{\pi}{4}$

$= -\sin 3x + \sin x,$

$\frac{\pi}{4} < x < \frac{\pi}{2}, -\frac{\pi}{4} < x < 0.$

Hence, $|\sin 3x + \sin x| + |\sin 3x - \sin x| = \sqrt{3}$ ...(1)

$\Rightarrow 2 \sin 3x = \sqrt{3}$ for $0 \le x \le \frac{\pi}{4}$

$\Rightarrow 3x = n\pi + (-1)^n \frac{\pi}{3}$

$\Rightarrow x = \frac{\pi}{9}, \frac{2\pi}{9}.$

For $\frac{\pi}{4} < x < \frac{\pi}{2}$, the equation (1) becomes

$2 \sin x = \sqrt{3}$

$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{3} \quad \Rightarrow \quad x = \frac{\pi}{3}$

Since, the function $|\sin 3x + \sin x| + |\sin 3x - \sin x|$ is an even function, the values $-\frac{2\pi}{9}, -\frac{\pi}{9}, -\frac{\pi}{3}$ will also satisfy the equation.

PROBLEM 1.32 Find a positive number such that its fractional part, its integral part and the number itself forms a geometrical progression.

SOLUTION Let the number be x $\{x\}$, $[x]$, x are in GP

$\Rightarrow [x]^2 = x \cdot \{x\} = x(x - [x])$

$\Rightarrow x^2 - [x]^x - [x]^2 = 0$

$\Rightarrow 2x = [x] \pm \sqrt{[x]^2 + 4[x]^2}$

$= t(1 + \sqrt{5})[x] \quad (x \ne 1 - \sqrt{5})$

$x = \frac{\sqrt{5}+1}{2}[x]$

If $0 < x < 1$, RHS = 0. This is not possible

If $1 \le x < 2$, $[x] = 1$ and this satisfies the equation.

Hence $x = \frac{\sqrt{5}+1}{2}$

$\Rightarrow [x] = 1 \quad \Rightarrow \quad \{x\} = \frac{\sqrt{5}-1}{2}.$

Now $\frac{\sqrt{5}-1}{2}, 1, \frac{\sqrt{5}+1}{2}$ are in GP with common ratio

$\frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}.$

PROBLEM 1.33 If $f(x) = \sin^{44} x - \cos^{44} x$ and $g(x) = \sin x + \cos x$, then find the general solution of $f(x) = \left[g\left(\frac{\pi}{10}\right)\right]$, (where, $[x]$ is the greatest integer less than or equal to x).

SOLUTION $f(x) = \left[g\left(\frac{\pi}{10}\right)\right]$

$\Rightarrow f(x) = \left[\sin\frac{\pi}{10} + \cos\frac{\pi}{10}\right]$

$= [\sin 18° + \cos 18°]$

$\Rightarrow f(x) = \left[\frac{\sqrt{5}-1}{4} + \frac{\sqrt{10+2\sqrt{5}}}{4}\right]$

$= [1.26] = 1$

$\Rightarrow \sin^{44} x - \cos^{44} x = 1$

$\Rightarrow \sin^{44} x = 1 + \cos^{44} x \ge 1$

which is possible if and only if $\cos^{44} x = 0$

i.e. $x \in (2n + 1)\frac{\pi}{2}.$

PROBLEM 1.34 If the equation

$$(a-2)(x-[x])^2 + 2(x-[x]) + a^2 = 0,$$

(where [x] denotes the greatest integer $\le x$) has exactly one solution in (2, 3), then find all possible values of a.

SOLUTION $(a-2)(x-[x])^2 + 2(x-[x]) + a^2 = 0$ ...(1)

Let $y = x - [x]$. Then (1) can be written as

$$f(y) = (a-2)y^2 + 2y + a^2 = 0 \quad ...(2)$$

As x cannot be an integer.

$$y = x - [x] \ne 0$$

Thus, $a \ne 0$

When $2 < x < 3$, $[x] = 2$

$\Rightarrow 0 < x - [x] < 1$ i.e. $0 < y < 1$.

Since (1) has exactly one solution in the interval (2, 3), (2) has exactly one solution in the interval (0, 1).

This is possible if $f(0)\ f(1) < 0$, otherwise the equation (2) has either no or two solution in (0, 1).

$\Rightarrow a^2\{a - 2 + 2 + a^2\} < 0$

$\Rightarrow a(a+1) < 0 \quad [\because a^2 > 0]$

$\Rightarrow -1 < a < 0$ or $a \in (-1, 0)$

PROBLEM 1.35 Solve $[x] + 2y = a, [y] + 2x = a$

SOLUTION Note that x and y must be half of integers.

Since $[x] + [y] + 2(x+y) = 2a$

and $x - \frac{1}{2} \le [x] \le x$

$y - \frac{1}{2} \le [y] \le y$

$\Rightarrow x + y - 1 \le [x] + [y] \le x + y$ we get

$\Rightarrow 2a \le 3(x+y) \le 2a + 1$

Since $x + y$ must be half of an integer.

it is one of the values $\frac{2a}{3}, \frac{2a + \frac{1}{2}}{3}, \frac{2a+1}{3}$

Let $2a = 3x$. Then $x + y = \frac{2a}{3}$,

$\Rightarrow x + y = 2m$

Here x and y cannot be half of odd integer. Both must be integer.

$\Rightarrow x + 2y = a; \quad y + 2x = a$

$\Rightarrow (x, y) = \left(\frac{a}{3}, \frac{a}{3}\right)$

Let $2a = 3m + 1$, then $x + y = \frac{2a + \frac{1}{2}}{3}$,

since $4a + 1$ is divisible by 3.

$(x, y) = \left(m + \frac{1}{2}, n\right)$

or, $\left(m + 1, n - \frac{1}{2}\right)$

Solving we get $\left(\frac{2a-1}{6}, \frac{a+1}{6}\right)$

and $\left(\frac{a+1}{3}, \frac{2a-1}{6}\right)$

Let $2a = 3m + 2$, $x + y = \frac{2a+1}{3}$

$x = [x] + \frac{1}{2}, \ y = [y] + \frac{1}{2}$

$\Rightarrow (x, y) = \left(\frac{2a+1}{6}, \frac{2a+1}{6}\right)$.

PROBLEM 1.36 The set of real values of 'x' satisfying the equality $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$ (where [ ] denotes the greatest integer function) belongs to the interval $\left(a, \frac{b}{c}\right]$ where $a, b, c \in N$ and $\frac{b}{c}$ is in its lowest form. Find the value of $a + b + c + abc$.

SOLUTION **Case 1:** If $x < 0$ then $\left[\frac{3}{x}\right]$ and $\left[\frac{4}{x}\right]$ are negative and hence $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right]$ can never be equal to 5.

Case 2: If $x > 0$, we have $\frac{3}{x} < \frac{4}{x}$

$\therefore \left[\frac{3}{x}\right] \leq \left[\frac{4}{x}\right]$

Since each of $\left[\frac{3}{x}\right]$ and $\left[\frac{4}{x}\right]$ is an integer

$\therefore$ There are 3 possibilities:

(1) $\left[\frac{3}{x}\right] = 0$ and $\left[\frac{4}{x}\right] = 5$

(2) $\left[\frac{3}{x}\right] = 1$ and $\left[\frac{4}{x}\right] = 4$

(3) $\left[\frac{3}{x}\right] = 2$ and $\left[\frac{4}{x}\right] = 3$

Now, if $\left[\frac{3}{x}\right] = 0$ then $0 \leq \frac{3}{x} < 1$

$\Rightarrow x > 3.$

and $\left[\frac{4}{x}\right] = 5 \Rightarrow 5 \leq \frac{4}{x} < 6$

$\Rightarrow \frac{1}{6} < \frac{x}{4} \leq \frac{1}{5} \Rightarrow \frac{2}{3} < x \leq \frac{4}{5}.$

These two equations are not possible. Hence, there is no solution in this case.

Now, if $\left[\frac{3}{x}\right] = 1$ then $1 \leq \frac{3}{x} < 2$

$\Rightarrow \frac{1}{2} < \frac{x}{3} \leq 1 \Rightarrow \frac{3}{2} < x \leq 3$

and $\left[\frac{4}{x}\right] = 4 \Rightarrow 4 \leq \frac{4}{x} < 5$

$\Rightarrow \frac{1}{5} < \frac{x}{4} \leq \frac{1}{4} \Rightarrow \frac{4}{5} < x \leq 1$

This is not possible simultaneously $\Rightarrow$ no solution.

Again, if $\left[\frac{3}{x}\right] = 2 \Rightarrow 2 \leq \frac{3}{x} < 3$

$\Rightarrow \frac{1}{3} < \frac{x}{3} \leq \frac{1}{2} \Rightarrow 1 < x \leq \frac{3}{2}$

and $\left[\frac{4}{x}\right] = 3 \Rightarrow 3 \leq \frac{4}{x} < 4$

$\Rightarrow \frac{1}{4} < \frac{x}{4} \leq \frac{1}{3} \Rightarrow 1 < x \leq \frac{4}{3}$

The common solution is $1 < x \leq \frac{4}{3}$.

Hence $x \in \left(1, \frac{4}{3}\right]$

$\Rightarrow a = b,\ b = 4, c = 3.$

$\Rightarrow a + b + c + abc = 1 + 4 + 3 + 12 = 20.$

Things to REMEMBER

1. Remainder theorem: The remainder of the division of the polynomial P(x) by the binomial $(x - \alpha)$ is equal to the value of the polynomial P(x) for $x = \alpha$, i.e. $r = P(\alpha)$

2. Factor theorem: The polynomial P(x) is exactly divisible by the binomial $(x - \alpha)$ if and only if the value of the polynomial is zero for $x = \alpha$,, i.e. $P(\alpha) = 0$

3. Irrational Inequalities:

(a) An inequation of the form $\sqrt[2n]{f(x)} < g(x)$, $n \in N$ is equivalent to the system

$$\begin{cases} f(x) \geq 0 \\ g(x) > 0 \\ f(x) < g^{2n}(x) \end{cases},$$

(b) An inequation of the form $\sqrt[2n]{f(x)} > g(x)$, $n \in N$ is equivalent to the collection of two systems of inequations

$$\begin{cases} g(x) \geq 0 \\ f(x) > g^{2n}(x) \end{cases} \text{ and } \begin{cases} g(x) < 0 \\ f(x) \geq 0 \end{cases}$$

4. Inequations containing absolute values:

(a) The inequation of the form $f(|x|) < g(x)$ is equivalent to the collection of systems
$$\begin{cases} f(x) < g(x), \text{ if } x \ge 0 \\ f(-x) < g(x), \text{ if } x < 0 \end{cases}$$

(b) The inequation of the form $|f(x)| < g(x)$ is equivalent to $\begin{cases} -g(x) < f(x) < g(x) \\ g(x) > 0 \end{cases}$

In particular $|f(x)| < a$

has no solution for $a \le 0$ and for $a > 0$ it is equivalent to $-a < f(x) < a$

(c) The inequation of the form $|f(x)| > g(x)$ is equivalent to the
$$\begin{cases} f(x) > g(x), \text{ if } g(x) < 0 \\ -f(x) > g(x), \text{ if } g(x) < 0 \end{cases}$$
$$\Rightarrow \begin{cases} f(x) < -g(x) \text{ or } f(x) > g(x), \\ \text{if } g(x) > 0 \text{ always true}, \\ \text{if } g(x) < 0 \end{cases}$$

In particular $|f(x)| > a$ has no solution for $a \le 0$ and for $a > 0$ is equivalent to $f(x) < -a$, $f(x) > a$

(d) The inequation of the form $|f(x)| \ge |g(x)|$ is equivalent to the system $f^2(x) \ge g^2(x)$

5. Properties of Modulus:

(a) $|a + b| \le |a| + |b|$

(b) $|a + b| = |a| + |b|$ only if $ab \ge 0$

(c) $|a - b| \ge ||a| - |b||$

(d) $|a - b| = ||a| - |b||$ only if $ab \ge 0$

(e) $|a + b + c| \le |a| + |b| + |c|$

(f) $|a + b + c| = |a| + |b| + |c|$ only if $ab \ge 0$, $bc \ge 0$ and $ca \ge 0$

(g) $|a \pm b \pm c| \le |a| + |b| + |c|$

6.

(a) $\max(f(x), g(x)) = \dfrac{f(x)+g(x)}{2} + \left|\dfrac{f(x)-g(x)}{2}\right|$

(b) $\min(f(x), g(x)) = \dfrac{f(x)+g(x)}{2} - \left|\dfrac{f(x)-g(x)}{2}\right|$

7. Laws of logarithm

(i) $\log_a MN = \log_a M + \log_a N$

(ii) $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

(iii) $\log_a (M^p) = p \log_a M$

(iv) $\log_{a^q} M = \dfrac{1}{q} \log_a M$

(v) $\log_b M = \dfrac{\log_a M}{\log_a b}$

(vi) $\log_b a = \dfrac{1}{\log_a b}$

(vii) $a^{\log_a x} = x$

(viii) $x^{\log_a b} = b^{\log_a x}$

8. Exponential Inequalities: Depending on the values of the parameters a and b, the set of solutions of the inequality $a^x > b$ can be in the following forms :

(i) $x \in (\log_a b, \infty)$ for $a > 1$, $b > 0$;

(ii) $x \in (-\infty, \log_a b)$ for $0 < a < 1$, $b > 0$;

(iii) $x \in R$ for $a > 0$, $b < 0$;

Depending on the values of a and b, the set of solutions of the inequality $a^x < b$ can be in the following forms

(i) $x \in (-\infty, \log_a b)$ for $a > 1$, $b > 0$;

(ii) $x \in (\log_a b, \infty)$ for $0 < a < 1$, $b > 0$;

(iii) $x \in \phi$ for $a > 0$, $b < 0$ (i.e. the inequality has no solutions)

9. Logarithmic Inequalities: An inequality of the form $\log_a f(x) > b$ is equivalent to the following systems of inequalities :

(i) $f(x) > 0$, $f(x) > a^b$ for $a > 1$

(ii) $f(x) > 0$. $f(x) < a^b$ for $a < 1$,

and an inequality of the form $\log_a f(x) < b$ is equivalent to the following systems of inequalities :

(i) $f(x) > 0$, $f(x) < a^b$ for $a > 1$;

(ii) $f(x) > 0$, $f(x) > a^b$ for $0 < a < 1$.

10. Summation of Trigonometric Series: Sum of sines or cosines of n angles in A.P.

(i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha + 2\beta) + \ldots\ldots$

$$+ \sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

$$\sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots\ldots$

$$+ \cos(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

$$\cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

11. General Solution of Trigonometric Equations:

(i) If $\sin\theta = \sin\alpha$

$\Rightarrow \quad \theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$

(ii) If $\cos\theta = \cos\alpha$

$\Rightarrow \quad \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

(iii) If $\tan\theta = \tan\alpha$

$\Rightarrow \quad \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

(v) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

(vi) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

Note that α is called the principal angle.

12. Properties of Greatest Integer Function :

(i) $[x] \le x < [x] + 1$

(ii) $x - 1 < [x] \le x$, $0 \le x - [x] < 1$

(iii) $[x + m] = [x] + m$ if m is an integer.

(iv) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

(v) $[-x] = -[x]$, $x \in I$

$= -[x] - 1$, $x \notin I$

(vi) $[x] \ge n \Rightarrow x \ge n$, $n \in I$

(vii) $[x] > n \Rightarrow x \ge n + 1$, $n \in I$

(viii) $[x] \le n \Rightarrow x < n + 1$, $n \in I$

(ix) $[x] < n \Rightarrow x < n$, $n \in I$

(x) $\left[\frac{[x]}{m}\right] = \left[\frac{x}{m}\right]$ if m is a positive integer

(xi) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots$

$.. + \left[x + \frac{n-1}{n}\right] = [nx]$

(xii) If n and a are positive integers, $[n/a]$ is the number of integers among 1,2....., n that are divisible by a.

(xiii) **de Polignac's Formula :** Let E be the largest exponent of p such that p^E divides n!.

Then $E = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$.

13. Properties of Fractional Part Function:

(i) $0 \le \{x\} < 1$

(ii) $\{x + m\} = \{x\}$ if m is an integer.

(iii) $\{-x\} = 1 - \{x\}$, $x \in I$

$= 0$, $x \in I$

14. Properties of Least Integer Function:

(i) $(x) = [x]$, if $x \in I$ and $(x) = [x] + 1$, if $x \notin I$, in fact $(x) = -[-x]$

(ii) $(x + n) = n + (x)$, $n \in I$

(iii) $(-x) = -(x)$, $x \in I$

$(-x) = -(x) + 1$, $x \notin I$

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. If $\sin\theta + \text{cosec}\,\theta = -2$, then the value of $\sin^8\theta + \text{cosec}^8\theta$ is equal to
   (A) 2 (B) 2^8 (C) 2^4 (D) 2^{16}

2. The solution set of the inequality $\dfrac{(e^x-1)(2x-3)(x^2+x+2)}{(\sin x-2)x(x+1)} \le 0$
   (A) $\left[\frac{3}{2},\infty\right)$ (B) $(-\infty,-1)\cup\left[\frac{3}{2},\infty\right)$
   (C) $(-1,0)\cup\left[\frac{3}{2},\infty\right)$ (D) $R-\{0,-1\}$

3. Number of integral solutions of the equation $|\tan x+\sec x| = |\tan x| + |\sec x|$ in $[0, 2\pi]$ is
   (A) 0 (B) 2 (C) 4 (D) 5

4. The solutions of $|\cos x| = \cos x - 2\sin x$ is
   (A) $x = n\pi$ (B) $x = n\pi + \frac{\pi}{4}$
   (C) $x = n\pi + (-1)^n\frac{\pi}{4}$ (D) $x = (2n+1)\pi + \frac{\pi}{4}$, $n \in I$

5. The set of values of a for which al the roots of $2^{\sin x} + a.2^{-\sin x} - 2 = 0$ are real and distinct $\left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$ is
   (A) $\left[\frac{3}{4},1\right)$ (B) $\left[\frac{3}{2},\frac{3}{2}\right]$ (C) $(-8, 1)$ (D) None of these

6. The solution set for $(\sqrt{2+\sqrt{2}})^x + (\sqrt{2+\sqrt{2}})^x = 2.2^{x/4}$ is
   (A) $\{2\}$ (B) $\{0\}$ (C) $[0, 2]$ (D) None of these

7. If all the solutions of $a^{\cos x} + a^{-\cos x} = 6$ $(a > 1)$ are real, then the set of values of a is
   (A) $(3+2\sqrt{2}, \infty)$ (B) $(6, 12)$ (C) $(1, 3+2\sqrt{2})$ (D) None of these

8 The number of ordered triplets (x, y, z) satisfying the inequality $(1+\sin^4 x)(2+\cot^3 y)(4+\sin 4z) \le 12\sin^2 x$, where $x, y, z \in [0, 2\pi]$ is
   (A) 20 (B) 12 (C) 8 (D) None of these

9. The equation $xa^{\frac{1}{x}} + \frac{1}{x}a^x = 2a$, ≥ 1; $a, x \in R$ has
   (A) only one solution as $x = 1$ (B) only positive solution as $x = 1$
   (C) infinite solutions (D) None of these

10. If $x_n < x_{n=1} > \ldots\ldots > x_2 > x_1 > 1$ then the value of $\log_{x_1} \log_{x_2} \log_{x_3} \ldots\ldots \log_{x_n} x_n^{x_{n-1}^{\cdot^{\cdot^{\cdot^{x_1}}}}}$

(A) 0 (B) 1 (C) 2 (D) None of these

11. The value of $\sum_{r=1}^{89} \log_{10}\left\{\tan\left(\frac{\pi\gamma}{180}\right)\right\}$ is equal to

(A) 10 (B) 1 (C) 0 (D) None of these

12. If a, b, c are distinct positive numbers different from 1 such that $(\log_b a \,.\, \log_c a - \log_a a) + (\log_a b \,.\, \log_c b - \log_b b) + (\log_a c.\, \log_b c - \log_c c) = 0$ then abc =

(A) 0 (B) e (C) 1 (D) None of these

13. If the expression, $2\cos 10° + \sin 100° + \sin 1000° + \sin 10000°$ is simplified, then it simplifies to

(A) $\cos 10°$ (B) $3 \cos 10°$ (C) $4 \cos 10°$ (D) $5 \cos 10°$

14. Number of positive integers x for which $f(x) = x^3 - 8x^2 + 20x - 13$, is a prime number, is

(A) 1 (B) 2 (C) 3 (D) 4

15. Let m denotes the number of digits in 2^{64} and n denotes the number of zeroes between decimal point and the first significant digit in 2^{-64}, then the ordered pair (m, n) is $(\log_{10}2 = 0.301)$

(A) (20, 21) (B) (20, 20) (C) (19, 19) (D) (20, 19)

16. Number of solutions of the equation $2 \sin^3x + 6 \sin^2x - \sin x - 3 = 0$ in $(0, 2\pi)$, is

(A) 6 (B) 4 (C) 3 (D) 2

17. If $y = \frac{\sqrt{1+\sin A} + \sqrt{1-\sin A}}{\sqrt{1+\sin A} - \sqrt{1-\sin A}}$, then one of the values of y is

(A) $\cot\frac{A}{2}$ if $2n\pi - \frac{\pi}{4} < \frac{A}{2} < 2n\pi + \frac{\pi}{4}$ (B) $\tan\frac{A}{2}$ if $2n\pi - \frac{\pi}{4} < \frac{A}{2} < 2n\pi + \frac{3\pi}{4}$

(C) $2 \cot A$ if $2n\pi + \frac{3\pi}{4} < \frac{A}{2} < 2n\pi + \frac{5\pi}{4}$ (D) none of these

18. The minimum value of the function $f(x) = (3\sin x - 4 \cos x - 10)(3 \sin x + 4 \cos x - 10)$, is

(A) 49 (B) 48 (C) 84 (D) 40

19. If a > b > 0 are two real numbers, then value of, $\sqrt{ab + (a-b)\sqrt{ab + (a-b)\sqrt{ab + (a-b)\sqrt{ab + \ldots}}}}$ is:

(A) independent of b (B) independent of a

(C) independent of both a and b (D) a + 2b

20. If $\ln^2 x + 3 \ln x - 4$ is non negative then x must lie in the interval

(A) $[e, \infty)$ (B) $(-\infty, e^{-4}] \cup [e, \infty)$ (C) $(1/e, e)$ (D) none of these

21. If $\log_{(5.2^x+1)} 2$, $\log_{(2^{1-x}+1)} 4$ and 1 are in harmonic progression then

(A) s is a positive real (B) x is a negative real

(C) x is rational which is not integral (D) x is an integer

22. The solution f the equation $\log_{\cos x^2}(3-2x) < \log_{\cos x^2}(2x-1)$ is
(A) (1/2, 1) (B) $(-\infty, 1)$
(C) (1/2, 3) (D) $(1, \infty) - \sqrt{2n\pi}$, $n \in N$

23. The real value of x for which the statement $\log_6 9 - \log_9 27 + \log_8 x = \log_{64} x - \log_6 4$ holds true, is
(A) 1/2 (B) 1/4 (C) 1/8 (D) 1/16

24. The value of $n \in N$ for which $y = ln((1-x)(x-n))$ can be defined is $2n - 11$. The value of n is
(A) 8 (B) 9 (C) 10 (D) 11

25. Number of integral values of parameter 'c' for which the inequality $1 + \log_2\left(2x^2 + 2x + \frac{7}{2}\right) > \log_2 (cx^2 + c)$, holds good $\forall$ $x \in R$, is
(A) 0 (B) 2 (C) 7 (D) infinite

26. For every $x \in R$ the value of the expression $y = \frac{x^2}{8} + x\cos x + \cos 2x$ is never less than
(A) –1 (B) 0 (C) 1 (D) 2

27. The solution set of the inequality $\log_{\frac{5}{8}}\left(2x^2 - x - \frac{3}{8}\right) \geq 1$ is
(A) $\left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$ (B) $\left[-\frac{1}{2}, 1\right]$
(C) $\left[-\frac{1}{2}, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right)$ (D) $\left(-\infty, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right)$

28. The number of solutions which satisfy the equation $\log_2 x \cdot \log_4 x \cdot \log_6 x = \log_2 x \cdot \log_4 x + \log_4 x \cdot \log_6 x + \log_6 x \cdot \log_2 x$ is
(A) 0 (B) 1 (C) 2 (D) infinite

29. The smallest integral x satisfying the inequality , is $\frac{1 - \log_4 x}{1 + \log_2 x} \leq \frac{1}{2}$
(A) $\sqrt{2}$ (B) 2 (C) 3 (D) 4

30. Given $a^2 + 2a + \operatorname{cosec}^2\left(\frac{\pi}{2}(a+x)\right) = 0$ then, which of the following holds goods?
(A) $a = 1$; $\frac{x}{2} \in I$ (B) $a = -1$; $\frac{x}{2} \in I$
(C) $a \in R$; $x \in \varphi$ (D) a, x are finite but not possible to find.

31. The sum of the roots of the equation $(x + 1) = 2\log_2(2^x + 3) - 2\log_4(1980 - 2^{-x})$ is
(A) 3954 (B) $\log_2 11$ (C) $\log_2 3954$ (D) indeterminate

32. Solution set of the inequality $\log_3 x - \log_3^2 x \leq \frac{3}{2}\log_{(1/2\sqrt{2})} 4$ is
(A) [3, 9] (B) $\left(0, \frac{1}{3}\right] \cup [9, 8)$ (C) $\left(-\infty, \frac{1}{3}\right] \cup [9, \infty)$ (D) $\left(\frac{1}{3}, 1\right] \cup (9, \infty]$

33. Which is the correct order for a given number $\alpha > 1$ in increasing order?
(A) $\log_2 \alpha, \log_3 \alpha, \log_e \alpha, \log_{10} \alpha$
(B) $\log_{10} \alpha, \log_3 \alpha, \log_e \alpha, \log_2 \alpha$
(C) $\log_{10} \alpha, \log_e \alpha, \log_2 \alpha, \log_3 \alpha$
(D) $\log_{10} \alpha, \log_e \alpha, \log_2 \alpha, \log_{10} \alpha$

34. Let $f_n(\theta) = \prod_{n=1}^{n}\left(\frac{1+\tan^2(2^n\theta)}{(1-\tan^2(2^n\theta))^2}\right)$ where $|\theta| < \frac{\pi}{2^{n+2}}$, then the value of $f_{10}\left(\frac{\pi}{2^{13}}\right)$ is
(A) $1+\cos\frac{\pi}{2^{10}}$ (B) $1+\cos\frac{\pi}{2^{11}}$ (C) $1+\cos\frac{\pi}{2^{12}}$ (D) $1+\cos\frac{\pi}{2^{13}}$

35. The value of the expression $\frac{2(\sin 1° + \sin 2° + \sin 3° + ...\sin 89°)}{2(\cos 1° + \cos 2° + ... + \cos 44°) + 1}$ equals
(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{1}{2}$ (D) 1

36. Let $y = \frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x}$ where $0 < x < \frac{\pi}{2}$, then the minimum value of y is
(A) 0 (B) 1 (C) 3/2 (D) 2

37. The set of all values of θ in $[0, 2\pi]$ for which the equation $(1 + \sin\theta)x^2 - (2\cos\theta)x + \sin\theta = 0$ has real roots -
(A) $\left(0, \frac{\pi}{3}\right)$
(B) $\left[\frac{\pi}{6}, \pi\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$
(C) $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$
(D) $\left[\frac{\pi}{3}, \frac{3\pi}{2}\right]$

38. If $\sqrt{\log_2 \cos(\sin x - 1)} = \cos x$, then the solution set for x is
(A) $\{x : x = (4n+1)\frac{\pi}{2}, n \in I\}$
(B) $\{x : x = (2n+1)\frac{\pi}{2}, n \in I\}$
(C) $\{x : x = n\pi, n \in I\}$
(D) none of these

39. The complete solution set of $|\cos 3x| + |\cos x| = |\cos 3x + \cos x|$ belonging to $[0, \pi]$ is
(A) $\left\{\frac{\pi}{2}\right\} \cup \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$
(B) $\left\{\frac{\pi}{2}\right\} \cup \left[0, \frac{\pi}{6}\right]$
(C) $[0, \pi]$
(D) $\left\{\frac{\pi}{2}\right\} \cup \left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, \pi\right]$

40. Consider $f(x) = e^{\cos(x^2+px+q)}$, $x \in R$ where p and q are are arbitray real number. The set of real number k for which the equation $f(x) = k$ has a solution depends on
(A) p but not q (B) q but not p (C) both p and q (D) neither p nor q

41. Solution of $2[x] = [2x]$ in $[1, 3]$ is (where [.] represent greatest integer function)
(A) $[1, 3]$
(B) $\left[1, \frac{3}{2}\right) \cup \left[2, \frac{5}{2}\right)$
(C) $\left[1, \frac{3}{2}\right) \cup \left[2, \frac{5}{2}\right) \cup \{3\}$
(D) $\left[\frac{3}{2}, 2\right] \cup \left[\frac{5}{2}, 3\right]$

42. If $|2 \sin \theta - \text{cosec}\,\theta| \geq 1$ and $\theta \neq \frac{n\pi}{2}$, $n \in I$, then

(A) $\cos 2\theta \geq \frac{1}{2}$ (B) $\cos 2\theta \geq \frac{1}{4}$ (C) $\cos 2\theta \leq \frac{1}{2}$ (D) $\cos 2\theta \leq \frac{1}{4}$

43. The range of k for which the inequality $k \cos^2 x - k \cos x + 1 \geq 0\ \forall\ x \in (-\infty, \infty)$, is

(A) $k < \frac{-1}{2}$ (B) $k > 4$ (C) $\frac{-1}{2} \leq k \leq 4$ (D) $\frac{1}{2} \leq k \leq 5$

44. All solutions of the equation $4x^2 - 40x + 51 + 40\{x\} = 0$ lies in the interval ({x} represents fractional part of x)

(A) $\left(\frac{23}{10}, \frac{83}{10}\right)$ (B) $\left(\frac{23}{10}, \frac{15}{2}\right)$ (C) $\left(7, \frac{83}{10}\right)$ (D) $\left(\frac{23}{10}, 7\right)$

45. The set of value(s) of x, satisfying $[\sin x] + [\cos x] = 1$, $x \in [0, 2\pi]$ is ([.] denotes the greatest integer fucntion)

(A) φ (B) $\{0, \pi/2, 2\pi\}$ (C) $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ (D) none of these

46. If 'x' satisfies the equation $[x + 0.19] + [x + 0.20] \ldots\ldots + [x + 0.91] = 542$ where [.] stands for G. I. function then the value of $[100x]$ is

(A) 802 (B) 739 (C) 719 (D) 791

47. If $0 < x, y \leq 1000$ where, $x, y \in N$ such that $\left[\frac{x}{2}\right] + \left[\frac{2x}{3}\right] + \left[\frac{y}{4}\right] + \left[\frac{4y}{5}\right] = \frac{7x}{6} + \frac{21y}{20}$ (where, [ ] denotes the greatest integer function) then number of ordered pair (x, y) is

(A) 8517 (B) 8300 (C) zero (D) None of these

48. Let $S = \{ a \in N, a \leq 100\}$ and $[\tan^2 x] - \tan x - a = 0$ has a real roots (where, [ ] denotes the greatest integer function), then number of elements in the set S is

(A) 2 (B) 4 (C) 9 (D) 0

49. If [x] and {x} denotes the greatest integer function less than or equal to x and fractional part function respectively, then the number of real x, satisfying the equation $(x - 2)[x] = \{x\} - 1$, is

(A) 0 (B) 1 (C) 2 (D) infinite

50. The solutions of the equation $kx - k[x] = 1$, where [x] denotes the greatest integer less than or equal to x and $k > 1$ are

(A) $x = n + \frac{1}{k}, n \in N$ (B) $x = n - \frac{1}{k}, n \in N$ (C) $x = n + \frac{1}{k}, n \in I$ (D) $x = n, n \in I$

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

51. Which of the following function (s) is/are transcendental?

(A) $f(x) = 5 \sin \sqrt{x}$ (B) $f(x) = \frac{2 \sin 3x}{x^2 + 2x - 1}$

(C) $f(x) = \sqrt{x^2 + 2x + 1}$ (D) $f(x) = (x^2 + 3).2^x$

52. Which of the following trigonometric equation(s) has/have no solution $\forall$ x $\in$ R?

(A) $2\cos^2\frac{x}{2}\sin^2 x = x^2 + \frac{1}{x^2}$

(B) $\sin^4 x - 2\sin^2 x - 1 = 0$

(C) $\cos e^x = 5^x + 5^{-x}$

(D) $\sin 2x = [1 + \sin 2x] + [1 - \cos 2x]$ where [ ] denotes greatest integer function.

53. Let $x = \frac{1+\sqrt{5}}{2}$ and $[n\,x] = p$ for some $n \in N$ where [.] represents G.I.F., then

(A) $\frac{p}{x} < n < \frac{p}{x} + 1$ (B) $px - p < n < px - p + 1$

(C) $[px] = n + p - 1$ (D) $[nx^2] = [[nx]\,x + 1]$

54. Suppose a, b, c, d are real number such that $|a - b| + |c - d| = 99$, $|a - c| + |b - d| = 1$, then the possible value(s) of $|a - d| + |b - c|$ is / are

(A) 1 (B) 98 (C) 99 (D) 100

55. Smallest integer function or ceiling function $\lceil x \rceil$ is the smallest integer greater than or equal to x, which of the following statement(s) is incorrect:

(A) $\lceil x \pm 1 \rceil = \lceil x \rceil \pm 1$ (B) $\lceil x^2 \rceil = \lceil x \rceil^2$

(C) $\lceil x^2 \rceil + \lceil -x \rceil = 1$ (D) $\lceil x \rceil^2 + \lceil -x \rceil^2 = 2\lceil x \rceil^2 + 1$

56. Let $(\log_2 x)^2 - 4\log_2 x - m^2 - 2m - 13 = 0$ be an equation in x and $m \in R$, then which of the following must be correct?

(A) For any $m \in R$, the equation has two distinct solutions.

(B) The product of the solutions of the equation does not depend on m.

(C) One of the solutions of the equation is less than 1 while the other is greater than 1 for $\forall$ m $\in$ R.

(D) The minimum value of the larger solution is 2^6 and maximum value of the smaller solution is 2^{-2}.

57. If {.} represents fractional part function and $\{x\} + \{-x\} = x^2 + x - 6$, then

(A) integral root are – 2 & 3 (B) number of non-integral roots is 2

(C) number of solutions is 4 (D) equation has exactly two integral roots

58. The inequality $\sqrt{x^{\log_2 \sqrt{x}}} \geq 2$is satisfied by

(A) only one value of x (B) $x \in \left(0, \frac{1}{4}\right]$

(C) $x \in [4, \infty)$ (D) $1 < x < 2$

59. The pair of equations which are equivalent is

(A) $x^2 - 2x = 8$ and $(x^2 - 2x)\,2^{\sqrt{x}} = 8.\,2^{\sqrt{x}}$

(B) $x^3 - \frac{4x(x+2)}{x+2} = 0$ and $x^3 - \frac{4x^2}{x} = 0$

(C) $|x+1|+|x-1| = -x^2+3$ and $2x = -x^2+3$
(D) None of these

60. The pair of inequalities which are equivalent is

(A) $\dfrac{\sqrt{x-1}\sqrt{x+4}}{x-2} > 0$ and $\dfrac{\sqrt{(x-1)(x+4)}}{x-2} > 0$ (B) $(1+x^2)^{\frac{x+4}{x-1}} < 1$ and $\dfrac{x+4}{x-1} < 0$

(C) $\dfrac{|x+14|(x-6)}{x+5} > 0$ and $\dfrac{x-6}{x+5} > 0$ (D) None of these

61. Which of the following real numbers when simplified are either terminating or repeating decimal ?

(A) $\sin\dfrac{3\pi}{8}\cos\dfrac{3\pi}{8}$ (B) $\log_2 112$
(C) $\log_3 2\ \log_4 3\ \log_8 4$ (D) $27^{-(\log_{125} 5)}$

62. For the function $f(x) = (x^2+bx+c)e^x$ and $g(x) = (x^2+bx+c)e^x + e^x(2x+b)$. Which of the following holds good?
(A) if $f(x) > 0$ for all real x $\not\Rightarrow$ $g(x) > 0$
(B) if $f(x) > 0$ for all real x $\Rightarrow$ $g(x) > 0$
(C) if $g(x) > 0$ for all real x $\Rightarrow$ $f(x) > 0$
(D) if $g(x) > 0$ for all real x $\not\Rightarrow$ $f(x) > 0$

63. Which of the following sets can be the subset of the general solution of the equation $1 + \cos 3x = 2\cos 2x$?

(A) $n\pi + \dfrac{\pi}{3}$ (B) $n\pi + \dfrac{\pi}{6}$ (C) $n\pi - \dfrac{\pi}{6}$ (D) $2n\pi$
where $n \in I$

64. Which of the following equations have no real solutions?
(A) $x^2 - 2x + 5 + \pi^x = 0$ (B) $\log_{1.5}\left(\cot^{-1}x - \operatorname{sgn}\left(e^x\right)\right) = 2$

(C) $x^4 - 2x^2\sin^2\dfrac{\pi x}{2} + 1 = 0$ (D) $\tan\left(x + \dfrac{\pi}{6}\right) = 2\tan x$

65. Which statement(s) is/are correct :
(A) area enclosed by the curve max $(|x+y|, |x-y|) = 1$ is 4
(B) area enclosed by the curve max $(|x|, |y|) = 1$ is 2
(C) area enclosed by the curve max $(|2x|, |2y|) = 1$ is 1
(D) area enclosed by the curve min $(|x+y|, |x-y|) = 1$ is 2

66. Consider the inequality, $(x+3)^n (x-2) < 0$, $n \in N$. Then the correct statement(s) is/are:
(A) the largest integral x satisfying the inequality is 1, if n is even
(B) the least integral x satisfying the inequality is -2, if n is odd
(C) number of integral x satisfying the inequality is 3, if n is odd
(D) number of positive integral x satisfying the inequality is 1, if n is even

67. If l is the least value of the expression $4\sec^2\theta + 25\,\mathrm{cosec}^2\theta$, and m is greatest value of the expression $4\cos^2\theta + \sin^2\theta + 4\sin\theta\cos\theta$ then
(A) $l = 5$ (B) $m = 49$ (C) $|l - m| = 44$ (D) $l + m = 54$

68. The value of θ which satisfies the equation $\dfrac{\cos^2\theta - 3\cos\theta + 2}{\sin^2\theta} = 1$ lies in the interval

(A) $\left(0, \dfrac{\pi}{3}\right)$ (B) $\left(\dfrac{\pi}{6}, \dfrac{\pi}{4}\right)$ (C) $\left(\dfrac{\pi}{6}, \dfrac{\pi}{2}\right)$ (D) $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

69. Number of real values of x satisfying the equation $3^{x+2} + 3^{2-x} = 82$ is not equal to

(A) number of real solutions of the equation $(|x|+1)^2 = 4|x|+9$, such that the quantity $\ln(5-2x)$ is a real number.

(B) the value of the expression $\dfrac{\log_5 250}{\log_{50} 5} - \dfrac{\log_5 10}{\log_{1250} 5}$ when simplified.

(C) number of real solutions of the equation, $2x \ln x + x - 1 = 0$.

(D) the value of 'm' if a line of gradient m passes through the points $(m, -9)$ and $(7, m)$

70. If p, q, r are positive rational numbers such that $p > q > r$ and the quadratic equation $(p + q - 2r)x^2 + (q + r - 2p)x + (r + p - 2q) = 0$ has a root in $(-1, 0)$ then which of the following statement hold(s) good?

(A) $\dfrac{r+p}{q} < 2$

(B) Both roots of the given quadratic equation are rational.

(C) The equation $px^2 + 2qx + r = 0$ has real and distinct roots.

(D) The equation $px^2 + 2qx + r = 0$ has no real roots.

Comprehension - 1

Let P(x) be the polynomial $P(x) = x^{15} - 2012x^{14} + 2012x^{13} \ldots\ldots - 2012x^2 + 2012x$

71. The value of P(2011) is

(A) $(2013)^2$ (B) 2012 (C) 2011 (D) 2010

72. The number of fixed points of the function P(x) (i.e. number of solutions of P(x) = x) is

(A) 15 (B) 5 (C) 3 (D) 1

73. In the interval $x \in (1, 2011)$,

(A) $P(x) > x$ (B) $P(x) = 2x$ has one solution

(C) $P'(x) = 1$ has atleast one solution (D) None of these

Comprehension - 2

For $x \in \left(0, \dfrac{\pi}{4}\right)$,

Let $S_n = \sum_{r=1}^{2n} \sin\left(\sin^{-1} x^{3r-2}\right)$,

$C_n = \sum_{r=1}^{2n} \cos\left(\cos^{-1} x^{3r-1}\right)$ and

$$T_n = \sum_{r=1}^{2n} \tan\left(\tan^{-1} x^{3r}\right)$$

where $n \in N$ and $n \geq 3$.

74. The correct order of S_n, C_n and T_n is given by
(A) $S_n > T_n > C_n$ (B) $S_n < C_n < T_n$ (C) $S_n < T_n < C_n$ (D) $S_n > C_n > T_n$

75. The value of $\lim\limits_{n\to\infty} (S_n + C_n + T_n)$ is equal to
(A) $\frac{1}{1-x}$ (B) $\frac{x}{1-x}$ (C) $\frac{1}{1-x}$ (D) $\frac{x}{1+x}$

76. The value of 'x' for which $S_n = C_n + T_n$, is
(A) $\sin\frac{\pi}{5}$ (B) $2\sin\frac{\pi}{5}$ (C) $2\sin\frac{\pi}{10}$ (D) $\sin\frac{\pi}{10}$

Comprehension - 3

Let $f(x) = \min\{x - [x], -x[-x]\}, -2 \leq x \leq 2$; $g(x) = |2 - |x - 2||, -2 \leq x \leq 2$ and $h(x) = \frac{|\sin x|}{\sin x}$, $-2 \leq x \leq 2$ and $x \neq 0$ (where [x] denotes the greatest integer function $\leq x$).

77. The number of solutions of the equation $x^2[f(x)]^2 = 1$ is $\{-1 \leq x \leq 1\}$
(A) 0 (B) 2 (C) 4 (D) 6

78. The sum of all the roots of the equation $g(x) - h(x) = 0$ is $\{-2 \leq x \leq 2\}$
(A) positive (B) negative (C) zero (D) None of these

79. The set of values of a such that the equation $f(x) - a = 0$ has exactly eight real and distinct roots
(A) $a \in \left(0, \frac{1}{2}\right)$ (B) $a \in \left(0, \frac{1}{2}\right)$ (C) $a \in [0, 1)$ (D) $a \in (0, 1)$

Comprehension - 4

a, b, c are the sides of ΔABC satisfying $\log\left(1 + \frac{c}{a}\right) + \log a - \log b = \log 2$. Also the quadratic equation $a(1 - x^2) + 2bx + c(1 + x^2) = 0$ has two equal roots.

80. a, b, c are in
(A) A.P. (B) G.P. (C) H.P. (D) None

81. Measure of angle C is
(A) 30° (B) 45° (C) 60° (D) 90°

82. The value of (sin A + sin B + sin C) is equal to
(A) $\frac{5}{2}$ (B) $\frac{12}{5}$ (C) $\frac{8}{3}$ (D) 2

Comprehension - 5

Let A denotes the sum of the roots 4of the equation $\frac{1}{5-4\log_4 x}+\frac{4}{1+\log_4 x}=3$. B denotes the value of the product of m and n, if $2^m = 3$ and $3^n = 4$. C denotes the sum of the integral roots of the equation $\log_{3x}\left(\frac{3}{x}\right)+(\log_3 x)^2 = 1$.

83. The value of A + B equals
(A) 10 (B) 6 (C) 8 (D) 4

84. The value of B + C equals
(A) 6 (B) 2 (C) 4 (D) 8

85. The value of A + C ÷ B equals
(A) 5 (B) 8 (C) 7 (D) 4

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

86. Consider the function $f(x)=\frac{1}{2\{-x\}}-\{x\}$ where { x } denotes the fractional part of x and x is not an integer.

Assertion (A) : The minimum value of f(x) is $\sqrt{2}-1$

Reason (R) : If the product of two positive numbers is a constant the minimum value of their sum is 2 times the square root of their product.

87. Consider two functions $f(x)=1+e^{\cot^2 x}$ and $g(x)=\sqrt{2|\sin x|-1}+\frac{1-\cos 2x}{1+\sin^4 x}$

Assertion (A) : The solutions of the equation f (x) = g (x) is given by $x=(2n+1)\frac{\pi}{2}\ \forall n\in I$.

Reason (R) : If $f(x)\ge k$ and $g(x)\le k$ (where $k\in R$) then solutions of the equation f(x) = g(x) is the solution corresponding to the equation f(x) = k.

88. **Assertion (A) :** If $c(b^2-a^2)-c^2(2a+c)>0$, then atleast one root of the quadratic equation $ax^2+bx+c=0$ must lie in the interval $[0,\infty)$ where $a,b,c\in R$.

Reason (R) : For any three real numbers α, β, γ; if $\alpha\beta\gamma<0$ then this condition is true when either all three are negative or any two positive and other negative.

89. **Assertion (A) :** If $\sin^2 x+\cos^2 y=\frac{3}{2}a$ and $\cos^2 x+\sin^2 y=\frac{1}{2}a^2$ then there are two possible values of a.

Reason (R) : On adding the two equations we get $a^2+3a-4=0$.

90. **Assertion (A) :** The system of equation $|x+y|=99$, $|x-y|=c$ has four real solutions (x, y) if $c\ge 0$.

Reason (R) : If c is positive, we have $x+y=\pm 99$, $x-y=\pm c$ which gives four solutions (x, y).

91. **Assertion (A) :** The value of $\{\ell n\ (1 + e)\} + \{\ell n\ (1 + e^2)\} + \{\ell n\ (1 + e^4)\} + \{\ell n\ (1 + e^8)\} + ...$ (where {x} is the fractional part of x), is $1 - \ell n\ (e - 1)$.

Reason (R) : Since $\ell n(1+e^{2^k})$ is just larger then 2^k, its fractional part is $\ell n(1+e^{2^k}) - \ell n\ e^{2^k}$ = $\ell n(1+e^{-2^k})$.

Now

$$\sum_{k=0}^{\infty} \ell n(1+e^{-2^k}) = \ell n\left(\prod_{k=0}^{\infty}(1+e^{-2^k})\right)$$

$$\sum_{k=0}^{\infty} \ell n(1+e^{-2^k}) = \ell n\left(\prod_{k=0}^{\infty}(1+e^{-2^k})\right).$$

92. **Assertion (A) :** The equation $x^2 + \left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] = 10$ has two solutions.

Reason (R) : Evidently x^2 must be an integer. Among positive x, $\sqrt{8}$ is too small and $\sqrt{9}$ is too big. Among negative x, $-\sqrt{15}$ is too small and $-\sqrt{13}$ is too big. One solution is $-\sqrt{14}$.

93. **Assertion (A) :** Given $\log_a b + \log_b c + \log_c a = 0$, the value of $(\log_a b)^3 + (\log_b c)^3 + (\log_c a)^3$ is 3.
Reason (R) : Let $x = \log_a b$ and $y = \log_b c$, then $\log_c a = -(x + y)$.
Now $x^3 + y^3 - (x + y)^3 = -3xy(x + y)$. On the other hand $xy\ (x + y) = \log_a b \log_b c \log_c a = 1$

94. **Assertion (A) :** The integral part of $\frac{2010^3}{2008 . 2009} - \frac{2008^3}{2009 . 2010}$ is 8.

Reason (R) : $= \left[\frac{8x^3 + 8x}{x^3 - x}\right] = \left[8 + \frac{16}{x^2 - 1}\right]$ 8 for $x \geq 5$. when [.] represents the greatest integer function.

95. **Assertion (A) :** $\frac{([x]-1)([x]-3)}{([x]-6)([x]-8)} > 0$

$\Rightarrow$ $x \in (-\infty, 1) \cup [4, 6) \cup [9, \infty]$ (where [.] represents greatest integer function)

Reason (R) : We can solve the inequality by ignoring greatest integer.

MATCH THE COLUMNS FOR JEE ADVANCED

96.

Column I	Column – II
(A) The number $n = xyzxyz$ will always be divisible by	(P) 7
(B) The numbers of the form $(6n \pm 1)$ in the domain of $f(x) = x\ \ln(x-1) + \frac{\sqrt{64 - \|x\|^2}}{\sin x}$ is /are	(Q) 11
(C) Maximum of max $(12\sin x - 5\cos x, -13-\{2x\})$ is (where {.} is fractional part function)	(R) 13
(D) Period of the function	(S) 5

$f(x) = \tan(\pi x) - \pi\cos\left(\frac{2\pi x}{7}\right) + \frac{\{x\}}{\sin\left(\frac{2\pi x}{7}\right)} + 11$ is

(where {.} is fractional part function) (T) 77

97.

Column I	Column – II
(A) $\sin x \cdot \cos^3 x > \cos x \cdot \sin^3 x$, $0 \le x \le 2\pi$, is	(P) $\left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$
(B) $4\sin^2 x - 8\sin x + 3 \le 0$, $0 \le x \le 2\pi$, is	(Q) $\left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$
(C) $\lvert \tan x \rvert \le 1$ and $x \in [-\pi, \pi]$ is	(R) $\left(0, \frac{\pi}{4}\right)$
(D) $\cos x - \sin x \ge 1$ and $0 \le x \le 2\pi$	(S) $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

98.

Column-I	Column-II
(A) If x and y are positive real numbers and p, q are any positive integers then the possible value which the expression $\frac{(1+x^{2p})(1+y^{2q})}{x^p y^q}$ can take is	(P) 3
(B) Given a and b are positive numbers not equal to 1 such that $\log_b\left(a^{\log_2 b}\right) = \log_a\left(b^{\log_4 a}\right)$ and $\log_a\left(p-(b-9)^2\right) = 2$ then the minimum integral value for p is	(Q) 5
(C) Possible values of x simultaneously satisfying the system of inequalities $\frac{(x-6)(x-3)}{x+2} \ge 0$ and $\frac{x-5}{x+1} \le 3$ is	(R) 9
(D) If the numbers, $(3^{1+x} + 3^{1-x})$, $(a/2)$, $(9^x + 9^{-x})$ form an A.P., then possible value(s) of 'a' is/are ($x \in R$)	(S) 10

99.

Column-I	Column-II
(A) The number of solutions of $3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2\sin^2 x} = 28$ in $[0, 2\pi]$, is	(P) 1
(B) The number of solutions of the equation $\sec x + \operatorname{cosec} x = 2\sqrt{2}$ in $[0, 2\pi]$, is	(Q) 2
(C) The number of roots of the equation $3\sin^2 x = 8\cos x$ in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, is	(R) 3
(D) The number of values of x in $[0, 2\pi]$ such that	(S) 4

$\sqrt{3}\sin x + \sqrt{2}\cos x = 0$, is

100. **Column - I** | **Column - II**

(A) The values of a such that the solution set of $(x - a) < 2$ is a subset of the solution set of $\frac{2x-1}{x+2} < 1$, are $a \in$	(P) $(-\sqrt{2}, \sqrt{2})$
(B) Let $a * b = ab + b$. The set of all real numbers a such that equation $x * (a * x) = \frac{-1}{4}$ has two distinct solutions in x, is $a \in$	(Q) $(-\infty, -3) \cup (-2, \infty)$
(C) Let $< a_n >$ be an A.P. where $a_1 = 6$, $a_2 = 8$, and $< b_n >$ be a sequence defined by $b_n = b_{n-1} + a_n$, $b_1 = 3$, then the solution set of $b_n + 9 > 0$ is $n \in$	(R) $(0, 1)$
(D) The values of a such that the graphs of $x^2 = y^2$ and $(x - a)^2 + y^2 = 1$ intersect at distinct points, are $a \in$	(S) $(-\infty, -1) \infty (0, \infty)$

REVIEW EXERCISES for JEE ADVANCED

1. If α, β are the roots of the equation, $(\tan^2 135°)x^2 - (\text{cosec}10° - \sqrt{3} \sec 10°)x + \tan^2 240° = 0$, then prove that the quadratic equation whose roots are $(2\alpha + \beta)$ and $(\alpha + 2\beta)$ is $x^2 - 12x + 35 = 0$.
2. Suppose that P(x) is a quadratic polynomial such that $P(0) = \cos^3 40°$, $P(1) = (\cos 40°)(\sin^2 40°)$ and $P(2) = 0$. Find the numerical value of P(3).
3. Consider the quadratic polynomial
   $f(x) = x^2 - 4ax + 5a^2 - 6a$.
   Let the smallest positive integral value of 'a' for which f(x) is positive for every real x be m
   Let the largest distance between the roots of the equation f(x) = 0 be n. Find m + n.
4. What conditions must be imposed on the number $d > 0$ for the inequality $0 < \frac{d^2 + R^2 - r^2}{2dR} \le 1$ to be valid for $R \ge r > 0$?
5. If $f(x) = 4x^2 + ax + (a - 3)$ is negative for atleast one negative x, find all possible values of a.
6. Find the set of values of a, for which the roots of the quadratic equation $x^2 - \frac{8a}{a+1}x + a^2 = 0$ are real and positive.
7. Solve $(x + 3)^4 + (x - 1)^4 \ge 82$, $x \in R$.
8. Solve the following equations:

   (i) $\sqrt{6x - x^2 - 5} = 2x - 6$

   (ii) $\sqrt{2x - 4} - \sqrt{x + 5} = 1$

   (iii) $\sqrt{x} + \sqrt{x - \sqrt{1 - x}} = 1$

9. Solve the following equations :

(i) $\sqrt[3]{12-x} + \sqrt[3]{14+x} = 2$

(ii) $\sqrt{x} + \sqrt{x - \sqrt{1-x}} = 1.$

(iii) $\sqrt{x+3} - 1 = \sqrt{x - \sqrt{x-2}}$

(iv) $\sqrt{x^2+2x+1} - \sqrt{x^2-4x+4} = 3$

10. Solve the equations :

(i) $\sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}} = \frac{3}{2}\sqrt{\frac{x}{x+\sqrt{x}}}$.

(ii) $\sqrt{y-2+\sqrt{2y-5}} + \sqrt{y+2+3\sqrt{2y-5}} = 7\sqrt{2}$

11. Solve the following equations :

(i) $\sqrt{x+\sqrt{x+11}} + \sqrt{x-\sqrt{x+11}} = 4$

(ii) $\sqrt{3x^2-2x+15} + \sqrt{3x^2-2x+8} = 7$

12. Find the domain of the function f(x) = $\sqrt{\frac{2}{x^2-x+1} - \frac{1}{x+1} - \frac{2x-1}{x^2+1}}$ is

13. Find the domain of the functions :

(i) $y = \sqrt{\frac{\sqrt{17-15x-2x^2}}{x+3}}$

(ii) $y = \sqrt{\frac{x^4-3x^2+x+7}{x^4-2x^2+1} - 1}$

14. Solve

(i) $(1-a)\sqrt{2x+1} < 1$

(ii) $\sqrt{x+2} \geq \sqrt{x-a}$

15. If number of integral values of 'x' satisfying the equation $|\,2 - x|[x] - 1|\,| \leq 2$ where [ ] stands for greatest integer function) is n, then find the value of $n^2 + n + 2$.

16. Find the values of x satisfying the equation $|\,|\,|\,x^2 - x + 4| - 2| - 3| = x^2 + x - 12$.

17. Find the solution set of the inequality $\left|\frac{3|x|-2}{|x|-1}\right| \geq 2$.

18. Prove that the equation $\cos(\log_3 x - 2) = \frac{1}{2}$ has only 2 solutions lying in the interval $[0, 2\pi]$.

19. Find the sum of the roots of the equation $2^{333x-2} + 2^{111x+1} = 2^{222x+2} + 1$.

20. Solve the following equations :

(i) $(\sqrt{x^2-4x+3}+1)\ \log_5 x/5 + 1/x\ (\sqrt{8x-2x^2-6}+1) = 0$

(ii) $2\log_{(x-2)}\sqrt{3} + (x-4)^2 \log_3 (x-2) = (x-4)^2 \log_{(x-2)}3 + 2\log_3\sqrt{x-2}$

(iii) $\frac{1+\log_2(x-4)}{\log_{\sqrt{2}}(\sqrt{x+3}-\sqrt{x-3})} = 1$

(iv) $\log_x (x^2 + 1) = \sqrt{\log_{\sqrt{x}}(x^2(1+x^2)) + 4}$

21. Solve the following equations :

(i) $3/2 \log_{1/4} (x + 2)^2 - 3 = \log_{1/5} (4 - x)^3 - \log_4 (x + 6)^3$

(ii) $\log_{1/2} \log_{1/4} (\sqrt{x^2+1} - x) + \log_{3/4} \log_8 (\sqrt{x^2+1} + x) = -2$

(iii) $\log_4 (6 + \sqrt{x} - |\sqrt{x} - 2|) = 1/2 + \log_2 |\sqrt{x} - |\sqrt{x} - 2||$

22. Solve the inequality, $2 \log_{1/2} (x - 1) \le \frac{1}{3} - \frac{1}{\log_{x^2-x} x}$.

23. Solve the inequality, $\sqrt{\log_{1/2}^2 x + 4\log_2 \sqrt{x}} < \sqrt{2}\,(4 - \log_{16}x^4)$

24. Solve the following inequations :

(i) $x^2 2^{2x} + 9 (x + 2)2^x + 8x^2 \le (x + 2)\, 2^{2x} + 9x^2 2^x + 8x + 16$.

(ii) $\log_{x/2} 8 + \log_{x/4} 8 < \frac{\log_2 x^4}{\log_2 x^2 - 4}$.

(iii) Solve $\log_x 2 \,.\, \log_{2x} 2 \,.\, \log_2 4x > 1$.

25. Find the solution of the inequality : $\log_{[(2/3)|x-2|]} 2^{(1-x^2)} \ge 0$

26. Solve the system of inequalities : $\begin{cases} \log_{(2-x)}(2-y) > 0 \\ \log_{(4-y)}(2x-2) > 0 \end{cases}$

27. Solve the following equations :

(i) Solve $\log_3 (\sqrt{x} + |\sqrt{x} - 1|) = \log_9 (4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$.

(ii) $\frac{\log_x (2a - x)}{\log_x 2} + \frac{\log_a x}{\log_a 2} = \frac{1}{\log_{(a^2-1)} 2}$.

28. (a) If $y = 10 \cos^2 x - 6 \sin x \cos x + 2 \sin^2 x$, then find the greatest and least value of y.

(b) If $y = 1 + 2 \sin x + 3 \cos^2 x$, find the maximum & minimum values of $y\ \forall\ x \in R$.

29. If θ is variable, then find the maximum / minimum value of the following :

(i) $4 \sin^2 \theta + \mathrm{cosec}^2\, \theta$ (ii) $8 + 8 \sec^2 \theta + 18 \cos^2 \theta$

30. Prove that $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$. Also find their exact numerical value.

31. If $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are real numbers, show that, $(\cos\alpha_1 + \cos\alpha_2 + \ldots\ldots + \cos\alpha_n)^2 + (\sin\alpha_1 + \ldots\ldots \sin\alpha_n)^2 \le n^2$.

32. Solve the following equations :

(i) $6 \tan x + 5 \cot 3x = \tan 2x$.

(ii) $\frac{1}{2}(\sin^4 x + \cos^4 x) = \sin^2 x \cos^2 x + \sin x \cos x$

(iii) $\cos^{2000}x - \sin^{2000} x = 1$.

(iv) $(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$

33. Solve for x, $\sqrt{13 - 18\tan x} = 6 \tan x - 3$, where $-2\pi < x < 2\pi$.

34. Solve the following inequalities :

(i) $\cos^2 x (\tan x + 1) > 1$

(ii) $3 \sin 2x > \sin x + \cos x + 1$

(iii) $x^{\log_{10} \sin x} \geq 1, \ x > 0$

(iv) $\tan\frac{x}{2} > \frac{\sin x - 2\cos x}{\sin x + 2\cos x}$ $(0 < x < \pi)$

35. If f is a function defined on the set of non-negative integers and taking values in the same set. Given that

(i) $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{19}\right]$ for all non negative integers.

(ii) $1900 < f(1990) < 2000$

Find the possible values of f(1990) can take.

TARGET EXERCISES for JEE ADVANCED

1. If $f(x) = \frac{x^2 + 3x - 40}{x^2 + 5x - 14}$, $x \neq -7$, $x \neq 2$ then find the least and the greatest integral value of x for which $f(x) < 0$.

2. Let $f(x) = \frac{x^3 - 3x + 1}{x(x-1)}$, $x \neq 0, 1$. Prove that $f(x) = f\left(\frac{1}{1-x}\right) = f\left(\frac{1}{1-x}\right)$. Also find the set of values of x for which $f(x) = \frac{a^3 - 3a^2 + 1}{a(1-a)}$, $a \neq 0, 1$ and a is a constant.

3. Prove that, given three line segments of length $a > 0$, $b > 0$ and $c > 0$, a triangle with these segments as sides can be constructed if and only if $pa^2 + qb^2 > pqc^2$ for any number p and q satisfying the condition $p + q = 1$.

4. Prove that for any real x, y and z we have the inequality $4x(x + y)(x + z)(x + y + z) + y^2z^2 \geq 0$.

5. Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be a polynomial of degree n. Prove each of the following :

(a) If $n \geq 1$ and $f(0) = 0$, then $f(x) = xg(x)$, where g is a polynomial of degree $n - 1$.

(b) For each real a, the function p given by $p(x) = f(x + a)$ is a polynomial of degree n.

(c) If $n \geq 1$ and $f(a) = 0$ for some real a, then $f(x) = (x - a)h(x)$, where h is a polynomial of degree $n - 1$. [Hint : Consider $p(x) = f(x + a)$.]

(d) If $f(x) = 0$ for $n + 1$ distinct real values of x, then every coefficient c_k is zero and $f(x) = 0$ for all real x.

(e) Let $g(x) = \sum_{k=0}^{m} b_k x^k$ be a polynomial of degree $m \geq n$. If $g(x) = f(x)$ for $m + 1$ distinct real values of x, then $m = n$, $b_k = c_k$ for each k, and $g(x) = f(x)$ for all real x.

6. For what values of 'a' does the equation $2\log_3^2 x - |\log_3 x| + a = 0$ possesses
   (i) no solution (ii) one (iii) two
   (iv) three (v) four solutions

7. Solve the following equations :
   (i) $\sqrt{x+4} + \sqrt{2x+6} = 7$
   (ii) $\sqrt{2x-4} - \sqrt{x+5} = 1$
   (iii) $\sqrt{x^2+x+4} + \sqrt{x^2+x+1} = \sqrt{2x^2+2x+9}$
   (iv) $\sqrt{x^2-9x+24} - \sqrt{6x^2-59x+149} = |5-x|$

8. Solve the following equations :
   (i) $\sqrt{2x-1} - x + a = 0$ (ii) $\sqrt{a^2 - x\sqrt{x^2+a^2}} = a - x$

9. Solve the inequality $\sqrt{\dfrac{3x-1}{a-2}} < 1$ for all real values of a.

10. Put $f_1(x) = x$; $f_2(x) = |1 - f_1(x)|$; $f_3(x) = |1 - f_2(x)|$; .... $f_n(x) = |1 - f_{n-1}(x)|$. Prove that the solutions of the equation $f_n(x) = 0$ are $\{\pm1, \pm3, ..., \pm(n-3), (n-1)\}$ if n is even and $\{0, \pm2, ..., \pm(n-3), (n-1)\}$ if n is odd.

11. Find the positive solutions of the equation $x^{(x^x)} = (x^x)^x$

12. Solve the following equations :
    (i) $\sqrt{\log_x (ax)^{1/5} + \log_a (ax)^{1/5}} + \sqrt{\log_a (x/a)^{1/5} + \log_x (a/x)^{1/5}} = a,\ a > 0$
    (ii) $\log_{10} \log_{1/\sqrt{1+x}} 10 \,.\, \log_{10}(x^2 - 3x + 2) = -2 + \log_{1/\sqrt{1+x}} 10 \,.\, \log_{10}(x-3)$
    (iii) $6/5\ a^{\log_a x . \log_{10} a . \log_a 5} - 3^{\log_{10}(x/10)} = 9^{\log_{100} x + \log_4 2}$

13. Solve the following equation for x if $a^A - 3^B = 9^C$ where $A = \log_a x \,.\, \log_{10} a.\ \log_a 5$, $b = \log_{10}(x/10)$ and $C = \log_{100} x + \log_4 2$.

14. If p, q are the roots of the quadratic equation $x^2 + 2bx + c = 0$, prove that $2\log\left(\sqrt{y-p} + \sqrt{y-q}\right) = \log 2 + \log\left(y + b + \sqrt{y^2 + 2by + c}\right)$

15. Prove that from the equalities, $\dfrac{x\,(y+z-x)}{\log x} = \dfrac{y\,(z+x-y)}{\log y} = \dfrac{z\,(x+y-z)}{\log z}$ follows $x^y y^x = y^z z^y = z^x x^z$.

16. Prove that if a and b are the lengths of the legs and c is the length of the hypotenuse of a right triangle, $c - b \neq 1$ and $c + b \neq 1$, then $\log_{c+b} a + \log_{c-b} a = 2\log_{c+b} a.\ \log_{c-b} a$

17. Prove that if a, b, c are successive (positive) terms of a geometric progression, then
$$\frac{\log_a N - \log_b N}{\log_b N - \log_c N} = \frac{\log_a N}{\log_c N}$$

18. If $\tan\left(\frac{\pi}{4}+\frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4}+\frac{x}{2}\right)$ then prove that $\sin y = \frac{\sin x(3+\sin^2 x)}{1+3\sin^2 x}$.

19. Prove that $\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} + \ldots\ldots + \sin\frac{n\pi}{3} = 2\sin\frac{n\pi}{6}\sin\frac{(n+1)\pi}{6}$ $\forall n \in N$.

20. Evaluate : $\sum_{n=1}^{\infty}\left(\frac{\tan\frac{\theta}{2^n}}{2^{n-1}\cdot\cos\frac{\theta}{2^{n-1}}}\right)$.

21. Prove that $(1+\sec 2x)(1+\sec 4x)(1+\sec 8x)\ldots\ldots(1+\sec 2^n x) = \tan 2^n x \cdot \cot x$ for every $n \in N$.

22. If $\theta = \frac{2\pi}{7}$, prove that $\tan\theta\tan 2\theta + \tan 2\theta\tan 4\theta + \tan 4\theta\tan\theta = -7$.

23. Find the minimum value of the function $f(x) = \frac{9x^2\sin^2 x + 4}{x\sin x}$ over the interval $(0, \pi)$.

24. Solve the following equations :

(i) $32\cos^6 x - \cos 6x = 1$ (ii) $\sin^8 x + \cos^8 x = \frac{17}{32}$

(iii) $8\sin^6 x + 3\cos 2x + 2\cos 4x + 1 = 0$ (iv) $\cos^{2001} x - \sin^{2001} x = 1$.

25. Determine the range of the values of the parameter λ for which the equation $\sec x + \operatorname{cosec} x = \lambda$ possesses a root x satisfying the inequality $0 < x < \frac{\pi}{2}$.

26. If $\sin x$, $\sin^2 2x$ and $\cos x \cdot \sin 4x$ form an increasing geometric sequence, find the numerical value of $\cos 2x$. Also find the common ratio of geometric sequence.

27. Solve the equation $\sin^{2n-1} x + 2\cos^{2n-1} x = 2$, where $n \in N$.

28. Determine all the values of a for which the equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ is solvable. Find the solutions.

29. Solve the following equations :

(i) $\sin^3 x + \sin^3 2x + \sin^3 3x = (\sin x + \sin 2x + \sin 3x)^3$.

(ii) $5\sin x + - 5 = 2\sin^2 x + \frac{1}{2\sin^2 x}$, if $x \in (0, \pi)$.

30. Prove that the equation $\cos\left(\left(\frac{3}{2}\right)^x - 1\right) = \frac{1}{2}$, has only 4 solutions lying in the interval $[0, 2\pi]$.

31. Find all the solutions of the equation $2-\sqrt{3}\cos 2x+\sin 2x=4\cos^2 3x$ which satisfy the inequality $\cos\left(2x-\frac{\pi}{4}\right)>0$.

32. Solve the following equations :

(i) $\log_{\sqrt{2}\sin x}(1+\cos x)=2$.

(ii) $\left|\cos\frac{x}{2}+1-\frac{3}{4\cos x/2}\right|^{\sqrt{x^2+3x-10}}=1$

33. Solve the following equations :

(i) $(\log_{\sin x} 2)^2<\log_{\sin x}(4\sin^3 x)$

(ii) $|3^{\tan \pi x}-3^{1-\tan \pi x}|\ge 2$

(iii) $\log_{\sin x-\cos x}(\sin x-5\cos x)\ge 1$

(iv) $\sqrt{\tan x-1}\,.\,[\log_{\tan x}(2+4\cos^2 x)-2]\ge 0$

(v) $\log_{\tan x}\sqrt{\sin^2 x-\frac{5}{12}}<-1$

34. Recall that a lattice point (x, y) in the plane is one whose coordinates are integers. Let f be a nonnegative function whose domain is the interval [a, b], where a and b are integers, a < b. Let S denote the set of points (x, y) satisfying $a\le x\le b$, $0<y\le f(x)$. Prove that the number of lattice points in S is equal to the sum $\sum_{n=a}^{b}[f(n)]$.

35. If a and b are positive integers with no common factor, we have the formula $\sum_{n=1}^{b-1}\left[\frac{na}{b}\right]=\frac{(a-1)(b-1)}{2}$. When b = 1, the sum on the left is understood to be 0.

(a) Derive this result by a geometric argument, counting lattice points in a right triangle.

(b) Derive the result analytically as follows : By changing the index of summation, note that $\sum_{n=1}^{b-1}[na/b]=\sum_{n=1}^{b-1}[a(b-n)/b]$.

Previous Years' Questions (JEE ADVANCED)

A. Fill in the Blanks

1. If the product of the roots of the equation $x^2+3kx+2e^{2\ln k}-1=0$ is 7, then the roots are real for k = ... **[IIT - 1984]**

2. The solution of equation $\log_7\log_5(\sqrt{x+5}+\sqrt{x})=0$ is .... **[IIT - 1986]**

3. The solution set of the system of equations $x+y=\frac{2\pi}{3}$, $\cos x+\cos y=\frac{3}{2}$, where x and y are real, is.......... **[IIT- 1986]**

4. The set of all x in the interval $[0,\pi]$ for which $2\sin^2 x-3\sin x+1\ge 0$, is......... **[IIT- 1987]**

5. If $x < 0, y < 0, x + y + \frac{x}{y} = \frac{1}{2}$ and $(x + y)\frac{x}{y} = -\frac{1}{2}$, then x = ...and y =......... **[IIT - 1990]**

6. General value of θ satisfying the equation $\tan^2\theta + \sec 2\theta = 1$ is.......... **[IIT - 1996]**

7. The sum of all the real roots of the equation $|x - 2|^2 + |x - 2| - 2 = 0$ is....... **[IIT - 1997]**

B. True/False

8. The equation $ex^2 + 3x + 1 = 0$ has an irrational root. **[IIT - 1983]**

9. There exists a value of θ between 0 and 2π that satisfies the equation $\sin^4\theta - 2\sin^2\theta - 1=0$ .... **[IIT- 1984]**

10. If $n_1, n_2, ...n_p$ are p positive integers, whose sum is an even number, then the number of odd integers among them is odd. **[IIT - 1985]**

C. Multiple Choice Questions with One Correct Answer

11. The entire graphs of the equation $y = x^2 + kx - x + 9$ is strictly above the x-axis if and only if **[IIT - 1979]**
(A) $k < 7$ (B) $-5 < k < 7$ (C) $k > -5$ (D) None of these

12. The least value of the expression $2\log_{10}x - \log_x(0.01)$, for $x > 1$, is **[IIT - 1980]**
(A) 10 (B) 2 (C) -0.01 (D) None of these

13. The expression $\frac{12}{3+\sqrt{5}+2\sqrt{2}}$ is equal to **[IIT - 1980]**
(A) $1 - \sqrt{5} + \sqrt{2} + \sqrt{10}$ (B) $1 + \sqrt{5} + \sqrt{2} - \sqrt{10}$
(C) $1 + \sqrt{5} - \sqrt{2} + \sqrt{10}$ (D) $1 - \sqrt{5} - \sqrt{2} + \sqrt{10}$

14. Given $A = \sin^2\theta + \cos^4\theta$, then for all real values of θ : **[IIT - 1980]**
(A) $1 \le A \le 2$ (B) $\frac{3}{4} \le A \le 1$ (C) $\frac{13}{16} \le A \le 1$ (D) $\frac{3}{4} \le A \le \frac{13}{16}$

15. The equation $2\cos^2\left(\frac{1}{2}x\right)\sin^2 x = x^2 + x^{-2}, x \le \frac{\pi}{9}$ has: **[IIT - 1980]**
(A) no real solution (B) one real solution
(C) more than one real solutions (D) None of these

16. The general solution of the trigonometric equation $\sin x + \cos x = 1$ is given by : **[IIT - 1981]**
(A) $x = 2n\pi; n = 0, \pm 1, \pm 2...$ (B) $x = 2n\pi + \pi/2; n = 0, \pm 1, \pm 2...$
(C) $x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}; n = 0, \pm 1, \pm 2...$ (D) None of these

17. If p, q, r are any real numbers, then **[IIT - 1982]**
(A) max (p, q) < max (p, q, r) (B) min (p, q) = 1/2(p + q – |p – q|)
(C) max (p, q) < min (p, q, r) (D) None of these

18. If x satisfies |x = 1| + |x – 2| + |x – 3| ≥ 6, then **[IIT - 1983]**
(A) $0 \le x \le 4$ (B) $x \le -2$ or $x \ge 4$
(C) $x \le 0$ or $x \ge 4$ (D) None of these

19. If f(x) = cos(ln x), then $f(x)\,f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value **[IIT - 1983]**
(A) –1 (B) 1/2 (C) –2 (D) None of these

20. The equation $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$ has **[IIT - 1984]**
(A) no root (B) one root (C) two equal roots (D) infinitely many roots

21. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is equal to : **[IIT- 1984]**
(A) $\frac{1}{2}$ (B) $\cos\frac{\pi}{8}$ (C) $\frac{1}{8}$ (D) $\frac{1+\sqrt{2}}{2\sqrt{2}}$

22. The expression $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$ is equal to **[IIT- 1986]**
(A) 0 (B) 1 (C) 3 (D) sin 4α + cos 6α

23. The number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2 \cos(2x) + a_3 \sin^2(x) = 0$ for all x is : **[IIT- 1987]**
(A) zero (B) one (C) three (D) infinite

24. The smallest positive root of the equation, tan x – x = 0 lies in : **[IIT- 1987]**
(A) $\left(0,\frac{\pi}{2}\right)$ (B) $\left(\frac{\pi}{2},\pi\right)$ (C) $\left(\pi,\frac{3\pi}{2}\right)$ (D) $\left(\frac{3\pi}{2},2\pi\right)$

25. The general solution sin x – 3 sin 2x + sin 3x = cos x – 3 cos 2x + cos 3x is : **[IIT- 1987]**
(A) $n\pi + \frac{\pi}{8}$ (B) $\frac{n\pi}{2} + \frac{\pi}{8}$ (C) $(-1)^n\frac{n\pi}{2} + \frac{\pi}{8}$ (D) $2n\pi + \cos^{-1}\frac{2}{3}$

26. The value of the expression $\sqrt{3}$ cosec 20° – sec 20° is equal to : **[IIT - 1988]**
(A) 2 (B) 2 sin 20°/ sin 40° (C) 4 (D) 4 sin 20° / sin 40°

27. The number of solution of the equation $\sin(e)^x = 5^x + 5^{-x}$ is **[IIT - 1990]**
(A) 0 (B) 1 (C) 2 (D) Infinitely many

28. The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ in the variable x, has real roots. then p can take any value in the interval : **[IIT- 1990]**
(A) (0, 2π) (B) (–π, 0) (C) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (D) (0, π)

29. In a triangle ABC, angle A is greater than angle B. If the measures of angles A and B satisfy the equation, $3 \sin x - 4 \sin^3 x - K = 0$, $0 < K < 1$, then the measure of angle C is **[IIT - 1990]**
(A) $\pi/3$ (B) $\pi/2$ (C) $2\pi/3$ (D) $5\pi/6$

30. Number of solutions of the equation $\tan x + \sec x = 2 \cos x$ lying in the interval $(0, 2\pi)$ is **[IIT - 1993]**
(A) 0 (B) 1 (C) 2 (D) 3

31. The number of points of intersection of two curves $y = 2 \sin x$ and $y = 5x^2 + 2x + 3$ is **[IIT - 1994]**
(A) 0 (B) 1 (C) 2 (D) ∞

32. Let $2 \sin^2 x + 3 \sin x - 2 > 0$ and $x^2 - x - 2 < 0$ (x is measured in radians). Then x lies in the interval **[IIT - 1994]**
(A) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (B) $\left(-1, \frac{5\pi}{6}\right)$ (C) $(-1, 2)$ (D) $\left(\frac{\pi}{6}, 2\right)$

33. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$ is equal to : **[IIT - 1995]**
(A) 11 (B) 12 (C) 13 (D) 14

34. The general value of θ satisfying the equation $2\sin^2 \theta - 3\sin \theta - 2 = 0$, is : **[IIT- 1995]**
(A) $n\pi + (-1)^n \frac{\pi}{6}$ (B) $n\pi + (-1)^n \frac{\pi}{2}$ (C) $n\pi + (-1)^2 \frac{5\pi}{6}$ (D) $n\pi + (-1)^n \frac{7\pi}{6}$

35. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has **[IIT - 1997]**
(A) no solution (B) one solution
(C) two solutions (D) more than two solutions

36. The graph of the function $\cos x \cos(x + 2) - \cos^2 (x + 1)$ is : **[IIT- 1997]**
(A) a straight line passing through $(0, -\sin^2 1)$ with slope 2
(B) a straight line passing through (0, 0)
(C) a parabola with vertex $(1, -\sin^2 1)$
(D) a straight line passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$ and parallel to the x-axis

37. Which of the following number is rational ? **[IIT - 1998]**
(A) sin 15° (B) cos 15° (C) sin 15° cos 15° (D) sin 15° cos 75°

38. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3 \sin^2 x - 7 \sin x + 2 = 0$ is **[IIT - 1998]**
(A) 0 (B) 5 (C) 6 (D) 10

39. The number of solution of $\log_4 (x - 1) = \log_2 (x - 3)$ is **[IIT - 2001]**
(A) 3 (B) 1 (C) 2 (D) 0

40. If $\alpha + \beta = \pi/2$ and $\beta + \gamma = \alpha$, then $\tan \alpha$ equals **[IIT - 2001]**
(A) $2 (\tan \beta + \tan \gamma)$ (B) $\tan \beta + \tan \gamma$ (C) $\tan \beta + 2\tan \gamma$ (D) $2\tan \beta + \tan \gamma$

41. The set of all real numbers x for which $| x^2 - |x + 2| + x > 0$, is **[IIT - 2002]**
(A) $(-\infty, 2) \cup (2, \infty)$ (B) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$
(C) $(-\infty, -1) \cup (1, \infty)$ (D) $(\sqrt{2}, \infty)$

42. The number of integral values of k for which the equation : $7 \cos x + 5 \sin x = 2k + 1$ has a solution is **[IIT -2002]**
(A) 4 (B) 8 (C) 10 (D) 12

43. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that min f(x) > max g (x), then the relation between b and c is **[IIT - 2003]**
(A) no real value of b & c
(B) $0 < c < b\sqrt{2}$
(C) $|c| < |b|\sqrt{2}$
(D) $|c| > |b|\sqrt{2}$

44. Given both θ and φ are acute angles $\sin\theta = 1/2$, $\cos\varphi = 1/3$, then the value of $\theta + \varphi$ belongs to **[IIT - 2004]**
(A) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right]$
(B) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$
(C) $\left(\frac{2\pi}{3}, \frac{5\pi}{6}\right]$
(D) $\left(\frac{5\pi}{6}, \pi\right]$

45. The number of ordered pairs (α, β), where $\alpha, \beta \in (-\pi, \pi)$ satisfying $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$ is **[IIT - 2005]**
(A) 0
(B) 1
(C) 2
(D) 4

46. The set of values of θ satisfying the inequation $2\sin^2\theta - 5\sin\theta + 2 > 0$ where $0 < \theta < 2\pi$, is **[IIT -2006]**
(A) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$
(b) $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$
(c) $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, 2\pi\right]$
(d) None of these

47. The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is **[IIT - 2007]**
(A) zero
(B) one
(C) two
(D) four

48. Let (x_0, y_0) be the solution of the following equation $(2x)^{ln 2} = (3y)^{ln 3}$, $3^{ln x} = 2^{ln y}$. Then x_0 is **[IIT - 2011]**
(A) $\frac{1}{6}$
(B) $\frac{1}{3}$
(C) $\frac{1}{2}$
(D) 6

49. Let $P = \{\theta: \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then **[IIT - 2011]**
(A) $P \subset Q$ and $Q - p \neq \varnothing$
(B) $Q \not\subset P$
(C) $P \not\subset Q$
(D) $P = Q$

D. Multiple Choice Questions with One or More Than One Correct Answer

50. If S is the set of all real x such that $\frac{2x-1}{2x^3 + 3x^2 + x}$ is positive, then S contains **[IIT - 1984]**
(A) $\left(-\infty, -\frac{3}{2}\right)$
(B) $\left(-\frac{3}{2}, -\frac{1}{4}\right)$
(C) $\left(-\frac{1}{4}, \frac{1}{2}\right)$
(D) $\left(\frac{1}{2}, 3\right)$

51. The equation $x^{3/4(\log_2 x)^2 + \log_2 x - 5/4} = \sqrt{2}$ has **[IIT - 1989]**
(A) at least one real solution
(B) exactly three solutions
(C) exactly one irrational solution
(D) complex roots

52. Let g(x) be a function defined on [–1 , 1]. If the area of the equilateral triangle with two of its vertices at (0, 0) and (x, g(x)) is $\sqrt{3}/4$, then the function g(x) is **[IIT - 1989]**

(A) $g(x) = \pm\sqrt{1-x^2}$ (B) $g(x) = \sqrt{1-x^2}$

(C) $g(x) = -\sqrt{1-x^2}$ (D) $g(x) = \sqrt{1+x^2}$

53. If $f(x) = \cos[\pi^2]\, x + \cos[-\pi^2]x$, where $[x]$ stands for the greatest integer function, then **[IIT - 1991]**

(A) $f(\pi/2) = -1$ (B) $f(\pi) = 1$ (C) $f(-\pi) = 0$ (D) $f(\pi/4) = 1$

54. For a positive integer n, let $f_n(\theta) = \left(\tan\frac{\theta}{2}\right)(1 + \sec\theta)(1 + \sec 2\theta)(1 + \sec 4\theta) \ldots\ldots\ldots (1 + \sec 2^n\theta)$ Then - **[IIT - 1999]**

(A) $f_2\left(\frac{\pi}{16}\right) = 1$ (B) $f_3\left(\frac{\pi}{32}\right) = 1$ (C) $f_4 = \left(\frac{\pi}{64}\right) = 1$ (D) $f_5 = \left(\frac{\pi}{128}\right) = 1$

E. Subjective Problems

55. Solve for x : $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$ **[IIT - 1978]**

56. Solve for x : $\sqrt{x+1} - \sqrt{x-1} = 1$. **[IIT - 1978]**

57. Solve the following equation for x: $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0,\ a > 0$ **[IIT - 1978]**

58. Show that the square of $\dfrac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$ is a rational number. **[IIT - 1978]**

59. Find all integers x for which $(5x - 1) < (x + 1)^2 < (7x - 3)$. **[IIT - 1978]**

60. Draw the graph of $y = |x|^{1/2}$ for $-1 \le x \le 1$. **[IIT - 1978]**

61. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, find f(6). **[IIT - 1979]**

62. If $x > y > 0$, then show that the expression $(\sqrt{2}\,(2x + \sqrt{x^2-y^2})\,(\sqrt{x-\sqrt{x^2-y^2}}))$ can be simplified to $\sqrt{(x+y)^3} - \sqrt{(x-y)^3}$ **[IIT - 1980]**

63. Let $y = \sqrt{\dfrac{(x+1)(x-3)}{(x-2)}}$. Find all real values of x for which y takes real values. **[IIT - 1980]**

64. For what values of m, does the system of equations

$3x + my = m$

$2x - 5y = 20$

has solution satisfying the conditions $x > 0,\ y > 0$. **[IIT - 1980]**

65. Show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real solution. **[IIT - 1982]**

66. Find the coordinates of the points of intersection of the curves $y = \cos x$, $y = \sin 3x$, if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. **[IIT - 1982]**

67. Find all real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \le 0$. **[IIT - 1983]**

68. Find all the solutions of $4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$ **[IIT - 1983]**

69. Find the values of x $(-\pi, +\pi)$ which satisfy the equation $2^{1+|\cos x|+|\cos^2 x|+\ldots} = 4$ **[IIT - 1984]**

70. Solve for x; $(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$ **[IIT - 1985]**

71. For $a \le 0$, determine all real roots of the equation $x^2 - 2a|x-a| - 3a^2 = 0$ **[IIT - 1986]**

72. Find the set of all x for which $\dfrac{2x}{(2x^2+5x+2)} > \dfrac{1}{(x+1)}$ **[IIT - 1987]**

73. Solve $|x^2 + 4x + 3| + 2x + 5 = 0$ **[IIT - 1988]**

74. If exp. $\{(\sin^2 x + \sin^4 x + \sin^6 x + \ldots\ \text{inf.}) \log_e 2\}$ satisfies the equation $x^2 - 9x + 8 = 0$, find the value of $\dfrac{\cos x}{\cos x + \sin x}$, $0 < x < \dfrac{\pi}{2}$. **[IIT - 1991]**

75. Determine the smallest positive values of x (in degrees) for which;
$\tan(x + 100°) = \tan(x + 50°) \tan x \tan(x - 50°)$. **[IIT - 1993]**

76. Let {x} and [x] denote the fractional and integral part of a real number x respectively. Solve $4\{x\} = x + [x]$ **[IIT - 1994]**

77. Find the smallest positive number p for which the equation $\cos(p \sin x) = \sin(p \cos x)$ has a solution $x \in [0, 2\pi]$ **[IIT - 1995]**

78. Find the set of all solutions of the equation $2^{|y|} - |2^{y-1} - 1| = 2^{y-1} + 1$. **[IIT - 1997]**

79. Prove that the values of the function $\dfrac{\sin x \cos 3x}{\sin 3x \cos x}$ do not lie between $\dfrac{1}{3}$ and 3 for any real x. **[IIT- 1997]**

80. In any triangle ABC, prove that, $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$. **[IIT - 2000]**

F. Integer Answer Type

81. The value of $6 + \log_{3/2}\left(\dfrac{1}{3\sqrt{2}}\sqrt{4 - \dfrac{1}{3\sqrt{2}}\sqrt{4 - \dfrac{1}{3\sqrt{2}}\sqrt{4 - \dfrac{1}{3\sqrt{2}}\ldots}}}\right)$ is **[IIT - 2012]**

ANSWERS

Practice Problems [A]

1. (i) $x \in \left[\dfrac{3}{2}, \infty\right)$ (ii) $x \in R - \{-1\}$ (iii) $x \in R - \{\pm 1\}$ (iv) $x \in (-\infty, 0)$

2. $\left\{\dfrac{a^2 - 3a + 9}{a - 3}\right\}$ for $a \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$; $\{c \mid c \in R\}$ for $a = -3$; φ for $a = 3$

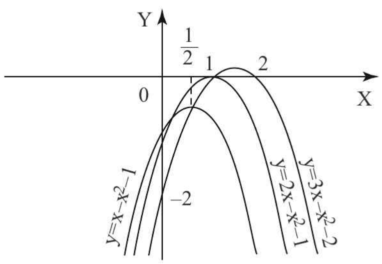
3. $\dfrac{1}{2}$

4. $[1/\alpha, \infty)$ for $\alpha \in (-\infty, 0)$, $(-\infty, \infty)$ for $\alpha = 0$, $(-\infty, 1/\alpha]$ for $\alpha \in (0, \infty)$.

5. Solve the inequality , $a^2 + ax < 1 - x$.

6. 4
7. $\left(-\infty, \frac{1}{2}\right]$.
8. (7, 0), (0, 7), (3, 4) and (4, 3)

PRACTICE PROBLEMS [B]

1. 

2. (i) $x \in (-\infty, -1) \cup (4, \infty)$ (ii) $x \in \left\{-\frac{1}{2}\right\}$ (iii) $x \in R$

3. (i) $x \in \left\{\frac{1}{2}\right\}$ (ii) {4}

4. (a) (–1.5, ∞), (b) (–∞, 2)

5. [5, ∞)

6. (i) $-\frac{3}{2}, \infty$ (ii) $-1, \infty$ (iii) $-\frac{3}{2}, 23$

7. 1, 4

8. $\left(-\infty, -\frac{a-\sqrt{a^2-4a}}{2}\right) \cup \left(\frac{\sqrt{a^2-4a}-a}{2}, \infty\right)$, $a \in (-\infty, 0) \cup (4, \infty)$; R, $a \in (0, 4)$; $R - \left\{-\frac{a}{2}\right\}$, $a \in \{0, 4\}$

9. a ∞ ∈ (5/3, ∞).
10. (–∞, –6)
11. (a) $y_{min} = y(1) = 7$, $y_{max} = y(2) = 15$ (b) $y_{max} = y(2) = -14$, $y_{min} = y(3) = -29$
12. $\frac{25}{8}$
13. $y \in \left(-\infty, -\frac{7}{8}\right]$
14. –7, 1 act
15. $\frac{6}{7}$

Practice Problems [C]

1. $3x^3 + 4x^2 - 5x - 2$
2. $2x$ 3. $[-4, -2] \cup \{1\}$
4. $(-\infty, -1] \cup \{1\} \cup [2, \infty)$
5. $(-5, -1)$
6. $(-\infty, -2] \cup [0, \infty)$
7. $-\frac{1}{2}$
8. (a) $p(x) = ax(x-1) + b$, a and b arbitrary (b) $p(x) = c$, c arbitrary
   (c) $p(x) = ax$, a arbitrary (d) $p(x) = c$, c arbitrary
9. (a) $(-\infty, \infty)$ (b) $(-(3 + \sqrt{33})/2, (\sqrt{33} - 3)/2)$
10. 18
11. No Solution
14. (i) -1 (ii) 1 (iii) -243 (iv) -121
    (v) 122

Practice Problems [D]

3. (i) $[1, 4) \in (4, \infty)$ (ii) $(-\infty, -3) \cup (2, 4)$ (iii) $\left\{-\frac{5}{2}, \frac{3}{2}\right\}$ (iv) $(-1, 0)$
   (v) $(-\infty, -2) \cup (-1, 1) \cup (2, 3) \cup (4, 6) \cup (7, \infty)$ (vi) $\left(-2, \frac{5}{2}\right) \cup \left(\frac{8}{3}, \infty\right)$
   (vii) $[-2, \infty)$ (viii) $(-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$
4. $x \in (-\infty, -2) \cup [2, 3)$
5. $\left(\frac{3}{2}, 4\right)$
6. $\{3, 4, 5, 6, 7\}$
7. $\frac{1}{2} < a < 1$
8. $m < \frac{11}{4}$
9. (i) $(-1, 2)$ (ii) $(-\infty, -1) \cup (0, 1/2] \cup (1, 2)$ (iii) $(-2, -1) \cup [0, 1] \cup [2, \infty)$

10. (i) $(-1/2, 1)$ (ii) $[-2, 1]$

11. (i) $(0, 1/3) \cup (3, \infty)$

(ii) $\left[-\dfrac{1+\sqrt{5}}{2}, -1\right] \cup \left[\dfrac{1-\sqrt{5}}{2}, 0\right) \cup \left[\dfrac{-1+\sqrt{5}}{2}, 1\right] \cup \left[\dfrac{1+\sqrt{5}}{2}, \infty\right)$

Practice Problems [E]

1. $\dfrac{x-1+\sqrt{x^2-1}+\sqrt{2x(x-1)}}{x-1}$

2. (a) $\dfrac{1}{\sqrt{2}}\left|\sqrt{\sqrt{2x+1}+\sqrt{x-4}}\right|$ (b) $\dfrac{1}{\sqrt{2}}\left|\sqrt{3a+x}-\sqrt{a-x}\right|$

3. (i) $3-\sqrt{5}$ (ii) $3-2\sqrt{2}$

4. (i) $x = a^2-1, \quad a \geq 0$; $\phi, \quad a < 0$

(ii) $x = \dfrac{a-3}{2}$ if $a \geq -3$; ϕ if $a < -3$ (iii) No solution (iv) $-7, 2$

5. (i) $(-\infty, -1] \cup [2, +\infty)$ (ii) $(1/2, 2]$ (iii) $[3, +\infty)$

6. (i) $(6/5, -1] \cup [2, 3)$ (ii) $(-\infty, 1)$ (iii) $(6, 8]$

7. $R - \{-1, 1\}$

9. (i) $[-4, -1] \cup [0, \infty)$ (ii) x-intercepts : $-4, -1, 0$; y-intercepts : 0

(iii) $a/4$ (iv) 0

10. (i) $\{3\}$ (ii) $\left\{3+\dfrac{2\sqrt{5}}{5}\right\}$ (iii) $\{5\}$ (iv) $\{2\}$

(v) $\{20\}$ (vi) $\{(-1-\sqrt{5})/2, (\sqrt{5}-1)/2, 1\}$

11. (i) $[0, 2]$ (ii) $(-\infty, -2) \cup [5, 74/13]$ (iii) $(-1, 0) \cup (3/5, 1]$

12. (i) $(1, 4]$ (ii) $(3, 5]$ (iii) $[3, 4) \cup (7, +\infty)$

13. (i) $[1/2, 2) \cup (5, +\infty)$ (ii) $(-2-2\sqrt{6}, -1) \cup [-2+2\sqrt{6}, 3]$ (iii) $[-1, \infty)$

14. (i) $\{-3, 1)$ (ii) $[(16+\sqrt{7})/2, 10]$

15. (i) $(-\infty, -1/2) \cup (3, \infty)$ (ii) $(-3, 0) \cup (0, 2) \cup (2, 4)$

(iii) $(-\infty, -1) \cup (4, \infty)$ (iv) $\{1\}$

16. (i) $\{(a-3)/2\}$ for $a \in [-3, \infty]$, φ or $a \in (-\infty, -3)$

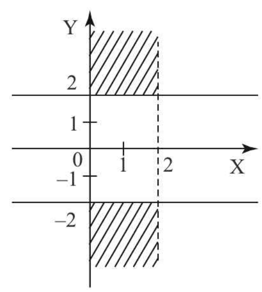
(ii) $\{5 \pm a\sqrt{8-a^2}/2\}$ for $a \in [2, 2\sqrt{2}]$; φ for $a \in (-\infty, 2) \cup (2\sqrt{2}, \infty)$

PRACTICE PROBLEMS [F]

1. (i) $\sqrt{3}-\sqrt{\sqrt{15}-2}$ (ii) $x-3$ (iii) $1-2x$ (iv) $-x+2$.

2. $-2a$ 3. $\{-2, -1/9\}$

4. (i) 1 (ii) ± 1 5. (i) 0 (ii) $-\frac{2}{5}, 2$

6. 

7. (i) $(-\infty, -1) \cup (3, \infty)$

(ii) $\left(\frac{3}{4}, 1\right) \cup (1, \infty)$ (iii) $(-\infty, -2) \cup (-1, \infty)$ (iv) $(-\infty, -4) \cup 0, \infty)$

8. (i) $(9/2, \infty)$ (ii) $(-7, -2) \cup (3, 4)$ (iii) $\left[\frac{2}{7}, \frac{2}{3}\right]$

9. $a \in (2, \infty)$ 10. 1 11. 15 12. 2

13. (i) -2 (ii) $x = -3, x \geq -1$ (iii) $x = -1, 0 \leq x \leq 1$

14. (i) $[0, \sqrt{2}]$ (ii) $(-\infty, 1] \cup [5, \infty)$ (iii) R (iv) No solution

15. $\pm\frac{a+\sqrt{a^2-16a+60}}{15-4a}$ for $a < \frac{15}{4}$; ϕ for $a \geq \frac{15}{4}$

16. (i) $(-\infty, -4) \cup (8/7, 2) \cup (2, \infty)$ (ii) $(-\infty, 11/4) \cup (7/2, \infty)$

PRACTICE PROBLEMS [G]

8. $[0, \infty)$

9. $[-2, -1] \cup [0, \infty)$ 10. $\left[-2, \frac{5-\sqrt{21}}{2}\right] \cup \left[\frac{5+\sqrt{21}}{2}, \infty\right)$

11. (i) $x \in \{0\} \cup (1. \infty)$ (ii) $[-1, 1]$ (iii) $(0, 2)$

12. (i) $x \in [-2, 7]$ (ii) $\left[-\frac{3}{4}, \infty\right) \cup \{-2\}$.

PRACTICE PROBLEMS [H]

1. 9
2. (i) 2 (ii) $\frac{1}{3}\log_5 6$ (iii) 1 (iv) 1
4. (i) 1, –2 (ii) 5, 2
5. 1.948445
7. $a = \frac{\sqrt{5}+1}{2}$
8. 2916
9. 2
10. $ab(a-b)^2$
11. (i) 1 (ii) $a^2 + a + 1$ (iii) $\frac{1}{\log_a b - 1}$ (iv) 6
12. 1398
13. 9
14. (i) 0.4409 (ii) 301

PRACTICE PROBLEMS [I]

1. 0
2. $\log_{3/2}2, 2\log_{3/2}2$
3. $1 \pm \sqrt{3}$
4. –2
5. {–2, 2}
6. –1; 1; $\sqrt{2}$; $-\sqrt{2}$
7. 1
8. –1; 1
9. 1; $1+\sqrt{2}$; $1-\sqrt{2}$
10. 2; $-1-\log_5 2$
11. –1; 1; 2

12. –3; 1; 2; 3; 4
13. 1
14. $[1, \infty)$
15. one
16. three

PRACTICE PROBLEMS [J]

1. $x \in R$
2. $(-5, 5))$
3. $(2, 6)$
4. $(-\infty, \log_2(-1+\sqrt{7})]$
5. $[-4, 2) \cup (0, \infty)$
6. $(-\infty, 0) \cup [1, +\infty)$
7. $(-\sqrt{7}, -\sqrt{3}) \cup (\sqrt{3}, \sqrt{7})$
8. $(2, +\infty)$
9. $x \in (a^2 \log_3^2 2, +\infty)$ for $a \in [0, +\infty)$, $x \in [0, +\infty)$ for $a \in (-\infty, 0)$
10. $(-\infty, 2)$
11. $(1, 2)$
12. $[0, 1)$

PRACTICE PROBLEMS [K]

1. 1, 3
2. $\left[\frac{1}{5}, \infty\right)$
3. $10^{-5}, 10^3$
4. $2^{-2\sqrt{2}}, 2^{2\sqrt{2}}$
5. 2
6. $\frac{1}{20}, \frac{1}{5}$

7. 2

8. $4 \log_3 2$

9. No Soln

10. $1, 8, 2^{-2/3}$

11. –5, 5

12. 17/12

13. 1/10 , 2, 1000, 4

14. $\pm \frac{1}{e}, \pm e^3, 2$

15. $\frac{1}{81}, 3$

16. $1, 4, \frac{1}{4}\sqrt[5]{8}$

17. 10

18. $\sqrt{(1+\sqrt{5})/2}$

19. $\{1/16\} \cup [4, \infty)$

21. 8

22. x = 6; y = 2 or x = 2; y = 6

23. $\log_2 11$

25. 5

PRACTICE PROBLEMS [L]

1. $[-7, -\sqrt{35}) \cup [5, \sqrt{35})$
2. (0, 10)
3. $[\log_3 0.9, 2)$
4. $(0, 1/2) \cup (2, 3)$

5. (i) $x \in (-\infty, -2) \cup (2, \infty) - \{3\}$ (ii) $x \in (0, 3) - \{1/2\}$.
6. $\{2, 3\}$
7. $(-\infty, -1)$
8. $\left(-\infty, \frac{-2}{3}\right) \cup (1, \infty)$
9. $(-1, 0) \cup (0, 2] \cup (3, 4)$
10. $(1, 2)$
11. $[28/3, \infty)$
12. $\left(-4, \left(1+\sqrt{17}\right)/2\right)$
13. $(1/a, a^4)$ for $a \in (1, \infty)$; $(a^4, 1/a)$ for $a \in (0, 1)$
14. $\left(1/\sqrt{5}, 1\right) \cup (3, \infty)$
15. $(-\infty, \log_2(\sqrt{2} - 1)] \cup \left[\frac{1}{2}, \infty\right)$
16. $(10 - \sqrt{43}, 4) \cup (10 + \sqrt{43}, \infty)$
17. $(0, 1/4] \cup [1, 4)$
18. $x \in (1, (1 + \sqrt{1+4a^2} / 2))$ for $a \in (0, 1)$; $x \in ((1 + \sqrt{1+4a^2})/2, \infty)$ for $a \in (1, \infty)$.
19. $(0, 2^{-1/3}] \cup [2, \infty)$
20. $\left(\frac{1}{2^{\sqrt{2}}}, \frac{1}{2}\right) \cup (1, 2^{\sqrt{2}})$
21. $(-\infty, -2/3] \cup [1/2, 2]$
22. $(-1, 0)$
23. $[\log_a (4\sqrt{a^2+16}), 3\log_a 2)$, $0 < a < 1$; $[\log_a (4 + \sqrt{16+a^2}), \infty)$, $a > 1$
24. $[-1, 1/2) \cup (1, 2) \cup (2, 7/2)$
25. $[\sqrt{6} - 1, 2) \cup (2, 5]$

PRACTICE PROBLEMS [M]

1. $\left[-1, \frac{\sqrt{3}}{2}\right]$
2. (i) $[-2, 8]$ (ii) $\left[\frac{3}{4}, 3\right]$
4. $\sqrt{3}$
5. $\sin\frac{\alpha}{2} = \frac{2\sqrt{2}}{3}$.
6. $\frac{2}{3}$
7. tan (sin 1)
9. $(1-\sqrt{5}, -1) \cup (3, 1+\sqrt{5})$
10. $[-1, 7/3]$
12. (i) max = 11, min = 1 (ii) max = 11, min = 1
13. 27
14. 0

PRACTICE PROBLEMS [N]

1. $x = 2n\pi + \frac{\pi}{6}, n \in I$ 2. $x = 1, y = n\pi, n \in I$
3. (i) $(2n+1)\pi, n \in I$ (ii) $2n\pi - \frac{\pi}{2}, n \in I$ or $n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in I$
4. (i) $\frac{n\pi}{3}, n \in I$ or $\frac{n\pi}{2} \pm \frac{\pi}{12}, n \in I$ (ii) $\frac{n\pi}{2}, n \in I$ or $2n\pi \pm \frac{2\pi}{3}, n \in I$

   (iii) $\frac{2n\pi}{3}, n \in I$ or $2n\pi - \frac{\pi}{2}, n \in I$ or $n\pi + \frac{\pi}{4}, n \in I$
5. (i) $n\pi - \frac{\pi}{4}, n \in I$ (ii) $n\pi - \frac{\pi}{4}, n \in I$

   (iii) $2n\pi + \frac{\pi}{2}, n \in I$ or $2n\pi, n \in I$ or $n\pi + \frac{\pi}{4}, n \in I$
6. (i) $(4p-3)\frac{\pi}{2}, p \in I$ (ii) $2m\pi + \frac{\pi}{2}, m \in I$ 7. $x = -\frac{\pi}{8} + k \cdot \frac{\pi}{2}$ $(k = 0, \pm 1, \pm 2, \ldots)$

8. $x_1 = -\frac{\pi}{66} + \frac{k\pi}{11}$, $x_2 = \frac{\pi}{36} + \frac{(2k+1)\pi}{12}$.

9. $x_1 = \frac{\pi}{8} + \frac{k\pi}{2}$, $x_2 = \frac{3\pi}{4} + k\pi$.

10. $x = 2n\pi + 2\alpha$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$, $n \in I$

11. $x = \frac{\pi}{2} + 2k\pi$.

12. no solution.

13. no solution.

14. $x = \left(2m + \frac{1}{2}\right)\pi$.

15. $x = \frac{5\pi}{16} + \pi k \quad (k \in Z)$.

PRACTICE PROBLEMS [O]

1. $\left(-\frac{5\pi}{8} + \pi n, \frac{\pi}{8} + \pi n\right)$
2. $\left[-\frac{5\pi}{6} + 10\pi n, \frac{35\pi}{6} + 10\pi n\right]$
3. $\left(\frac{1}{3} - \frac{\pi}{6} + \frac{\pi n}{3}, \frac{1}{3} + \frac{\pi}{18} + \frac{\pi n}{3}\right)$
4. $\left(-\frac{\pi}{2} + \pi n, -\frac{\pi}{3} + \pi n\right], \left[\frac{\pi}{3} + \pi n, \frac{\pi}{2} + \pi n\right)$
5. $\left(-\frac{\pi}{2} + \pi n, -\frac{\pi}{6} + \pi n\right], \left[\frac{\pi}{6} + \pi n, \frac{\pi}{2} + \pi n\right)$
6. $\left(-\pi + 2\pi n, -\frac{\pi}{3} + 2\pi n\right), \left(\frac{\pi}{3} + 2\pi n, \pi + 2\pi n\right)$
7. $\left(\frac{\pi}{4} + \pi n, \frac{3\pi}{4} + \pi n\right)$
8. ϕ
9. $x \neq \frac{\pi n}{2}$
10. $\left(\frac{\pi}{4} + \pi n, \frac{5\pi}{12} + \pi n\right)$
11. $-\frac{\pi}{3} + 2\pi k \le x < 2\pi k$; $\frac{\pi}{3} + 2\pi k \le x < \pi + 2\pi k$.
12. $\frac{\pi}{12} + 2\pi k < x < \frac{3\pi}{4} + 2\pi k$; $\frac{17\pi}{12} + 2\pi k < x < \frac{7\pi}{4} + 2\pi k$.
13. $-\frac{\pi}{8} + \pi k < x < \pi k$; $\frac{\pi}{2} + \pi k < x < \frac{5\pi}{8} + \pi k$; $\frac{\pi}{8} + \pi k < x < \frac{3\pi}{8} + \pi k$.
14. $-\frac{7\pi}{12} + \pi k \le x < -\frac{\pi}{2} + \pi k$; $-\frac{\pi}{2} + \pi k < x \le \frac{\pi}{12} + \pi k$.

15. $-\frac{\pi}{3} + 2\pi k < x < \frac{\pi}{3} + 2\pi k$

16. $-\frac{\pi}{3} + 2\pi k < x < \frac{\pi}{3} + 2\pi k$; $\frac{8\pi}{3} + 4\pi k < x < \frac{10\pi}{3} + 4\pi k$

17. $(-\pi/4 + 2\pi n, \pi/6 + 2\pi n) \cup (5\pi/6 + 2\pi n, 5\pi/4 + 2\pi n), n \in I$

18. $\left(\tan^{-1}(\sqrt{2}-1), \frac{\pi}{4}\right) \cup \left(\pi + \tan^{-1}\left(\sqrt{2}-1\right), \frac{5\pi}{4}\right)$

19. $[n + 1/4 , n + 3/4], n \in I$

20. $2n\pi + \frac{\pi}{3}$, $2n\pi + \cos{-1}\ \frac{\sqrt{13}-1}{4}$; $n \in I$

21. $\left(2n\pi + \frac{\pi}{12}, 2n\pi + \frac{5\pi}{12}\right), n \in I$

22. $\left(4n\pi, \frac{12n+1}{3}\pi\right) \cup \left(\frac{12n+5}{3}\pi, 4n\pi + 2\pi\right), n \in I$

23. $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

PRACTICE PROBLEMS [P]

1. 75
2. 276 4.0, $n + \frac{1}{2}$ where $n \in I$
5. No re l solution.
6. The formula $f(x) = \frac{1}{(x) - [x]}$ is not define for $x \in Z$. If $x \in R - Z$ then $\frac{1}{(x) - [x]} = 1$. Thus the graph consists of the horizontal line of equation $y = 1$ with punctures at the point $(n, 1), n \in Z$.
7. 2
9. 2
10. $x \in [-1, 0] \cup [1, 2]$
11. $x = 0, 1, 2, 3, ...$
12. no re l solution.
13. $A = \left\{n + \frac{1}{n} \middle| n \in N - \{1\}\right\}$
15. infinite
17. 2
18. $[3, \pi) \cup \left[\bigcup_{n=1}^{\infty} (2n\pi, 2n\pi + \pi)\right]$
19. $2n\pi \pm \frac{\pi}{4}$
20. $0 < x < 1$
21. $\{-1, 0, 1\}$

22. 2

23. (iv) 24. $[0, \sqrt{2})$ 25. 2 26. 739

Objective Exercises

1. A	2. B	3. C	4. D
5. D	6. B	7. A	8. D
9. A	10. B	11. C	12. C
13. A	14. C	15. D	16. B
17. A	18. A	19. A	20. D
21. B	22. A	23. C	24. B
25. B	26. A	27. A	28. C
29. B	30. B	31. B	32. B
33. B	34. B	35. A	36. B
37. C	38. A	39. D	40. D
41. C	42. A	43. C	44. A
45. B	46. B	47. B	48. C
49. D	50. C	51. A,B,D	52. All
53. All	54. C	55. B,C,D	56. All
57. B,C,D	58. B, C	59. D	60. A
61. C,D	62. A,C	63. B,C,D	64. A,B,D
65. B	66. A,B,C	67. C,D	68. C,D
69. A,C,D	70. A,B,C	71. C	72. C
73. C	74. D	75. B	76. C
77. B	78. A	79. A	80. A
81. D	82. B	83. C	84. A
85. B	86. A	87. C	88. D
89. D	90. D	91. A	92. D
93. C	94. A	95. C	

96. A–(P),(Q),(R),(T) ; B–(R),(S); C–(P),(Q); D–(P),(T)
97. (A)–(R) ; (B)–(S) ; (C)–(P) ; (D)–(Q)
98. (A)–(Q),(R) (S) ; (B)–(R); (C)–(P),(R),(S) ; (D)–(R),(S)
99. (A)–(S) ; (B)–(R) ; (C)–(Q) ; (D)–(P)
100. (A)–(R), (B)–(S), (C)–(Q), (D)–(P)

Review Exercises for JEE Advanced

2. $-\frac{1}{2}$
3. 13
4. $R - r \le d \le R + r$
5. $(-\infty, 4) \in (12, \infty)$
6. $[-5, -1) \cup (0, 3]$
7. $(-\infty, -2] \cup [0, \infty)$
8. (i) $\left\{3+\frac{2\sqrt{5}}{5}\right\}$ (ii) $\{20\}$
   (iii) $\{16/25\}$
9. (i) $\{-15, 13\}$ (ii) $\{16/25\}$
   (iii) $\{8\}$ (iv) $[2, \infty)$
10. (i) 25/16 (ii) 15
11. (i) 5 (ii) $1, -\frac{1}{3}$
12. $(-\infty, -1) \cup (-1, 2]$
13. (i) $(-3, 1]$
    (ii) $[-2, -1) \cup (-1, 1) \cup (1, 3]$
14. (i) $[-0.5, \infty)$ for $a \in [1, \infty)$, $[-0.5, -0.5(1 - 1/(1 - a)^2)$ for $a \in (-\infty, 1)$
    (ii) $[-2, \infty)$ for $a \in [-2, \infty)$, ϕ for $a \in (-\infty, -2)$
15. 92
16. $1\frac{1}{2}$
17. $(1, \infty) \cup \left[\frac{4}{5}, 1\right) \cup (-\infty, -1) \cup \left(-1, -\frac{4}{3}\right] \cup \{0\}$
19. 2/111

20. (i) 1 (ii) 7/3, 5

(iii) 5 (iv) $\sqrt{\dfrac{1+\sqrt{5}}{2}}$

21. (i) $2, 1-\sqrt{33}$ (ii) 15/32

(iii) $\{1/16\} \cup [4, \infty)$

22. $x > 1$

23. $[1, 4) \cup (0, \frac{1}{2}]$

24. (i) $[-1, 0] \cup [2, 3]$

(ii) $0 < x < 2, x > 4$ (iii) $\left(\dfrac{1}{2^{\sqrt{2}}}, \dfrac{1}{2}\right) \cup (1, 2^{\sqrt{2}})$

25. $x \in [-1, 1/2) \cup [1, 2) \cup (2, 7/2)$

26. $3/2 < x < 2;\ 1 < y < 2$

27. (i) $\{4\} \cup [0, 1]$

(ii) $x \in \phi$ for $a \in (-\infty, 1]$; $x = a \pm 1$ for $a \in (1, \sqrt{2}) \in (\sqrt{2}, 2) \cup (2, \infty)$, $x = a + 1 = 3$ for $a = 2$.

28. (a) 11, 1 (b) $\dfrac{13}{3}, -1$

30. $\dfrac{3}{2}$

32. (i) $x = \pm\dfrac{1}{2}$ arc cos $\dfrac{1}{3} + k\pi;\ \pm\dfrac{1}{2}$ arc cos $\left(-\dfrac{1}{4}\right) + k\pi$.

(ii) $x = (-1)k\dfrac{1}{2}$ arc sin $\dfrac{\sqrt{5}-1}{2} + \dfrac{k\pi}{2}$

(iii) $x = n\pi, n \in I$

(iv) $x = \pm\dfrac{\pi}{3} + 2\pi k, k \in I$.

33. $\alpha - 2\pi;\ \alpha - \pi,\ \alpha,\ \alpha + \pi$, where $\tan \alpha = \dfrac{2}{3}$

34. (i) $2k\pi < x < 2k\pi + \dfrac{\pi}{4}$, $k \in I$

(ii) $2k\pi - \dfrac{\pi}{4} + \sin^{-1}\dfrac{2\sqrt{2}}{3} < x < 2k\pi + \dfrac{3\pi}{4} - \sin^{-1}\dfrac{2\sqrt{2}}{3},\ 2k\pi - \pi < x < 2k\pi - \dfrac{\pi}{2}$

(iii) $0 < x < 1$

(iv) $0 < x < \frac{\pi}{2}$, $2 \tan^{-1} \frac{1+\sqrt{5}}{2} < x < \pi$

35. 1904 and 1994

TARGET EXERCISES FOR JEE ADVANCED

1. 3, 4 respectively
2. $\left\{\frac{1}{a}, \frac{a}{a-1}, 1-a\right\}$

6. (i) $a > \frac{1}{8}$ (ii) ϕ

 (iii) $a < 0$, $a = \frac{1}{8}$ (iv) $a = 0$

 (v) $0 < a < \frac{1}{8}$

7. (i) {5} (ii) {20}

 (iii) {–1; 0} (iv) {5}

8. (i) $\{a + 1 + \sqrt{2a}, a + 1 - \sqrt{2a}\}$ for $a \in [0, 1/2]$, $\{a + 1 + \sqrt{2a}\}$ for $a \in (1/2, \infty)$, φ for $a \in (-\infty, 0)$

 (ii) $(-\infty, 0)$ for $a = 0$; $\{0, 3a/4\}$ for $a \in (0, \infty)$, φ for $a \in (-\infty, 0)$

9. φ for a=2; $\left[\frac{1}{3}; \frac{a-1}{3}\right)$ for a>2; $\left(\frac{a-1}{3}; \frac{1}{3}\right]$ for $a < 2$

11. 1, 2

12. (i) $a^{5a^2/4}$, $a^{4/5a^2}$ if $a \geq 2/\sqrt{5}$

 (ii) 5 (iii) 100

13. 100

20. $\dfrac{2}{\sin 2\theta} - \dfrac{1}{\theta}$

23. 12

24. (i) $x_1 = \left(k + \dfrac{1}{2}\right)\pi$; $x_2 = \pm \dfrac{1}{2}$ arc $\cos\left(-\dfrac{1}{4}\right) + k\pi$.

(ii) $x = \dfrac{2k+1}{8}\pi$.

(iii) $x = \dfrac{\pi}{4} + k\dfrac{\pi}{2}$.

(iv) $x = 2n\pi, 2n\pi - \dfrac{\pi}{2}, n \in I$

25. $\lambda \geq 2\sqrt{2}$.

26. $\sqrt{2}$

27. $x = 2k_1\pi$ or $x = 2k_2\pi + 2\tan^{-1}\dfrac{1}{2}, k_1, k_2 \in I$.

28. The equation is solvable if $|a| \leq \sqrt{2}$, and its solutions are $x = \pm\dfrac{1}{2}$ arc cos $(3-2\sqrt{3-a^2}) + k\pi$.

29. (i) $x = \dfrac{2n_1}{3}\pi;\ \dfrac{2n_2}{2}\pi;\ \dfrac{2n_3}{5}\pi$,

(ii) $x \in \left\{\dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}\right\}$

31. $(2m+1)\dfrac{\pi}{2} - \dfrac{5\pi}{24}, m\pi + \dfrac{5\pi}{8}, (4m+1)\dfrac{\pi}{4} + \dfrac{5\pi}{48}, m \in I$

32. (i) $x = (6k + 1)\pi/3$

(ii) $x_1 = 2, x_2 = -5, x_3 = (2\pi/3) + 4k\pi$,

$x_4 = (-2\pi/3) + 4k\pi\ (k \neq 0)$,

$x_5 = 2\arccos\left(\dfrac{\sqrt{7}-2}{2}\right) + 4k\pi$,

$x_6 = -2\arccos[2\sqrt{3} - 3)/2] + 4k\pi(k \neq 0)$

33. (i) $2k\pi < x < 2k\pi + \frac{\pi}{6}$, $2k\pi + \frac{\pi}{6} < x < (2k+1)\pi$

(ii) $k + \frac{1}{4} \le x < k + \frac{1}{2}$, $k - \frac{1}{2} \le x < k$

(iii) $k\pi + \tan{-15} < x < 2k\pi + \pi$

(iv) $k\pi + \frac{\pi}{4} < x \le k\pi + \frac{\pi}{3}$

(v) $k\pi + \frac{\pi}{4} < x < k\,\pi + \frac{\pi}{3}$

Previous Year's Questions (JEE Advanced)

1. 2 2. 4 3. no solution

4. $x \in \left[0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, 2\pi\right] \cup \left\{\frac{\pi}{2}\right\}$ 5. $-\frac{1}{4}, -\frac{1}{4}$

6. $\theta = m\pi, n\pi \pm \frac{\pi}{3}$

7. 4	8. False	9. False	10. False
11. B	12. D	13. B	14. B
15. A	16. C	17. B	18. C
19. D	20. A	21. C	22. B
23. D	24. C	25. B	26. C
27. A	28. D	29. C	30. C
31. A	32. D	33. C	34. D
35. A	36. D	37. C	38. C
39. B	40. C	41. B	42. B
43. D	44. B	45. D	46. A
47. C	48. C	49. D	50. AD
51. ABC	52. BC	53. AC	54. AB
55. 3/2	56. 5/4	57. $a^{-1/2}, a^{-4/3}$	59. 3

63. $[-1, 2) \cup [3, \infty)$

64. $m \in \left(-\infty, \frac{-15}{2}\right) \cup (30, \infty)$

66. $\left(\frac{\pi}{8}, \cos\frac{\pi}{8}\right)\left(\frac{\pi}{4}, \cos\frac{\pi}{4}\right)\left(\frac{3\pi}{8}, \cos\frac{3\pi}{8}\right)$

67. $[-1, 1) \cup (2, 4]$

68. $\{n\pi\} \cup \left\{n\pi + (-1)^n \frac{\pi}{10}\right\} \cup \left\{n\pi + (-1)^n \left(\frac{-3\pi}{10}\right)\right\}$

69. $\left\{\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}\right\}$

70. $\pm 2, \pm\sqrt{2}$

71. $\{a \pm a\sqrt{2}, -a \pm a\sqrt{6}\}$

72. $(-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$

73. $-4, -1 - \sqrt{3}$

74. $\frac{\sqrt{3}-1}{2}$

75. 30°

76. x = 0 or 5/3

77. $\frac{\pi}{2\sqrt{2}}$

78. $\{-1\} \cup [1, \infty)$

81. 4

CHAPTER 2

GRAPHS

2.1 | INTRODUCTION

The graph of a function is a plot of values of the function against the independent variable x. The value of a function changes in accordance with the function rule as x changes. The graph depicts these changes pictorially.

Plotting a graph is an extremely powerful technique to know properties of a function such as domain, range, periodicity, discontinuity and other features which involve derivative of a function.

Subsequently, we shall see that plotting enables us to know these properties more elegantly and easily as compared to other analytical methods.

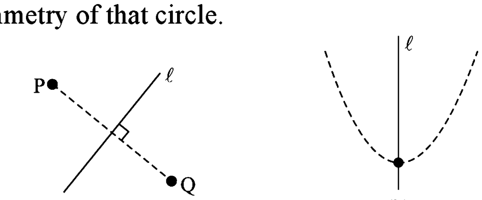
2.2 | SYMMETRY OF GRAPH

Symmetry

We say that two points P and Q are symmetric with respect to a line l if l is the perpendicular bisector of the line segment connecting P and Q.

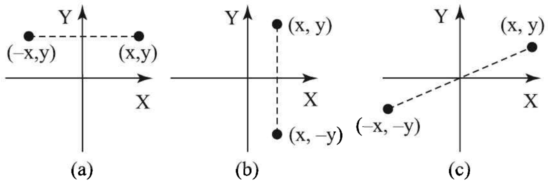
We say that two points P and Q are symmetric with respect to a point B if B is the midpoint of the segment connecting P and Q.

A curve is said to be symmetric with respect to a line l (respectively, point B) if, for any point P on the curve, there is another point Q on the curve such that P and Q are symmetric with respect to l (respectively, B). If a curve is symmetric with respect to a line l, then l is called an axis of symmetry of the curve. For example, any line through the centre of a circle is an axis of symmetry of that circle.



(a) (b)

Points (x,y) and (–x, y) are symmetric with respect to the y-axis, and points (x, y) and (x, –y) are symmetric with respect to the x axis. Points (x,y) and (–x, –y) are symmetric with respect to the origin.



Symmetry of graph of F(x, y) = 0

Consider the graph of an equation $F(x, y) = 0$ in the x-y plane.

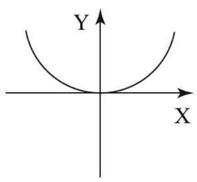
1. The graph of $F(x, y) = 0$ is symmetric about the y-axis if on replacing x by $-x$, the equation of the curve does not change.

i.e. $F(x, y) = 0$ implies $F(-x, y) = 0$.

If F represents a function, then it is said to be even.

For example, the parabola $y = x^2$ is symmetric with respect to the y-axis.

Graph of $y = x^2$

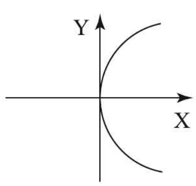


2. The graph of $F(x, y) = 0$ is symmetric about the x-axis if on replacing y by $-y$, the equation of the curve does not change.

i.e. $F(x, y) = 0$ implies $F(x, -y) = 0$.

For example, the parabola $y^2 = x$ is symmetric with respect to the x-axis.

Graph of $y^2 = x$



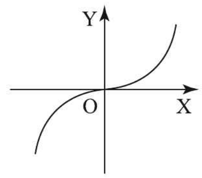
3. The graph of $F(x, y) = 0$ is symmetric about the origin if on replacing x by $-x$ and y by $-y$, the equation of the curve does not change.

i.e. $F(x, y) = 0$ implies $F(-x, -y) = 0$.

If F represents a function, then it is said to be odd.

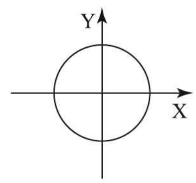
For example, the graph of $y = x^3$ is symmetric with respect to origin.

Graph of $y = x^3$

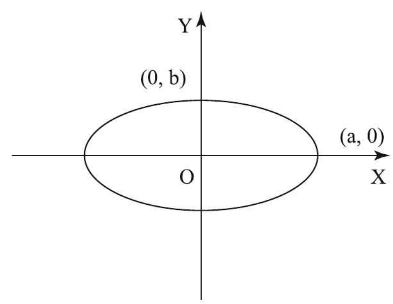


Further, a circle $x^2 + y^2 = r^2$, an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are symmetric with respect to the y-axis, the x-axis and the origin.

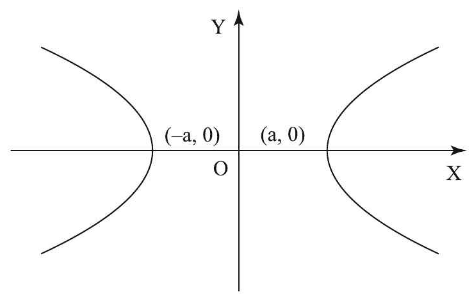
Graph of $x^2 + y^2 = r^2$



Graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

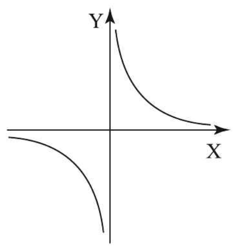


4. The graph of $F(x, y) = 0$ is symmetric about the line $y = x$ if on interchanging x and y, the equation of the curve does not change.

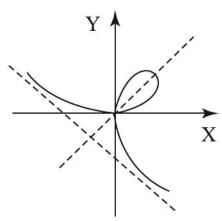
i.e. $F(x, y) = 0$ implies $F(y, x) = 0$.

For example, the graph of $xy = c^2$ is symmetric with respect to the line $y = x$.

Graph of $xy = c^2$



Graph of $x^3 + y^3 = 3axy$, $a > 0$

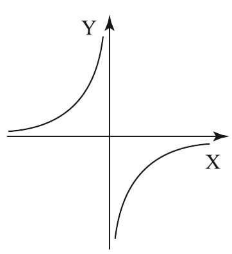


5. The graph of $F(x, y) = 0$ is symmetric about the line $y = -x$ if on replacing x by $-y$ and y by $-x$, the equation of the curve does not change.

i.e. $F(x, y) = 0$ implies $F(-y, -x) = 0$.

For example, the graph of $xy = -c^2$ is symmetric with respect to the line $y = -x$.

Graph of $xy = -c^2$



EXAMPLE 2.1 Investigate which of the following functions have graphs symmetrical about x-axis or about origin.

(i) $a : R \to R$, $a(x) = \dfrac{x^3}{x^2 + 1}$

(ii) $b : R \to R$, $b(x) = \dfrac{|x|}{x^2 + 1}$

(iii) $c : R \to R$, $c(x) = |x| + 2$.

(iv) $d : R \to R$, $d(x) = |x + 2|$.

(v) $f : [-4, 5] \to R$, $f(x) = |x| + 2$.

SOLUTION

(i) $a(-x) = \dfrac{(-x)^3}{(-x)^2 + 1} = -\dfrac{x^3}{x^2 + 1} = -a(x)$.

Hence a is odd and its graph is symmetric about origin.

(ii) $b(-x) = \dfrac{|-x|}{(-x)^2 + 1} = \dfrac{|x|}{x^2 + 1} = b(x)$.

Hence b is even and its graph is symmetric about x-axis.

(iii) $c(-x) = |-x| + 2 = |x| + 2 = c(x)$.

Its graph is symmetric about x-axis.

(iv) $d(-1) = |-1 + 2| = 1$, but $d(1) = 3$. This function is neither even nor odd. Its graph is not symmetric about x-axis or origin.

(v) The domain of f is not symmetric, so its graph is not symmetric about x-axis or origin.

Note

For graphs of algebraic equations, the symmetry is judged as follows:

(a) If all the powers of y in the equation are even, the curve is symmetric about thex- axis.

(b) If all the powers of x are even, the curve is symmetric about the y-axis.

(c) If all powers of x and y are even, the curve is symmetric about the x-axis as well as y-axis.

EXAMPLE 2.2 Find whether the following equations have graphs symmetrical about x-axis, y-axis, line y = x or about origin.

(i) $y = \sqrt{x^2+1} - \sqrt{x^2-1}$ (ii) $y^2 = 8x^2 - x^4$

(iii) $y^2 = (x-1)(x+1)^{-1}$ (iv) $y^3 = x(x^4+1)$

(v) $x^3 + y^3 = 1$.

SOLUTION

(i) $y = \sqrt{x^2+1} - \sqrt{x^2-1}$

Since all the powers of x are even, the curve is symmetric about the y-axis.

(ii) $y^2 = 8x^2 - x^4$.

Since, all powers of x and y are even, the curve is symmetric about the x-axis as well as y-axis. Also, the curve is symmetric with respect to origin.

(iii) $y^2 = (x-1)(x+1)^{-1}$

Since all the powers of y in the equation are even the curve is symmetric about the x- axis.

(iv) $y^3 = x(x^4+1)$

On replacing x by – x and y by – y, the equation of the curve does not change. Hence, the curve is symmetric with respect to origin.

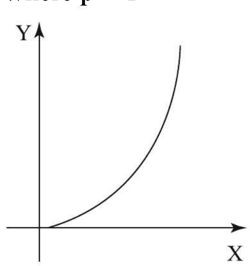
(v) $x^3 + y^3 = 1$

On interchanging x and y, the equation of the curve does not change. Hence, the curve is symmetric with respect to the line y = x.

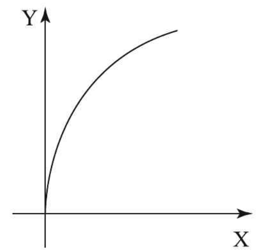
2.3 | GRAPH OF POWER FUNCTIONS

When we draw the graph of $y = x^p$ in the first quadrant we have three kinds of graph :

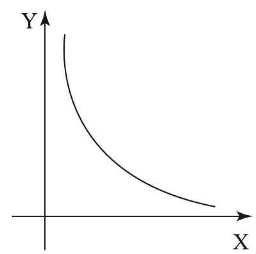
(a) $y = x^p$ where $p > 1$



(b) $y = x^p$ where $0 < p < 1$

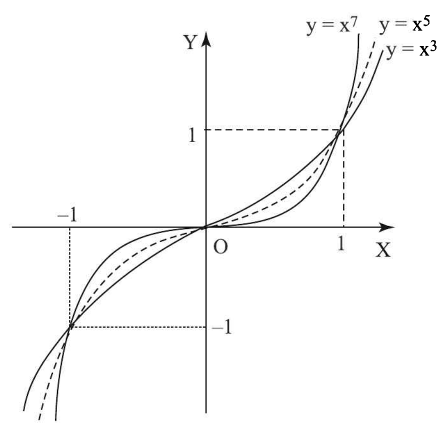


(c) $y = x^p$ where $p < 0$



Further, the graph of $y = x^{m/n}$ is extended in the third quadrant if both m and n are odd, while it is extended in the second quadrant if m is even and n is odd. It is not extended in either second or third quadrant if n is even.

(i) Graph of $f(x) = x^{2n-1}$; $n \in N$



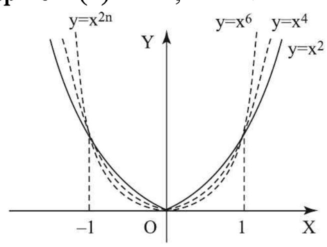
$f(x) = x^{2n+1}$, $n \in N$ is an odd function. The graph is symmetric about origin.

Note

For $x \in (-\infty, -1] \cup [0, 1]$, $\quad x^3 \geq x^5 \geq x^7$

For $x \in (-1, 0) \cup (1, \infty)$, $\quad x^3 < x^5 < x^7$

(ii) Graph of $f(x) = x^{2n}$; $n \in N$



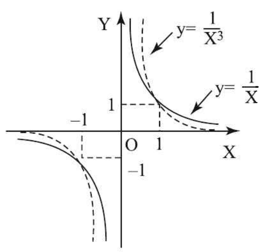
$f(x) = x^{2n}$ is an even function. The graph is symmetric about y-axis.

Note

For $|x| \leq 1$, $x^6 \leq x^4 \leq x^2$

For $|x| > 1$, $x^6 > x^4 > x^2$

(iii) Graph of $f(x) = \dfrac{1}{x^{2n-1}}$; $n \in N$



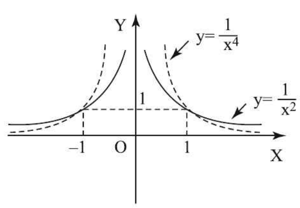
Note

For $0 < x < 1$, $\dfrac{1}{x^3} > \dfrac{1}{x}$

For $x > 1$, $\dfrac{1}{x^3} < \dfrac{1}{x}$.

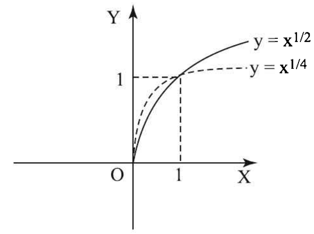
The graph is symmetric about origin.

(iv) Graph of $f(x) = \dfrac{1}{x^{2n}}$; $n \in N$



The graph is symmetric about y-axis.

(v) Graph of $f(x) = x^{1/2n}$; $n \in N$



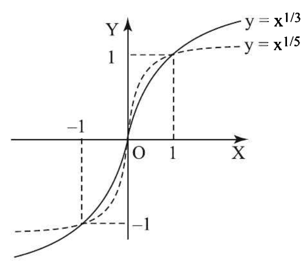
Note

For $0 < x < 1$, $\quad x^{1/2} > x^{1/4}$

For $x > 1$, $\quad x^{1/2} < x^{1/4}$

Domain is $x \in [0, \infty)$.

(vi) Graph of $f(x) = x^{1/2n-1}$, when $n \in N$



Note

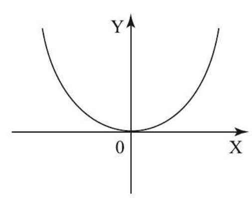
For $x \in (-\infty, -1) \cup (0, 1)$, $\quad x^{1/3} > x^{1/5}$

For $x \in (-1, 0) \cup (1, \infty)$, $\quad x^{1/3} < x^{1/5}$

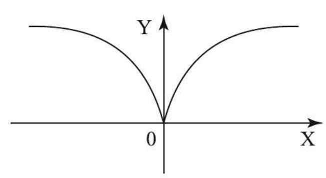
The graph is symmetric about origin.

Now consider some examples :

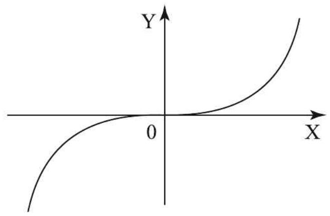
(i) $y = x^{4/3}$



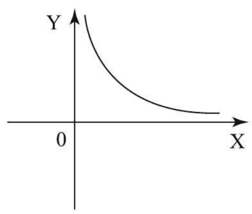
(ii) $y = x^{2/3}$



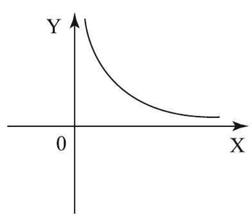
(iii) $y = x^{5/3}$



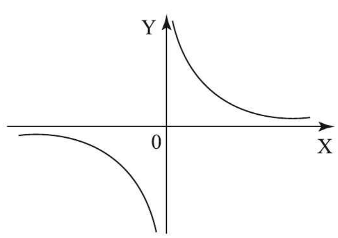
(iv) $y = \frac{1}{\sqrt{x}} = x^{-1/2}$



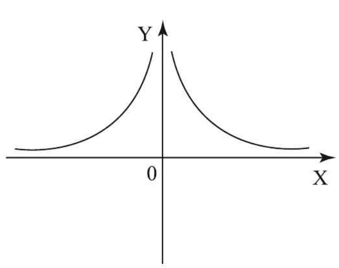
(v) $y = x^{-3/2}$



(vi) $y = x^{-3/5}$



(vii) $y = x^{-2/3}$



EXAMPLE 2.1 A function $y = f(x)$ is defined parametrically as: $y = t^2 + t|t|$, $x = 2t - |t|$, $t \in R$. Find the number of solutions of the equation

$(f(x))^3 = x$.

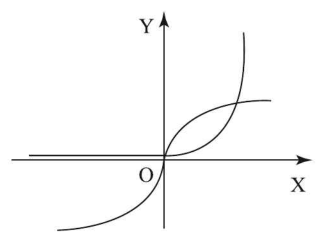
SOLUTION If $t \ge 0$, $x = t$, $y = 2t^2$

$\Rightarrow y = 2x^2 \; \forall \; x \ge 0$

If $t < 0$, $x = 3t$, $y = 0 \Rightarrow y = 0$, $x < 0$

Hence, $f(x) = \begin{cases} x^2, & x \ge 0 \\ 0, & x < 0 \end{cases}$

We express the given equation as $f(x) = x^{1/3}$ and draw the graphs of $y = f(x)$ and $y = x^{1/3}$.

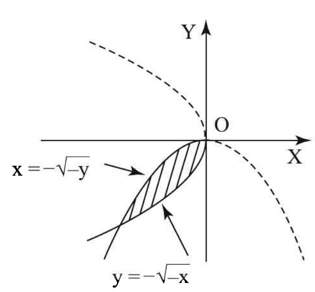


We can see that there are two solutions.

EXAMPLE 2.2 Sketch the area bounded by the curves $y = -\sqrt{-x}$ and $x = -\sqrt{-y}$ where $x, y \le 0$

SOLUTION $y = -\sqrt{-x} \Rightarrow y^2 = -x$, where x & y are both negative.

$x = -\sqrt{-y} \Rightarrow x^2 = -y$ where x & y are both negative.



PRACTICE PROBLEMS [A]

1. Draw the graph of the following functions on the same axes :

   (i) $y = x^{1/2}$; $y = x^{3/4}$ (ii) $y = x$; $y = x^{1/4}$; $y = x^{1/5}$

   (iii) $y = \frac{1}{x}$; $y = \frac{1}{x^2}$; $y = \frac{1}{x^3}$

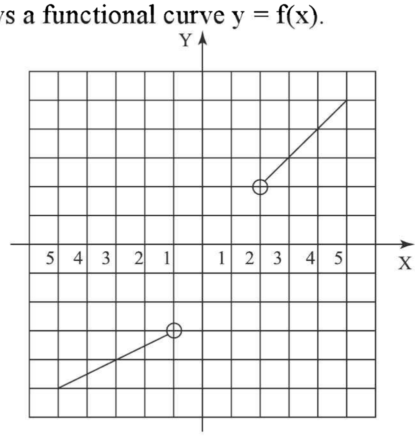
2. Draw the graph of the following functions on the same axes:

   (i) $y = x^{-1/2}$; $y = {}^{1/x}$ (ii) $y = x^{5/3}$; $y = x^{3/2}$

3. Draw the graph of the following functions :

   (i) $f(x) = \begin{cases} x^3 , & x^2 < 1 \\ x , & x^2 \ge 1 \end{cases}$ (ii) $f(x) = \begin{bmatrix} x^{3/5} & \text{if } x \le 1 \\ -(x-2)^3 & \text{if } x > 1 \end{bmatrix}$

4. The following figure shows a functional curve $y = f(x)$.

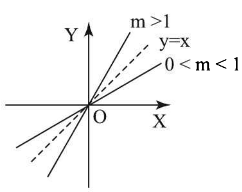


Find (i) f(3) (ii) the domain of the function (iii) the range of the function.

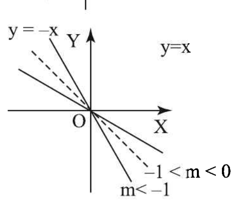
2.4 | GRAPH OF GEOMETRICAL CURVES

Graph of lines y = mx

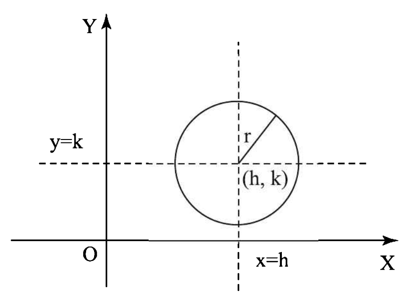
(i)



(ii)

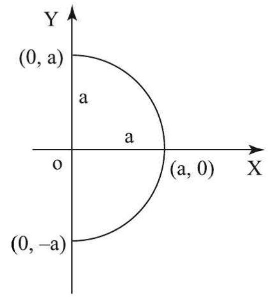


Graph of circle $(x - h)^2 + (y - k)^2 = r^2$

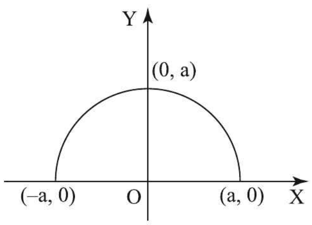


Graph of semi-circles

(i) $x = \sqrt{a^2 - y^2}$

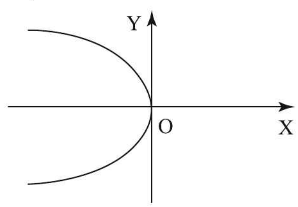


(ii) $y = \sqrt{a^2 - x^2}$

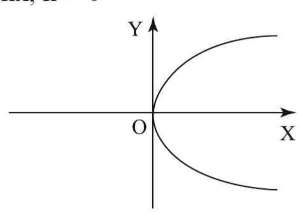


Graph of parabolas

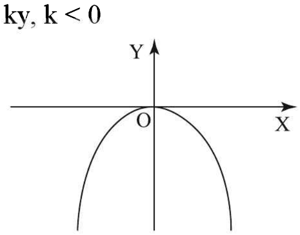
(i) $y^2 = kx,\ k < 0$



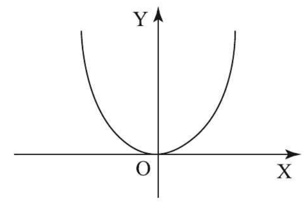
(ii) $y^2 = kx,\ k > 0$



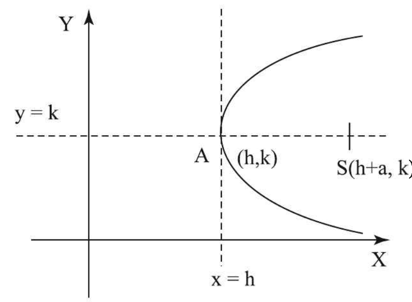
(iii) $x^2 = ky,\ k < 0$



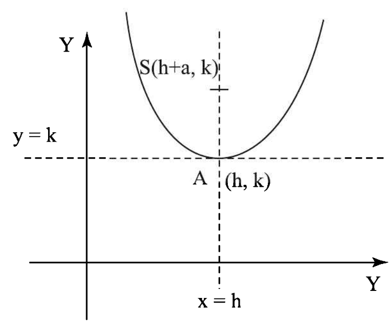
(iv) $x^2 = kx,\ k > 0$



(v) $(y - k)^2 = 4a(x - h),\ a > 0$

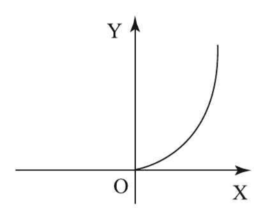


(vi) $(x - h)^2 = 4a(y - k),\ a > 0$

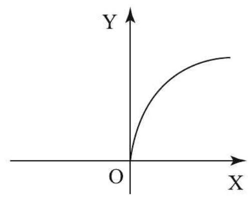


Graphs of $y = k\sqrt{x}$ and $x = k\sqrt{y}$

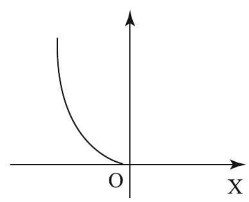
(i) $x = k\sqrt{y},\ k > 0$



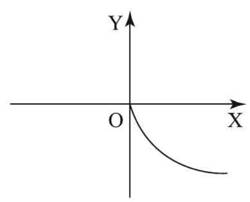
(ii) $y = k\sqrt{x}$, $k > 0$



(iii) $x = k\sqrt{y}$, $k < 0$

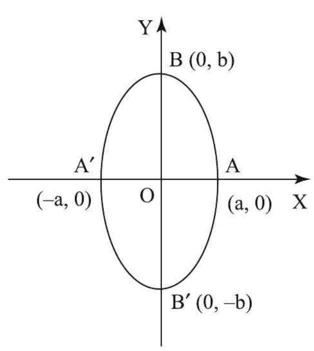


(iv) $y = k\sqrt{x}$, $k < 0$

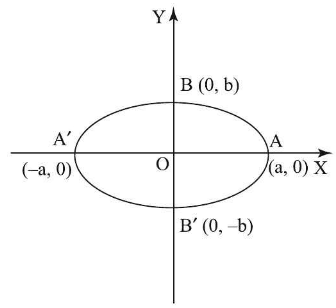


Graph of ellipses

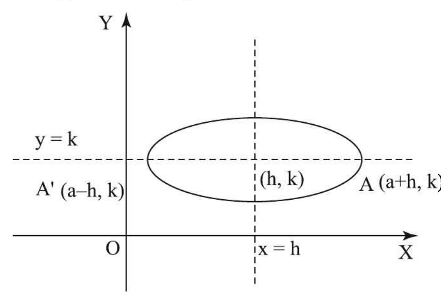
(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\quad a < b$



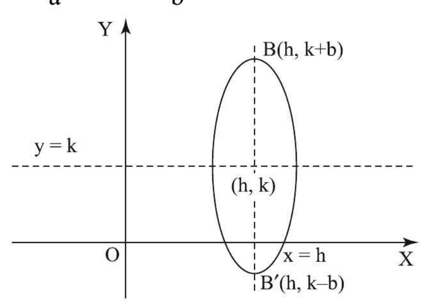
(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\quad a > b$



(iii) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b$

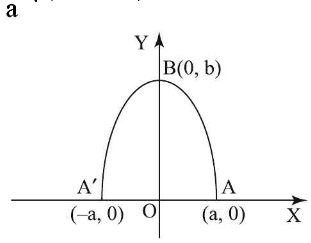


(iv) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a < b$

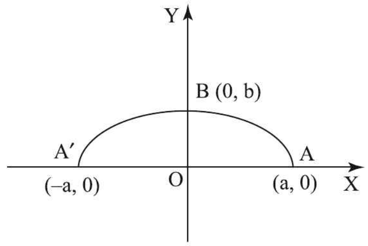


Graph of semi-ellipses

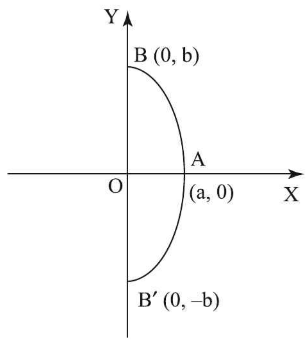
(i) $y = \frac{b}{a}\sqrt{(a^2 - x^2)}$, $a < b$



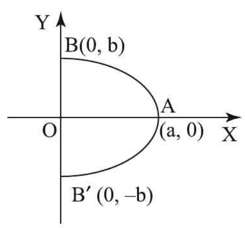
(ii) $y = \frac{b}{a}\sqrt{(a^2 - x^2)}$, $a > b$



(iii) $x = \frac{a}{b}\sqrt{(b^2 - y^2)}$, $a < b$

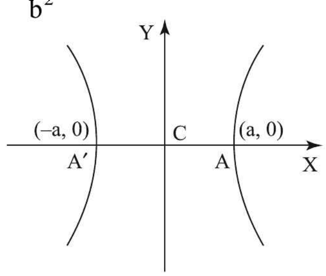


(iv) $x = \frac{a}{b}\sqrt{(b^2 - y^2)}$, $a > b$

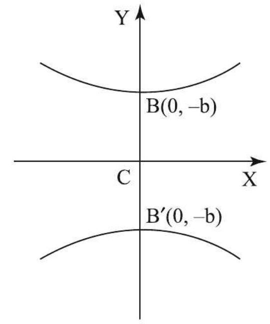


Graph of hyperbolas

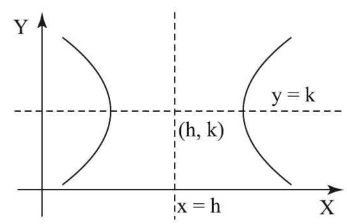
(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



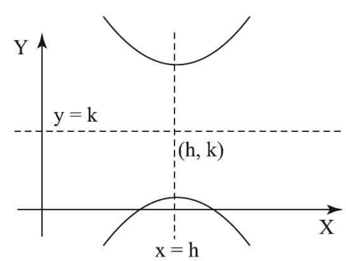
(ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$



(iii) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$



(iv) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$



2.5 | TRANSFORMATION OF GRAPHS

Transformation of graphs means changing graphs. This generally allows us to draw graphs of more complicated functions from graphs of basic or simpler functions by applying different transformation techniques.

Many functions in applications are built up from simple functions by inserting constants in various places. It is important to understand the effect such constants have on the appearance of the graph.

Graphing of such a function involves modifying graph of a core function. We modify core function and its graph, applying various mathematical operations on the core function.

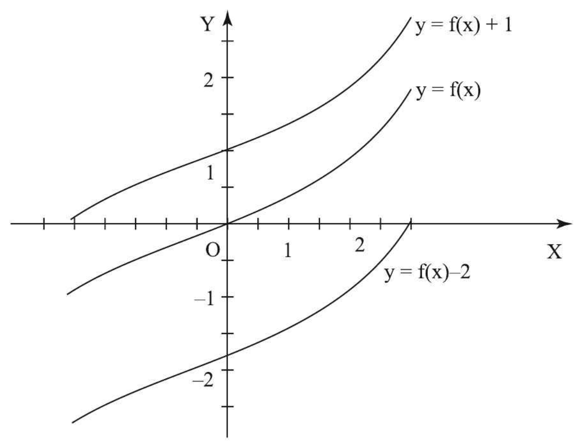
Effect of arithmetic operations

Addition/subtraction operation on independent variable results in shifting of core graph along x-axis i.e. horizontally. Similarly, product/division operations results in scaling (shrinking or stretching) of core graph horizontally. The change in graphs due to negation is reflected as mirroring (across y-axis) horizontally. Clearly, modifications resulting from modification to input modifies core graph horizontally. Another important aspect of these modification is that changes takes place opposite to that of operation on independent variable. For example, when "2" is added to independent variable, then core graph shifts left which is opposite to the direction of increasing x. A multiplication by 2 shrinks the graph horizontally by a factor 2, whereas division by 2 stretches the graph by a factor of 2. On the other hand, modification in the output of function is reflected in change in graphs along y-axis i.e. vertically. Effects such as shifting, scaling (shrinking or stretching) or mirroring across x-axis takes place in vertical direction. Also, the effect of modification in output is in the direction of modification as against effects due to modifications to input. A multiplication of function by a positive constant greater than 1, for example, stretches the graph in y-direction as expected. These aspects will be clear as we study each of the modifications mentioned here.

1. Graph of y = f(x) + a

Addition and subtraction operation with function

In order to understand this type of transformation, we need to explore how output of the function changes as we add constant value to the output. If we add 1 unit to the function, then each value of function is increased by 1 unit. In notation, we would say that the graph of "f(x) + 1" is same as the graph of f(x), which has been moved up by 1 unit. Alternatively, we can also pull down the x-axis by 1 unit, by saying that vertical reference of measurement has been moved down by 1 unit.



Similarly, if we subtract 1 unit from the function, then each value of function is decreased by 1 unit. In notation, we would say that the graph of "f(x) – 1" is same as the graph of f(x), which has been moved down by 1 unit. Alternatively, we can also describe this transformation by saying that vertical reference of measurement i.e. x-axis has moved up by 1 unit.

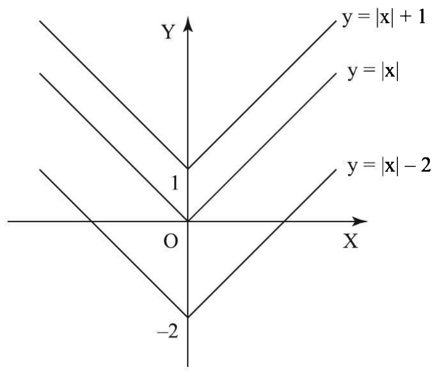
We conclude :

The plot of $y = f(x) + |a|$, $|a| > 0$ is the plot of $y = f(x)$ shifted up by $|a|$ units.

The plot of $y = f(x) - |a|$, $|a| > 0$ is the plot of $y = f(x)$ shifted down by $|a|$ units.

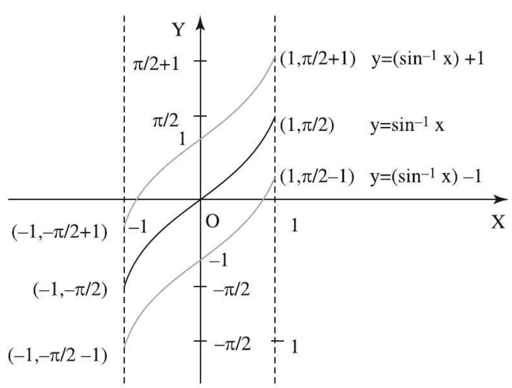
We use these facts to draw the plot of transformed function f(x) ± |a| by shifting the plot f(x) by |a| unit along y-axis. Each point forming the plot is shifted parallel to y-axis. In the figure below, the plot depicts modulus function y= |x|. When it is shifted "1" unit up then the function representing the shifted plot is y= |x| + 1.

Note that corner of plot at x = 0 is also shifted by 1 unit along y-axis. Further, when the plot is shifted "2" units down then the function representing the shifted plot is |x| – 2. In this case, the corner of the plot is shifted by 2 units down along y-axis.



EXAMPLE 2.1 Plot $y = \sin^{-1} x$ and use transformation to draw the graphs of $y = (\sin^{-1} x) + 1$ and $y = (\sin^{-1} x) – 1$.

SOLUTION



Note

If we replace y by y – a, then the graph moves up "a" units. (If "a" is negative, then this means that the graph moves down |a| units.) If the formula is written in the form y = f(x) and if y is replaced by y – a to get y – a = f(x), we can equivalently move "a" to the other side of the equation and write y = f(x) + a. Thus, this principal can be stated as : to get the graph of y = f(x) + a, take the graph of y = f(x) and move it "a" units up.

2. Graph of y = f(x + a)

Addition and subtraction to independent variable leads to horizontal shifts.

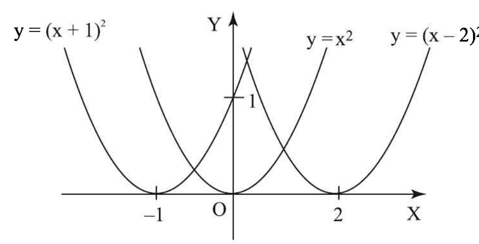
The plot of y= f(x – |a|), |a| > 0 is the plot of y = f(x) shifted to the right by |a| units.

The plot of y= f(x + |a|), |a| > 0 is the plot of y = f(x) shifted to the left by |a| units.

We use these facts to draw the plot of transformed function f(x ± |a|) by shifting the plot f(x) by |a| unit along x-axis.

For example, the graph of $y = (x – 2)^2$ is the parabola $y = x^2$ shifted to the right so as to have its vertex at the point (2, 0) on the x-axis. Alternatively, we can also shift the y-axis to the left by 2 units, by saying that horizontal reference of measurement has been moved to the left by 2 units.

The graph of $y = (x + 1)^2$ is the parabola shifted to the left so as to have its vertex at (–1, 0) on the x-axis.



Note

Each element of the graph is shifted horizontally by the same value.

Also, the function $y = x^2 – 4x = (x – 2)^2 – 4$ can be obtained from $y = (x – 2)^2$ by moving the graph 4 units down. The result is the x^2–parabola shifted 2 units to the right and 4 units down so as to have its vertex at the point (2, – 4).

CAUTION Do not confuse $f(x) + a$ and $f(x + a)$. For example, if $f(x)$ is the function x^2, then $f(x) + 2$ is the function $x^2 + 2$, while $f(x + 2)$ is the function $(x + 2)^2 = x^2 + 4x + 4$.

EXAMPLE 2.2 Compare the graph of each of the following with the graph of $f(x) = 3^x$. Identify the domain and range of each function :

(i) $g(x) = 3^{x+1}$

(ii) $h(x) = 3^x - 2$

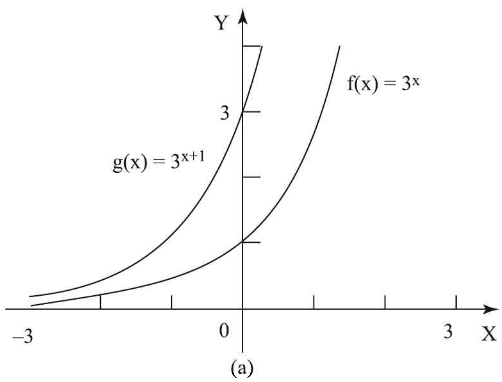
(iii) $k(x) = -3^x$

SOLUTION

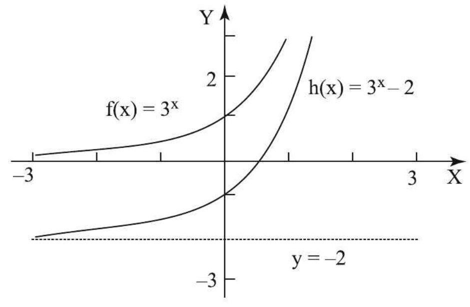
(i) Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the left, as shown in figure (a). The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

(ii) Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f down two units, as shown in Figure (b). The domain is $(-\infty, \infty)$ and the range is $(-2, \infty)$.

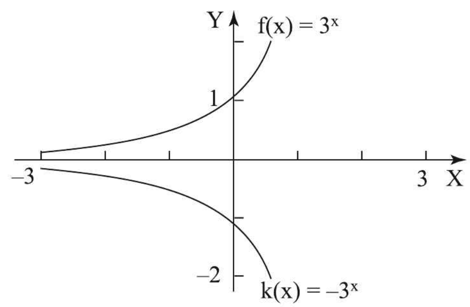
(iii) Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by reflecting the graph of f in the x-axis, as shown in Figure (c). The domain is $(-\infty, \infty)$ and the range is $(-\infty, 0)$.



(a)



(b)



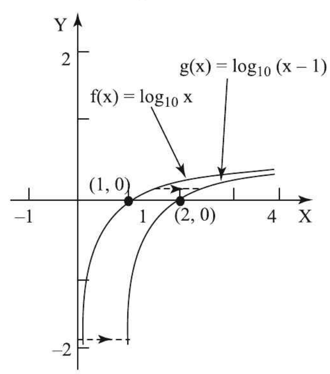
(c)

EXAMPLE 2.3 Compare the graphs of the following functions with the graph of $f(x) = \log_{10} x$.

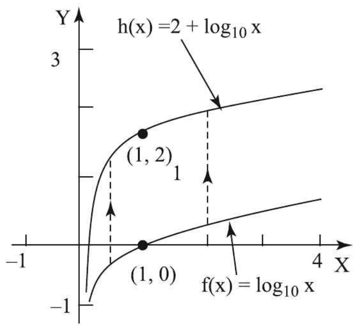
(i) $g(x) = \log_{10}(x - 1)$

(ii) $h(x) = 2 + \log_{10} x$

SOLUTION The graph of each of these functions is shown in the figure.



(a) Right shift of 1 unit



(b) Upward shift of 2 units

(i) Because $g(x) = \log_{10}(x-1) = f(x-1)$, the graph of g can be obtained by shifting the graph of f one unit to the right.

(ii) Because $h(x) = 2 + \log_{10}x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units up.

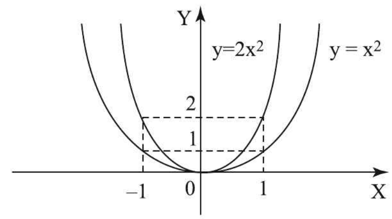
3. Graph of y = af(x)

Multiplication and division of function

Multiplication and division of a function by a constant scales (stretches or shrinks) the core graph in the vertical i.e. y-direction. The modification due to either of these two arithmetic operations has no scaling impact in the x-direction. If we multiply output of the function by a positive constant greater than 1, then graph of core function is stretched vertically by the factor, which is equal to the constant being multiplied. The magnification of graph i.e. stretching in y-direction is more noticeable in nonlinear graphs like sine and cosine graphs, whose values are bounded in the interval [–1, 1].

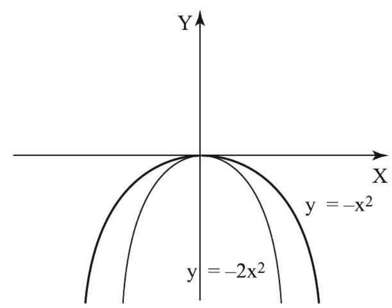
An important thing to note about vertical scaling (shrinking or stretching) is that the behaviour of graph at y = 0 remains unchanged. It means that the x-intercept of the graph remains same and is not affected by scaling resulting from multiplication or division of a function. Also, the domain of the function remains same while the range may change.

Let us now compare the functions $y = 2x^2$ and $y = x^2$. For the same value of x, the function $y = 2x^2$ is twice the value of the function $y = x^2$. Consequently, the graph of $y = 2x^2$ is obtained by stretching the graph of $y = x^2$, two times along the y–axis.

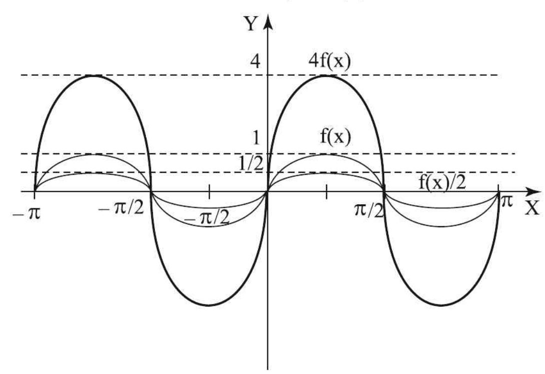


In general, the graph of $y = ax^2$ for $a > 0$ can be obtained by stretching the parabola $y = x^2$ a times along y axis (more precisely, by stretching for $a > 1$ and by compressing for $0 < a < 1$).

Similarly, the graph of $y = ax^2$ for $a < 0$ can be obtained by stretching the parabola $y = -x^2$, |a| times along y-axis.



Let us consider function, $y = f(x) = 4 \sin x$



The amplitude of the function "4sinx" is 4 times that of the core graph "sinx". In the same fashion,

a division by a positive constant greater than 1 results in shrinking of the core graph by the factor, which is equal to constant being multiplied. Let us consider division of function :

$$y = \frac{1}{2}\sin x$$

The amplitude of the graph "sin x" changes from 1 to 1/2 in the graph of "1/2 sin x".

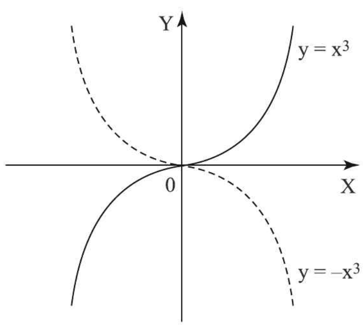
Graph of y = – f(x)

Negation of function

What would happen if we negate output of a function? The answer is easy. All positive values will turn negative and all negative values will turn positive. It means that the graph of the core function which is being negated will be swapped across x-axis in the transformation. The graph of "–f(x)", therefore, is the mirror image of the graph f(x), in the x-axis. In other words, we would need to flip the graph of f(x) across x-axis to draw the graph "– f(x)".

Let us compare the function $y = -x^3$ and $y = x^3$. For the same value of x, the values of these functions are equal in magnitude but opposite in sign.

Consequently, the graph of the function $y = -x^3$. can be obtained by reflection of $y = x^3$ about the x-axis.

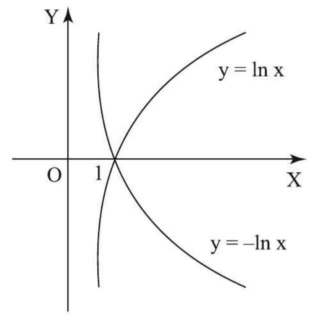


EXAMPLE 2.4 Draw the graph of $y = \ln\frac{1}{x}$.

SOLUTION We simplify the given function as :

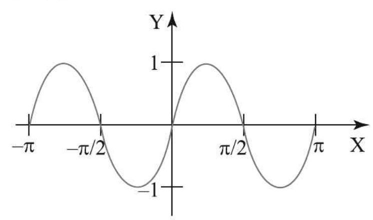
$$y = \ln\frac{1}{x} = -\ln x$$

Here, the core function is f (x) = ln(x). Clearly, the given function is a transformed function of the type y = –f(x). We obtain its graph by taking mirror image of the graph of y = f(x) about x-axis.



EXAMPLE 2.5 Graph f(x) = sin 2x.

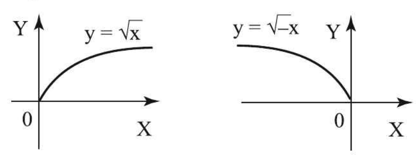
SOLUTION y = f(2x) is a scaling of the graph of y = f(x) in the direction of x-axis, in which the x-coordinate is halved. Since $-1 \le \sin 2x \le 1$, the amplitude of sin 2x is $\frac{1-(-1)}{2} = 1$. The period of sin 2x is $2\pi/2 = \pi$. The graph of f(x) = sin 2x is shown below.



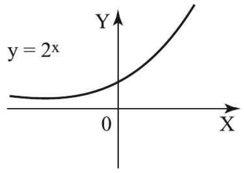
Graph of y = f(–x)
Negation of independent variable

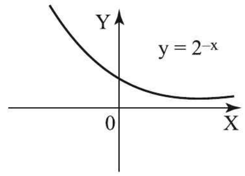
The graph of y = f(–x) can be obtained by taking mirror image of the graph of y = f(x) in y-axis.

For example, the function $y = \sqrt{-x}$, which has domain $x \le 0$, is obtained by taking the graph of $\sqrt{x}$ and flipping it around the y-axis into the second quadrant.



We reflect the graph of $y = 2^x$ about y-axis to get the graph of $y = 2^{-x}$.



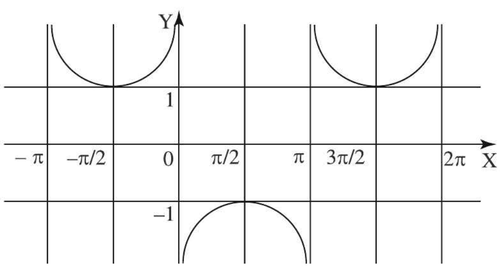


While using this transformation, we should know about even function. For even function $f(x) = f(-x)$.

As such, this transformation will not have any implication for even functions as they are already symmetric about y-axis. It means that two parts of the graph of even function across y-axis are images of each other. For this reason, $y = \cos(-x) = \cos(x)$, $y = |-x| = |x|$ etc. The graphs of these even functions are not affected by change in sign of independent variable.

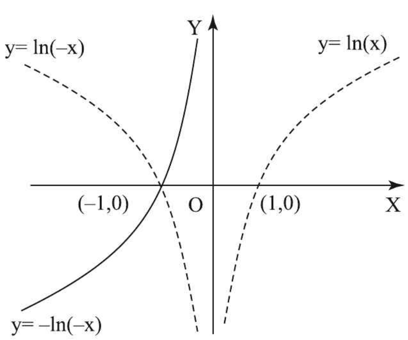
EXAMPLE 2.6 Draw the graph of $y = \text{cosec}\,(-x)$.

SOLUTION The plot is obtained by plotting image of core graph $y = \text{cosec}\, x$ in the y axis.



EXAMPLE 2.7 Draw the graph of $y = -ln(-x)$.

SOLUTION



Note

If x is replaced by x/a in a formula and a > 1, then the effect on the graph is to expand it by a factor of "a" in the x-direction (away from the y-axis). If a is between 0 and 1 then the effect on the graph is to contract by a factor of 1/a (towards the y-axis).

For example, replacing x by $x/0.5 = 2x$ has the effect of contracting toward the y-axis by a factor of 2. If a is negative, we expand by a factor of $|a|$ and then flip about the y-axis. Thus, replacing x by $-x$ has the effect of taking the mirror image of the graph with respect to the y-axis.

Certain function are derived from the core function as a result of multiple arithmetic operations on independent variable. Consider an example:

$$f(x) = -2x - 2$$

We can consider this as a graph determined by applying the following operations on $f(x) = x$:

(i) $f(2x)$ i.e. shrink the graph horizontally by half.

(ii) $f(-2x)$ i.e. flip the graph across y-axis.

(iii) $f(-2x - 2)$ i.e. subtract 2 from $-2x$ i.e. shift the graph to the right by 2 units.

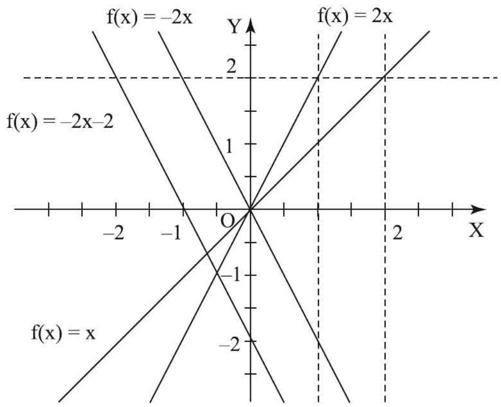
This sequence of operation is not correct for the reason that the third operation is a subtraction operation to $-2x$ not to independent variable x, whereas we have defined transformation for subtraction from the independent variable.

CAUTION The horizontal shift for the function $f(bx + c)$ w.r.t. $f(bx)$ is not "c". We rearrange the argument of the function,

$$f(bx + c) = f\left\{b\left(x + \frac{c}{b}\right)\right\}$$

Hence, the horizontal shift is c/b.

Let us rework the third transformation step in the example : $f(-2x - 2) = f(-2(x + 1))$ i.e. shift the graph of $f(-2x)$ to the left by 1 unit.



Alternative:

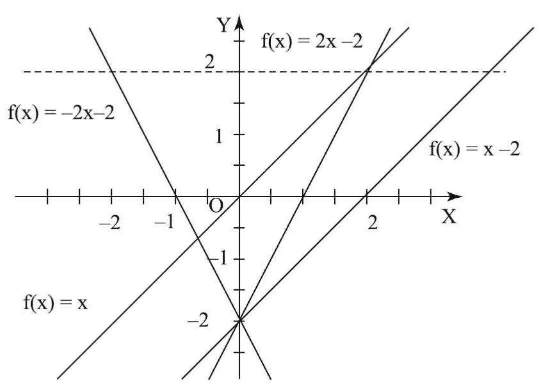
We can decompose a given function in more than one way so long as the transformations are valid.

(i) f(x–2) i.e. subtract 2 from independent variable x i.e. shift the graph right by 2 units.

(ii) f(2x–2) i.e. multiply independent variable x by 2 i.e. shrink the graph horizontally by half.

(iii) f(–2x–2) i.e. negate independent variable i.e. flip the graph across y-axis.

The resulting graph is shown in the figure below:



It is important the way graph is shrunk horizontally towards origin. Important thing is to ensure that y-intercept is not changed. It can be seen that function before being shrunk is $f(x) = x - 2$.
Its y-intercept is 2. When the graph is shrunk by a factor of 2, the function is $f(x) = 2x - 2$.

The y-intercept is again 2. The graph moves 1 unit half of x-intercept towards origin. Further, we can verify validity of points like x and y intercepts to ensure that transformation steps are indeed correct. Here,

$$x = 0,\ y = -2\,.\,0 - 2 = -2$$

$$y = 0,\ x = -\frac{(y+2)}{2} = -\frac{2}{2} = -1.$$

Combined input and output operations
The combined input and output is symbolically represented as:

$$y = af(bx + c) + d, \quad a, b, c, d \in R$$

Carrying out output operation before input operation does not make sense. There will be two different outputs which are not connected to each other. Hence, the logical order is that we first carry out input operations then follow it with output operations.

EXAMPLE 2.8 Draw $y - 1 = \ln(x - 2)$

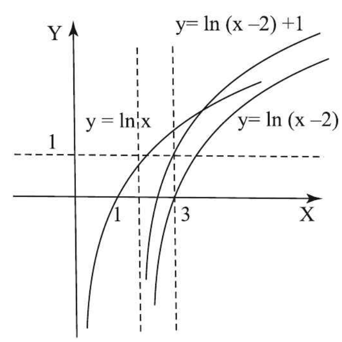
SOLUTION We rewrite the function :

$$y = \ln(x - 2) + 1$$

In order to plot this function, we plot the graph of the core function $y = \ln x$.

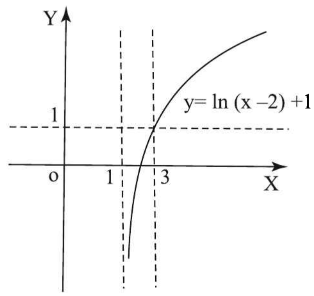
Now, the plot of $y = \ln(x - 2)$ is plot of $y = \ln x$ shifted to the right by 2 units.

The plot of $y = \ln(x - 2) + 1$ is plot of $y = \ln(x - 1)$ shifted up by 1 unit.



There is yet another alternative to obtain graph of the transformed function by shifting the axes themselves instead of the plot. The transformation is affected by shifting the axes in the

direction opposite to that required for the graph. In the above example, we need to move y-axis by 2 units towards left and move x-axis by 1 unit downwards.

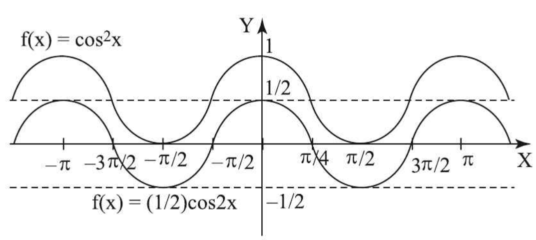


EXAMPLE 2.9 Draw the plot $y = \cos^2 x$.

SOLUTION We know that

$$y = \cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{\cos 2x}{2}$$

Here, the core graph is y = cos x . Multiplying independent variable by 2 shrinks the core graph horizontally. As a result its period is reduced from 2π to π. Division of cos 2x by 2 is division operation on function. This operation shrinks the graph cos2x by 2 vertically. Note that amplitude of graph is reduced to 1/2 due to this operation. In the figure, lower graph corresponds to (cos 2x)/2. Once we draw the graph of (cos 2x)/2, we draw the given function $y = \cos^2 x$ by shifting the graph of (cos 2x)/2 by 1/2 units up.

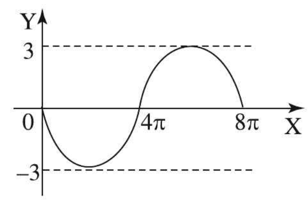


EXAMPLE 2.10 Draw the graph $f(x) = -3\sin\frac{x}{4}$. What is the amplitude, period, and where is the first positive real zero of this function?

SOLUTION Since $-3 \le -3\sin x \le 3$, the amplitude is $\frac{3-(-3)}{2} = 3$. The period is $2\pi / (1/4) = 8\pi$, and the first positive zero occurs when

$\frac{x}{4} = \pi$, i.e. at $x = 4\pi$.

A portion of the graph is shown below.

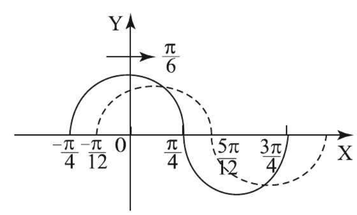


EXAMPLE 2.11 Graph $y = \cos(2x - \pi/3)$.

SOLUTION The sequence of operations are :

$y = \cos x$
$y = \cos 2x$
$y = \cos(2x - \pi/3) = \cos 2(x - \pi/6)$

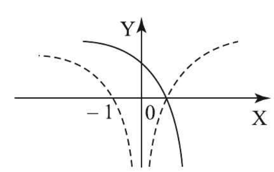
Note that we shift the graph by $\pi/6$ units to the right.



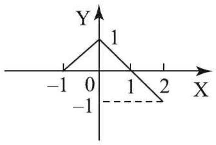
EXAMPLE 2.12 Graph $y = \log_2(-x + 2)$.

SOLUTION The sequence of operations are:

$y = \log_2 x$
$y = \log_2(-x)$
$y = \log_2(-x + 2)$
$y = \log_2(-(x - 2))$



EXAMPLE 2.13 Given the graph of y = f(x) as



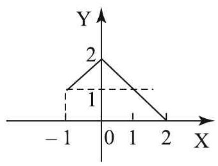
draw the graph of the following functions :

(i) y = f(x) + 1 (ii) y = f(x + 1)

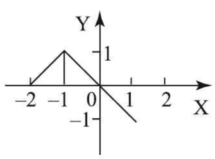
(iii) y = f(2x) (iv) y = 3 – 1/2 f(x)

SOLUTION

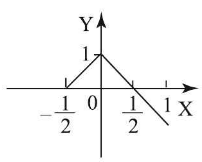
(i) We raise the graph vertically by 1 unit.



(ii) We shift the graph to the left by 1 unit.

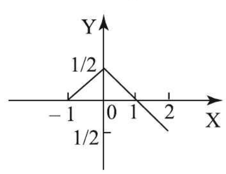


(iii) We compress the graph by a factor of 2 along x-axis.

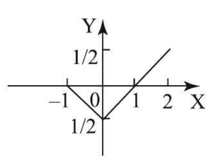


(iv) We follow a sequence of steps:

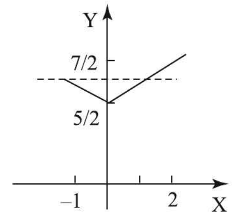
y = 1/2 f(x)



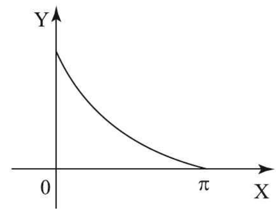
y = – 1/2f(x)



y = – 1/2f(x) + 3



EXAMPLE 2.14 The graph



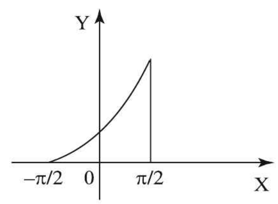
shows the plot of y = f(sin x). Plot the curve y = f(cos x).

SOLUTION We have

$$y = f(\cos x) = f\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}$$

$$= f\left\{\sin\left[-\left(x - \frac{\pi}{2}\right)\right]\right\}$$

Thus, the graph of f(cos x) is drawn by first taking the mirror image of the graph of f(sin x) in the y-axis and then translating it to the right by $\frac{\pi}{2}$ units along the x-axis, as shown below.



EXAMPLE 2.15 Let f : R – {0} → R with

$$f(x) = x + \frac{2}{x} - 1.$$

The curve y = f(x) experiences the following successive transformations:

(i) A reflection about the x-axis.

(ii) A translation 3 units left.

(iii) A reflection about the y-axis.

(iv) A vertical stretch by a factor of 2.

Find the equation of the resulting curve. Note also how the domain of the function is affected by these transformations.

SOLUTION

(i) A reflection about the x-axis gives the curve

$$y = -f(x) = 1 - \frac{2}{x} - x = a(x), \text{ say,}$$

with domain $D_a = R - \{0\}$.

(ii) A translation 3 units left gives the curve

$$y = a(x+3) = 1 - \frac{2}{x+3} - (x+3)$$

$$= -2 - \frac{2}{x+3} - x = b(x) \text{ say,}$$

with $D_b = R - \{-3\}$.

(iii) A reflection about the y-axis gives the curve

$$y = b(-x) = -2 - \frac{2}{x+3} + x = c(x), \text{ say,}$$

with $D_c = R - \{-3\}$.

(iv) A vertical stretch by a factor of 2 gives the curve

$$y = 2c(x) = -4 + \frac{4}{x+3} + 2x = d(x), \text{ say,}$$

with $D_d = R - \{-3\}$.

Notice that the resulting curve is

$$y = d(x) = 2c(x) = 2b(-x) = 2a(-x+3)$$

$$= -2f(-x+3).$$

PRACTICE PROBLEMS [B]

1. Draw the graphs of the following functions:

   (i) $y = (x-1)^2$ (ii) $y = 2(x-1)^2$

   (iii) $y = 2(x-1)^2 - 1$

2. Draw the graph of the following functions:

   (i) $y = \sqrt{x} - 2$ (ii) $y = \sqrt{x-2}$; $y = \sqrt{x+1}$

   (iii) $y = \sqrt[3]{x+1}$

3. Draw the graph of the following functions:

   (i) $f(x) = 2 \sin x$ (ii) $f(x) = \sin 4x$

   (iii) $f(x) = \ln(3x+4)$ (iv) $f(x) = \log_2(-3x)$

4. Plot $y = \sin^{-1}x$ and use transformation to draw the graphs of $y = \sin^{-1}(x-1)$ and $y = \sin^{-1}(x+1)$.

5. Construct the graph of the following functions :

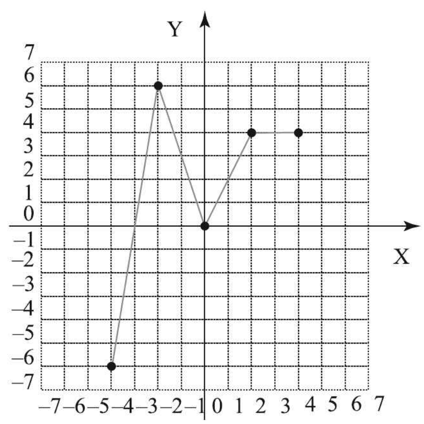
   (i) $y = \dfrac{1}{1-x}$ (ii) $y = \dfrac{1}{2+x}$

   (iii) $y = \dfrac{1}{x+1} - \dfrac{1}{2}$

6. Draw the graph of the following functions:

   (i) $y = \dfrac{x+3}{x-1}$ (ii) $y = \dfrac{2x+1}{x-1}$

7. Starting with the graph of $y = \sqrt{x}$, sketch the graph of each of the following functions:
   (i) $f(x) = \sqrt{x-2}$ (ii) $f(x) = 4 + \sqrt{x+2}$
   (iii) $f(x) = -\sqrt{-x}$ (iv) $f(x) = -4 + \sqrt{1-(x-1)}$
8. Starting with the graph of $y = 1/x$, and the graph of $y = \sqrt{1-x^2}$, sketch the graph of each of the following functions:
   (i) $f(x) = -1 - 1/(x + 2)$ (ii) $f(x) = 2 + \sqrt{1-(x-1)^2}$
   (iii) $f(x) = 2\sqrt{1-(x/3)^2}$ (iv) $f(x) = 4 + 2\sqrt{1-(x-5)^2/9}$
9. Draw the graph of the following functions:
   (ii) $y = \log_2(2 - x)$ (ii) $y = \log_{1/2}(1 - x)$
   (iii) $y = \log_x \sqrt{x}$ (iv) $y = 3 \,.\, 2^x - 2$
10. Draw the graph of the following functions:
   (i) $y = \sin x + \cos x$ (ii) $y = \sin^2 x$
11. Draw the graph of the following functions:
   (i) $y = \begin{cases} 4^x - 1, & \text{if } x < 0 \\ \sqrt{4x - x^2}, & \text{if } x \geq 0 \end{cases}$ (ii) $y = \begin{cases} 1 - \sqrt{1-x^2}, & \text{if } x \leq 1 \\ 1 + \log_{1/2} x, & \text{if } x > 1 \end{cases}$
12. Given the graph of $y = f(x)$, draw the graphs of
   (i) $y = f(x)/2$ (ii) $y = f(2x)$.



13. Find an equation of the new curve C_1 that is obtained when the graph of the curve C with the equation $x^2 - 3xy + 2y^2 = 1$ is reflected in:
   (a) the x-axis (b) the y-axis
   (c) the origin.
14. The graphs of the following curves suffer the following successive transformations :
   (i) a vertical translation of 2 units down,

(ii) a reflection about the y-axis, and finally,

(iii) a horizontal translation of 1 unit to the left.

Find the resulting equations after all the transformations have been exerted.

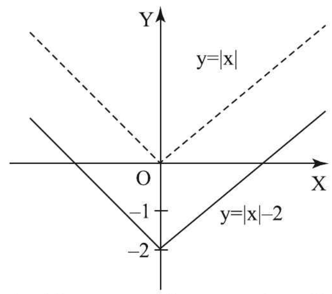
(a) $y = x(1 - x)$ (b) $y = 2x - 3$ (c) $y = |x + 2| + 1$

2.6 | GRAPHS OF FUNCTIONS INVOLVING MODULUS

Let us first draw the graphs of simple functions involving modulus.

EXAMPLE 2.1 Draw the graph of $y = |x| - 2$

SOLUTION



EXAMPLE 2.2 Compare the graph of the following functions with the graph of $f(x) = |x|$:

(i) $h(x) = 3|x|$ (ii) $g(x) = \frac{1}{3}|x|$

SOLUTION

(i) Relative to the graph of $f(x) = |x|$, the graph of

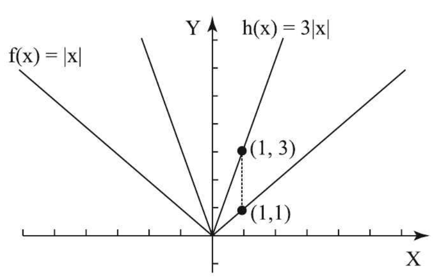
$h(x) = 3|x| = 3f(x)$

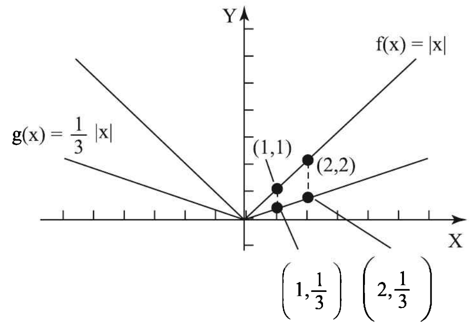
is a vertical stretch (multiply each y-value by 3) of the graph of f.

(ii) Similarly, the equation

$g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$

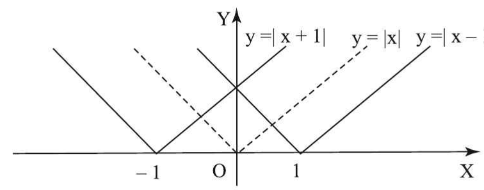
indicates that the graph of h is a vertical shrink of the graph of f.





EXAMPLE 2.3 Draw the graphs of $y = |x - 1|$ and $y = |x + 1|$.

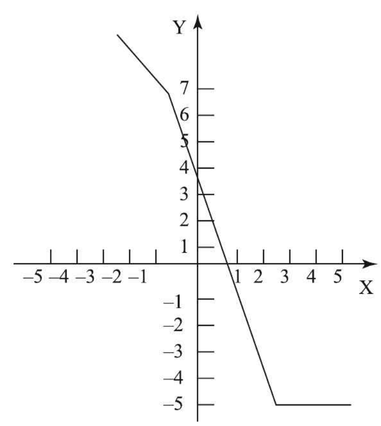
SOLUTION



EXAMPLE 2.4 Sketch the graph of the function $f(x) = 2|x - 2| - |x + 1| - x$

SOLUTION

$$f(x) = \begin{cases} -2(x-2) + (x+1) - x & x < -1 \\ -2(x-2) - (x+1) - x & -1 \le x < 2 \\ 2(x-2) - (x+1) - x & x \ge 2 \end{cases}$$

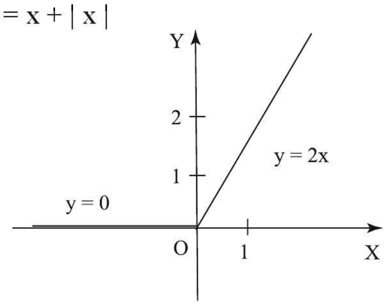


$$\Rightarrow f(x) = \begin{cases} 5-2x, & x<-1 \\ 3-4x, & -1 \le x < 2 \\ -5, & x \ge 2 \end{cases}$$

EXAMPLE 2.5 Sketch the graph of the function $f(x) = x + \sqrt{x^2}$.

SOLUTION

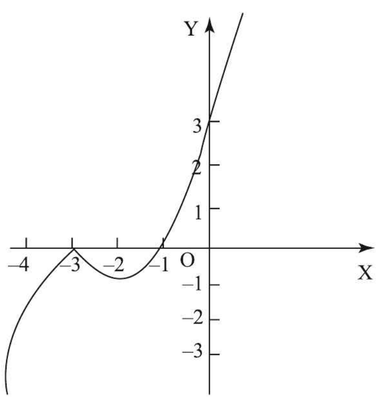
$f(x) = x + |x|$



EXAMPLE 2.6 Sketch the graph of the function $f(x) = |x + 3|(x + 1)$

SOLUTION The given function can be written as

$$f(x) = \begin{cases} -(x+3)(x+1), & x<-3 \\ (x+3)(x+1), & x \ge -3 \end{cases}$$



EXAMPLE 2.7 Construct the graph of the function $f(x) = x\left(|x-1| + \frac{x}{3} - 1\right)$.

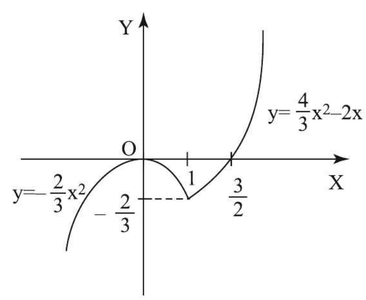
SOLUTION The function f(x) is defined for all x. It follows from the definition of the modulus that

$$f(x) = \begin{cases} x\left(\frac{4}{3}x - 2\right), \text{ if } x \ge 1 \\ -\frac{2}{3}x^2, \text{ if } x < 1. \end{cases}$$

The graph of the function $y = -\frac{2}{3}x^2$ is a parabola whose vertex lies at the origin and which is concave down.

The graph of the function $y = x\left(\frac{4}{3}x - 2\right)$ is a parabola which cuts the x axis at the points x = 0 and x = 3/2. The abscissa of the vertex of the parabola is equal to 3/4 and the ordinate equal to – 3/4. If follows that the function f(x) increases for $x \ge 1$.

The graph of the function f(x) is shown below.



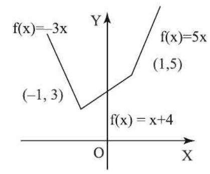
EXAMPLE 2.8 Find the values of k so that the line y = k can intersect the graph of

$y = x + |2(x + 1)| + 2|x - 1|$ at exactly one point.

SOLUTION We have

$$y = f(x) = \begin{cases} x-2(x+1)-2(x-1), & x<-1 \\ x+2(x+1)-2(x-1), & -1 \le x \le 1 \\ x+2(x+1)+2(x-1), & x>1 \end{cases}$$

$$f(x) = \begin{cases} -3x, & x < -1 \\ x + 4, & -1 \le x \le 1 \\ 5x, & x > 1 \end{cases}$$



The graph of y = f(x) is as shown alongside.

Clearly y = k can intersect y = f(x) at exactly one point only if k = 3.

PRACTICE PROBLEMS [C]

1. Draw the graph of the following functions:
   (i) $y = (|x+1| - |x-1|)/2$ (ii) $y = |x-1| + |x-2| + x$
   (iii) $y = x|x| - 4x - 5$ (iv) $y = x|x-3|$.
2. Draw the graph of the following functions:
   (i) $y = 2^{(|x|+x)/x}$ (ii) $y = \dfrac{\cos\left(|x| + \dfrac{\pi}{2}\right)}{\sin x}$
   (iii) $y = \dfrac{2|x|-1}{x-3}$ (iv) $y = 2^{|\log_2 x|}$.
3. Draw the graphs of the following functions:
   (i) $y = |x|(x-2)$ (ii) $y = (3-x)|x+1|$.
4. Draw $y = |x^2 - 4| - |x^2 - 9|$.
5. Draw the graph of the following functions :
   (i) $y = x^2 - 2|x+1| - 1$ (ii) $y = (|1-x| + 2)(x+1)$
   (iii) $y = |2x^2 - 3x + |x-1||$.
6. Draw $f(x) = \dfrac{1}{2}\left(\dfrac{|\sin x|}{\cos x} + \dfrac{|\cos x|}{\sin x}\right)$.

2.7 | TRANSFORMATION OF GRAPHS BY MODULUS FUNCTION

A function like y = f(x) has different elements. We can apply modulus operator to these elements of the function. There are the following different possibilities:

1. $y = f(|x|)$ 2. $y = |f(x)|$
3. $|y| = |f(x)|$ 4. $|y| = f(x)$

1. **Drawing the graph of y = | f(x)|, from the graph of y = f(x):**

$|f(x)| = f(x)$ if $f(x) \ge 0$ and
$|f(x)| = -f(x)$ if $f(x) < 0$.

The output of the function is now either zero or a positive number. This has the implication that the part of the graph y = f(x) corresponding to negative function values is not present in the graph of y = |f(x)|. Rather, a negative function value of f(x) is converted to a positive function value. This change in the sign of function takes place without changing magnitude of the value.

It implies that we can obtain function values, which correspond to negative function value in

$y = f(x)$ by taking image of negative function values about the x-axis.

To construct the graph of $y = |f(x)|$, we need to modify the graph of $y = f(x)$ as:

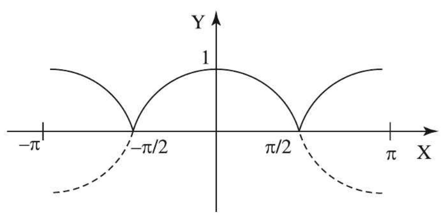
(i) take the mirror image of the lower half of the graph about the x-axis.

(ii) remove the lower half of the graph.

It means that the graph of $f(x)$ and $|f(x)|$ would coincide if $f(x) \geq 0$ and the portions where $f(x) < 0$ would get reflected above the x-axis.

EXAMPLE 2.1 Draw graph of $y = |\cos x|$.

SOLUTION We first draw the graph of $y = \cos x$. Then, we take the mirror image of lower half of the graph in x-axis and remove lower half of the graph to complete the graph of $y = |\cos x|$.

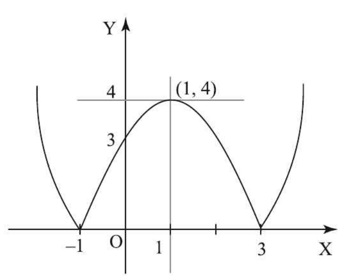


EXAMPLE 2.2 Draw graph of $y = |x^2 - 2x - 3|$.

SOLUTION We first draw the graph

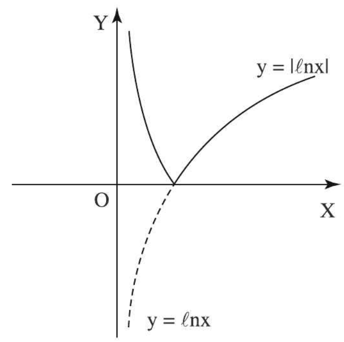
$y = x^2 - 2x - 3$.

The roots of the corresponding quadratic equation are –1 and 3. After plotting the graph of quadratic function, we take the mirror image of the lower half of the graph in the x-axis and remove the lower half of the graph to complete the graph of $y = |x^2 - 2x - 3|$.



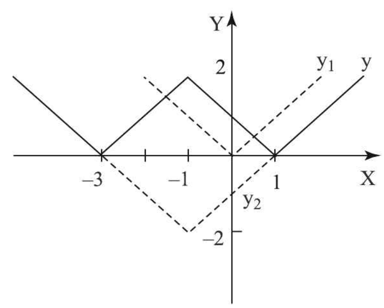
EXAMPLE 2.3 Draw the graph of $y = |ln\ x|$.

SOLUTION Take the image of the negative part of f(x) considering the x–axis as plane mirror.



EXAMPLE 2.4 Construct the graph of the function $y = ||x + 1| - 2|$.

SOLUTION Taking the graph of the function $y_1 = |x|$, translate it one unit leftwards along the x–axis and two units downwards along the y-axis.



This yields the graph of the function

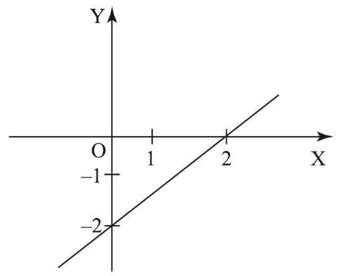
$y_2 = |x + 1| - 2$.

Then replace the portion of the graph below the x–axis corresponding to $-3 \leq x \leq 1$, by the portion symmetric to it about the x–axis. The resulting polygonal line is the graph of the function y.

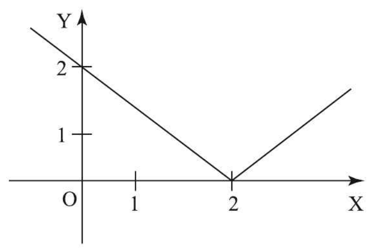
EXAMPLE 2.5 Draw the graph of the function

$y = | 2 - | x - 2 | |$.

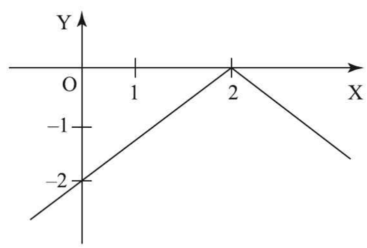
SOLUTION Graph of $y = x - 2$



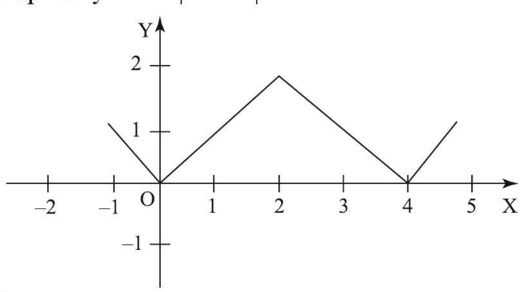
Graph of $y = |x - 2|$



Graph of $y = -|x - 2|$



Graph of $y = 2 - |x - 2|$

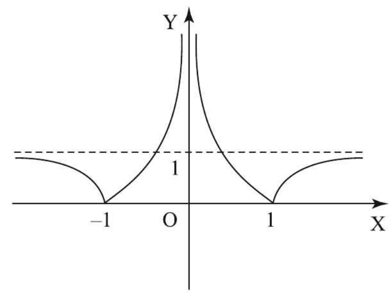


EXAMPLE 2.6 Draw the graph of

$f(x) = \left|\left|\frac{1}{x}\right| - 1\right|$.

SOLUTION We draw $y = 1/x$ first and then draw $y = |1/x|$. For this we take image of the lower half in the x-axis and remove the lower half. To draw $y = |1/x| - 1$, we shift down the graph of $|1/x|$ by 1 unit.

To draw $||1/x| - 1|$, we take image of the lower half of the graph of $|1/x| - 1$ in the x-axis and remove the lower half.



2. **Drawing the graph of $y = f(|x|)$, from the graph of $y = f(x)$**

Consider $y = f(|x|)$.

It can be seen that modulus operator here modifies the independent variable of the function. The input to the function is now either zero or positive number. This has the implication that part of the graph $y = f(x)$ corresponding to negative value of x is not present in the graph of $y = f(|x|)$. Rather, negative value of x is passed as positive value to the function. This means that negative value of independent variable x yields function value which is equal to function value obtained for corresponding positive x whose magnitude is same as that corresponding negative x. It implies that we can obtain function value for negative x by taking image of positive x across y-axis. This is image in y-axis.

It is clear that, $f(|x|) = f(x), \forall x \geq 0$ and

$f(|x|) = f(-x), \forall x < 0$.

Thus, the graph of function $y = f(|x|)$ is obtained from the graph of the function $y = f(x)$ as follows:

(i) first we remove the left half (graph for $x < 0$) of the graph

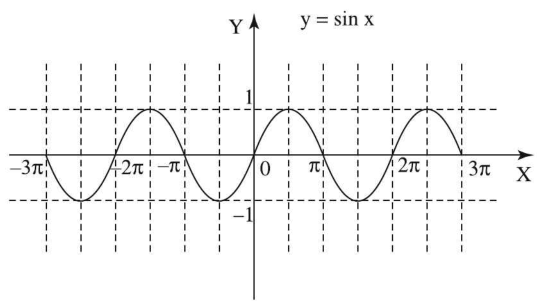
(ii) then we take the mirror image of right half (graph for $x \geq 0$) of the graph in the y-axis.

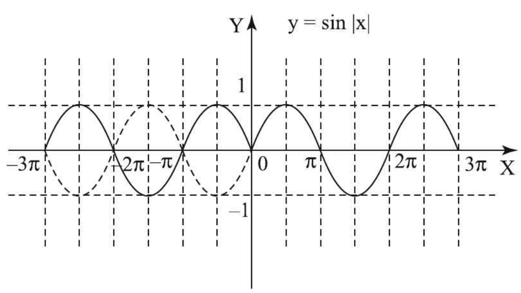
Note

(i) Since $f(|x|)$ is an even function, the graph of $f(|x|)$ would be symmetrical about the y–axis.
(ii) The graphs of $f(|x|)$ and $f(x)$ would be identical in the first and the fourth quadrants (as $x \geq 0$).
(iii) The part of $f(x)$ lying on the left of the y–axis (if it exits) is omitted.

EXAMPLE 2.7 Draw graph of $y = \sin|x|$.

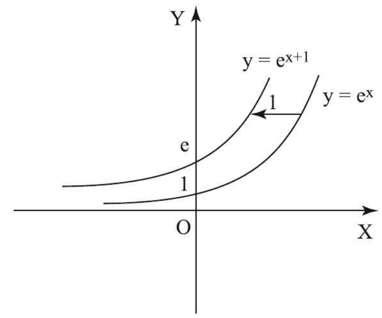
SOLUTION First we draw graph of sinx. In order to obtain the graph of $y = \sin|x|$, we remove left half of the graph and take the mirror image of right half of the graph of in the y-axis.

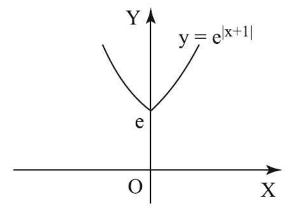




EXAMPLE 2.8 Draw graph of $y = e^{|x| + 1}$.

SOLUTION We first draw the graph of $y = e^x$. Then, we shift the graph left by 1 unit to obtain the graph of e^{x+1}. At $x = 0$, $y = e^{0+1} = e$. In order to obtain the graph of $y = e^{|x|+1}$, we remove the left part of the graph and take the mirror image of right half of the graph of $y = e^{x+1}$ in the y-axis.





EXAMPLE 2.9 Draw graph of

$y = x^2 - 2|x| - 3$.

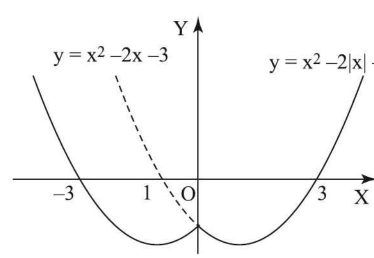
SOLUTION The given expression

$f(x) = x^2 - 2|x| - 3$ is obtained by taking modulus of the independent variable of the corresponding quadratic polynomial in x as given here,

$f(x) = x^2 - 2x - 3$. Hence, we first draw

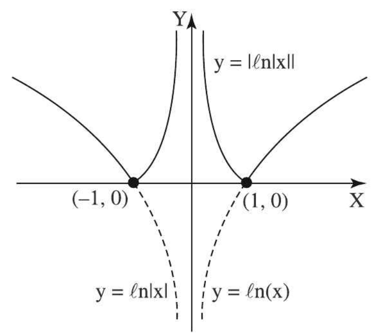
$f(x) = x^2 - 2x - 3 = 0$. It has real roots – 1 and 3. The coefficient of x^2 is positive. Hence, its plot is a parabola which opens upward and intersects x-axis at $x = -1$ and $x = 3$.

In order to draw the graph of $f(x) = |x|^2 - 2|x| - 3 = x^2 - 2|x| - 3$, we remove the left half of the graph and take the mirror image of the right half the in y-axis.



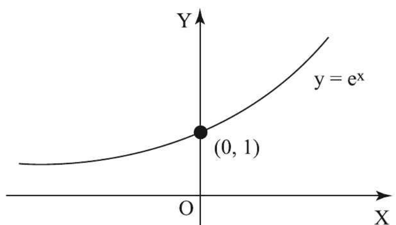
EXAMPLE 2.10 Draw the graph of the function $y = |\ln|x||$

SOLUTION

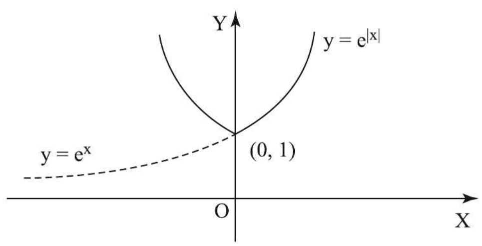


EXAMPLE 2.11 Draw the graph of the function $y = |e^{|x|} - 3|$

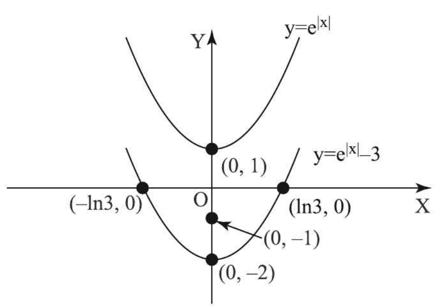
SOLUTION

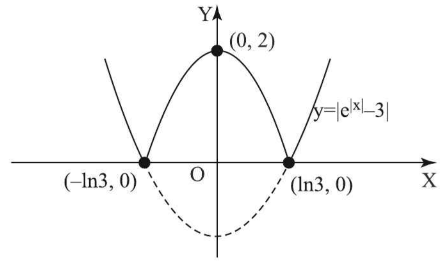


(a)



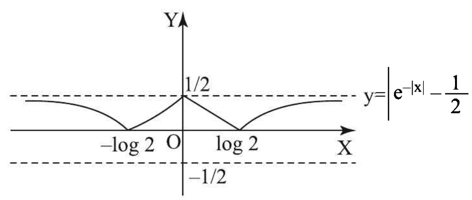
(b)





EXAMPLE 2.12 Plot the function $y = \left|e^{-|x|} - \frac{1}{2}\right|$

SOLUTION The graph is shown below:



EXAMPLE 2.13 Draw the graphs of the curves

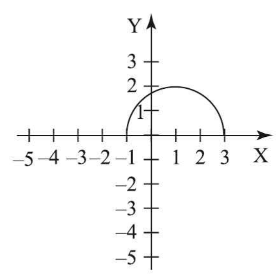
$y=\sqrt{-x^2+2|x|+3}$, $y=\sqrt{-x^2-2|x|+3}$.

SOLUTION Observe that $y = \sqrt{-x^2 + 2x + 3}$ is an upper semicircle and that

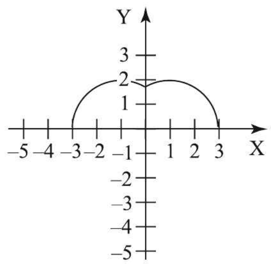
$$y = \sqrt{-x^2 + 2x + 3} \Rightarrow x^2 - 2x + y^2 = 3$$

$$\Rightarrow (x - 1)^2 + y^2 = 4.$$

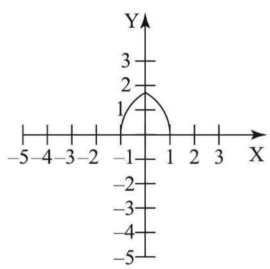
from where the semicircle has radius 2 and centre at (1, 0). Its graph is shown below.



The graph of $y = \sqrt{-x^2 + 2|x| + 3}$ is shown below.



The graph of $y=\sqrt{-x^2-2|x|+3}$ is obtained by reflecting the left half of the graph in the y-axis.

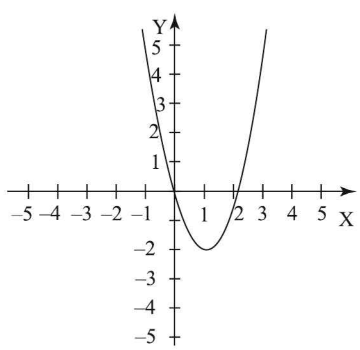


EXAMPLE 2.14 Draw the following graphs in succession.

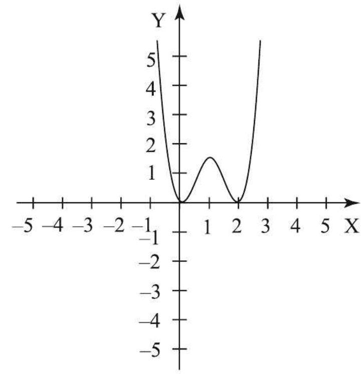
(i) $y = (x - 1)^2 - 2$ (ii) $y = |(x - 1)^2 - 2|$
(iii) $y = (|x| - 1)^2 - 2$ (iv) $y = (1 + |x|)^2 - 2$

SOLUTION Here is the graph of

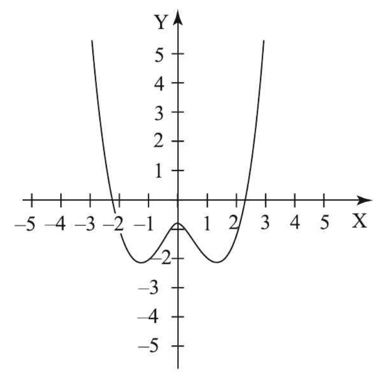
$y = (x - 1)^2 - 2$



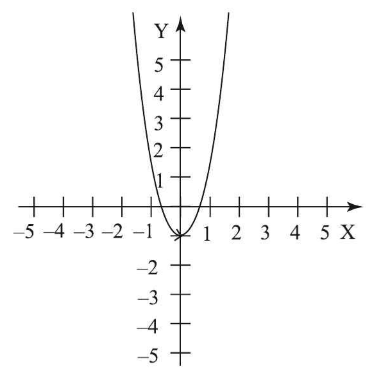
Here is the graph of $y = |(x - 1)^2 - 2|$



Here is the graph of $y = (|x| - 1)^2 - 2$



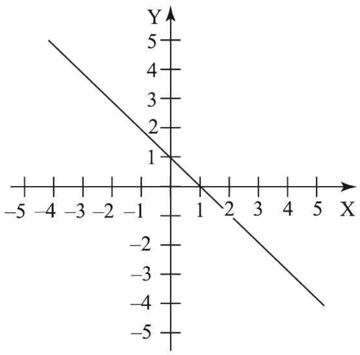
Here is the graph of $y = (|x| + 1)^2 - 2$.



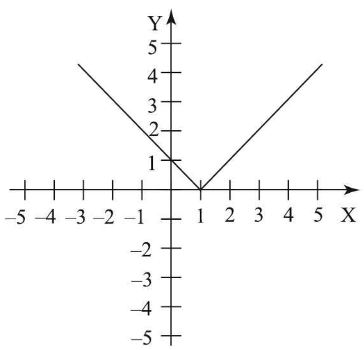
EXAMPLE 2.15 Draw the following graphs in succession:

(i) $y = 1 - x$ (ii) $y = |1 - x|$
(iii) $y = 1 - |1 - x|$ (iv) $y = |1 - |1 - x||$
(v) $y = 1 - |1 - |1 - x||$ (vi) $y = |1 - |1 - |1 - x|||$
(vii) $y = 1 - |1 - | 1 - |1 - x|||$

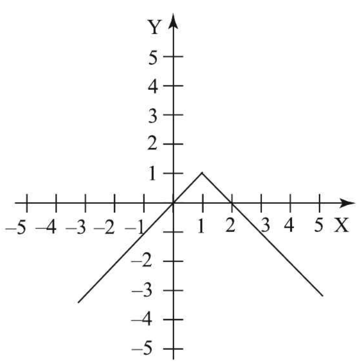
SOLUTION Here is the graph of $y = 1 - x$.



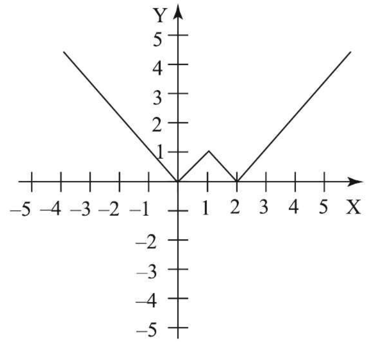
Here is the graph of $y = |1 - x|$.



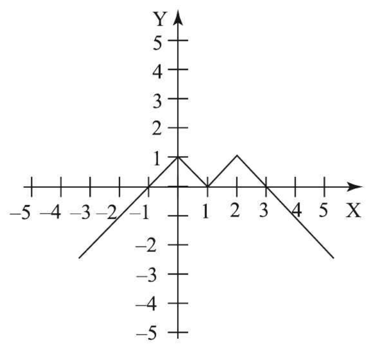
Here is the graph of $y = 1 - |1 - x|$.



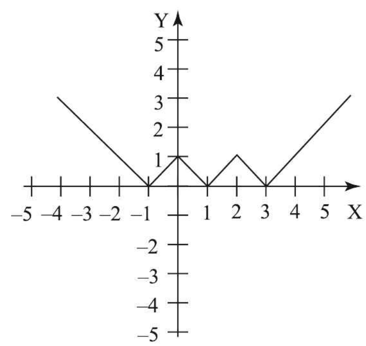
Here is the graph of $y = |1 - |1 - x||$.



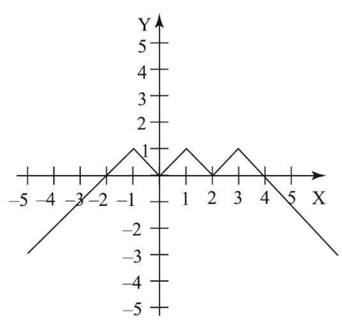
Here is the graph of $y = 1 - |1 - |1 - x||$.



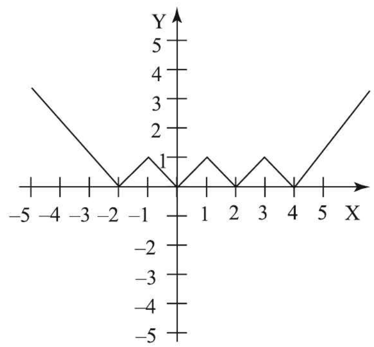
Here is the graph of $y = |1 - |1 - |1 - x|||$.



Here is the graph of $y = 1 - |1 - |1 - |1 - x||$.



Here is the graph of $y = |1 - |1 - |1 - |1 - x||||$.



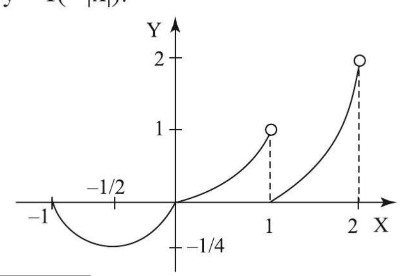
Note

The graph of function $y = f(-|x|)$ is obtained from the graph of the function $y = f(x)$ as follows:

(i) first we remove the right half (graph for $x > 0$) of the graph.

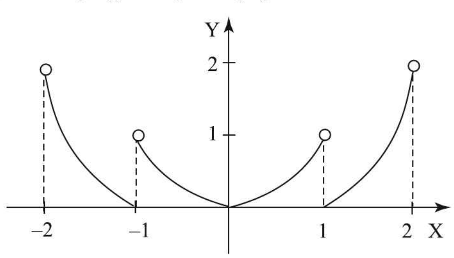
(ii) the graph of the function is retained for $x \le 0$, and this retained portion of the graph is reflected symmetrically about the y–axis, thus determining the graph of the function for $x > 0$.

EXAMPLE 2.16 The graph of $y = f(x)$ is shown below. Draw the graphs of (i) $y = f(|x|)$ and (ii) $y = f(-|x|)$.

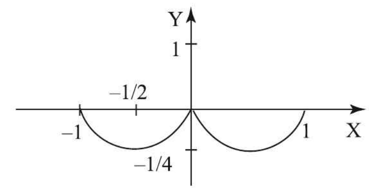


SOLUTION

(i) The graph of $y = f(|x|)$ is shown below :



(ii) The graph of $y = f(-|x|)$ is shown below :

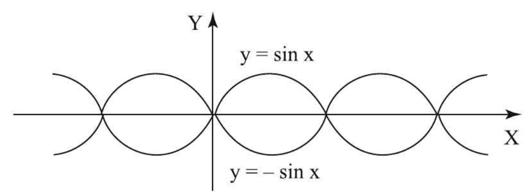


3. Drawing the graph of $|y| = |f(x)|$, from the graph of $y = f(x)$:

The graph of $|y| = |f(x)|$ is obtained from the graph of the function $y = f(x)$ as follows :

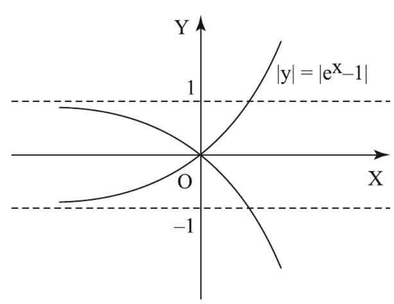
First draw the graph of $y = f(x)$. Then take the image of the graph of $f(x)$ about the x–axis considering it as a plane mirror. The union of the graphs is the required graph. Note that the graph of $|y| = |f(x)|$ does not represent a function.

Let us draw the graph of $|y| = |\sin x|$.



EXAMPLE 2.17 Draw the graph of $|y| = |e^x - 1|$.

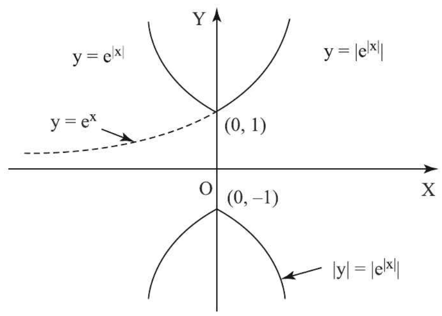
SOLUTION We first draw $y = e^x - 1$. Then we reflect this graph about x-axis. Now the graph of $|y| = |e^x - 1|$ is the union of both the graphs.



EXAMPLE 2.18 Draw the graph of

$|y| = |e^{|x|}|$

SOLUTION



4. Drawing the graph of |y| = f(x), from the graph of y = f(x):

Consider $|y| = f(x)$.

In this case, the value of a function evaluated at some x is assigned to modulus function |y|. We interpret equality of the modulus function |y| to a value as follows:

Let $f(a) = b$, then $|y| = f(x) \Rightarrow |y| = b$

If $b > 0$, then $y = \pm b$

If $b = 0$, then $y = 0$

If $b < 0$, then we have no solution, since modulus cannot be equated to a negative value.

From the point of view of construction of plot, for a single positive value of f(x), say $f(x) = 4$, we have two values of the dependent variable i.e. –4 or 4. Hence, there are two points (4, 4) and (4, –4) on the graph corresponding to one value of the independent variable 4.

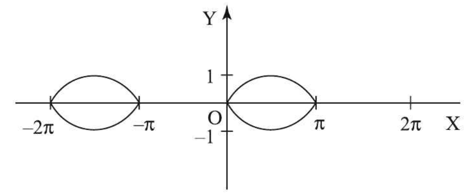
In the equation $|y| = f(x)$, clearly $|y| \geq 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$.

The graph of $|y| = f(x)$ is obtained from the graph of the function $y = f(x)$ as follows:

First delete its portion located below the x–axis. The portion of the graph of the function $y = f(x)$ lying above the x–axis remains unchanged. Then take the image of the positive part of f(x) about the x–axis considering it as a plane mirror.

Hence, the graph of $|y| = f(x)$ would exist only in the regions where $f(x) \geq 0$ and will be reflected about x–axis only in those regions. Regions where $f(x) < 0$ will be neglected.Note that the graph of $|y| = f(x)$ does not represent a function.

Let us draw the graph of $|y| = \sin x$.

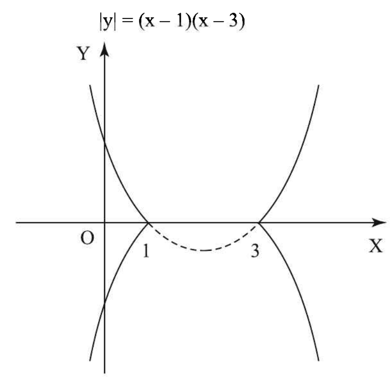


EXAMPLE 2.19 Draw the graph of

$|y| = (x – 1)(x – 3)$.

SOLUTION We fist draw the graph of quadratic function given by $y = (x – 1)(x – 3)$. Then, we remove the lower half of the graph and take mirror image of upper half of the graph in x-axis to complete the construction of graph of

$|y| = (x – 1)(x – 3)$



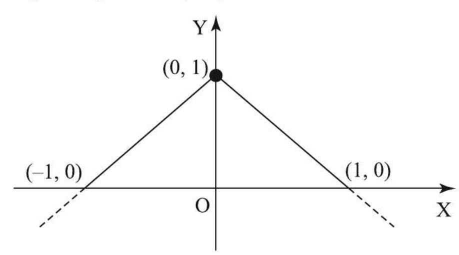
EXAMPLE 1.20 Draw the graph of the function

$|x| + |y| = 1$.

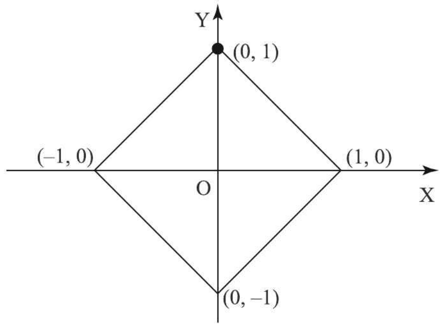
SOLUTION The given curve can be rewritten as

$|y| = 1 – |x|$

Graph of $y = 1 – |x|$:



Now graph of $|y| = 1 - |x|$:

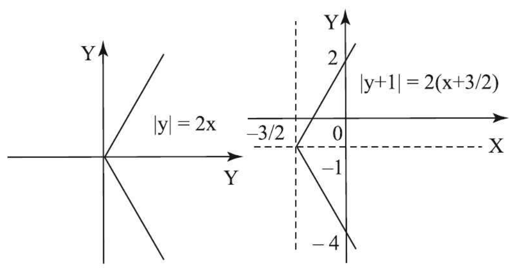


EXAMPLE 2.21 Plot $|y + 1| = 2x + 3$

SOLUTION We have to plot

$$|y + 1| = 2\left(x + \frac{3}{2}\right)$$

Shifting the origin to $\left(\frac{-3}{2}, -1\right)$, the given curve can be drawn similar to $|y| = 2x$.



The entire curve lies on the right of $x = -3/2$ and cuts the y-axis at point given by

$|y + 1| = 3$ [putting x = 0]

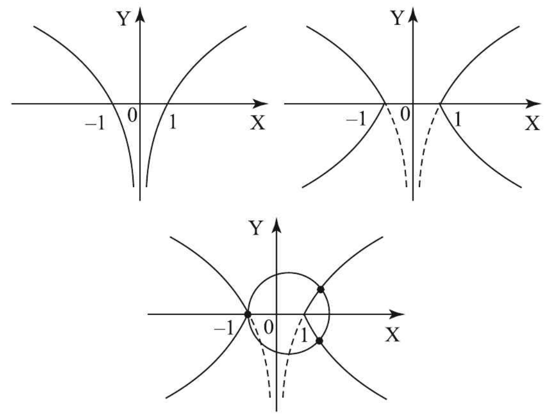
$\Rightarrow\ y + 1 = \pm 3$

$\Rightarrow\ y = 2, -4.$

EXAMPLE 2.22 Determine the number of points where the graphs of $|y| = \ln|x|$ and $(x - 1)^2 + y^2 - 4 = 0$ intersect each other.

SOLUTION The curve $|y| = \ln|x|$ is obtained by transforming $y = \ln x$. To draw $y = \ln|x|$, we need to remove left half (but here there is no left half) and take image of right half in y-axis. To draw $|y| = \ln|x|$, we transform the graph of $y = \ln|x|$. For this, we remove the lower half and take image of upper half in x-axis.

On the other hand, $(x - 1)^2 + y^2 - 4 = 0$ is a circle with centre at 1, 0 having radius of 2 units. Finally, drawing the two graphs together, we determine the intersection points.

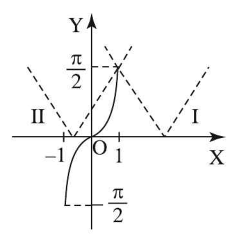


Clearly, there are three intersection points as shown by solid circles.

EXAMPLE 2.23 Find the complete set of values of 'a' for which the equation $\sin^{-1}x = |x - a|$ has atleast one solution.

SOLUTION

$$\sin^{-1}x = |x - a|$$



Consider the limiting case when $a > 0$.

In this case, $y = (a - x)$ must pass through $\left(1, \frac{\pi}{2}\right)$.

$\Rightarrow\ \frac{\pi}{2} = a - 1 \quad \Rightarrow \quad a = 1 + \frac{\pi}{2}.$

Now, consider the limiting case when $a < 0$.

In this case, $y = (x - a)$ must pass through $\left(1, \frac{\pi}{2}\right)$.

$\Rightarrow \quad \frac{\pi}{2} = 1 - a \quad \Rightarrow \quad a = 1 - \frac{\pi}{2}$.

Thus, for atleast one solution, we must have

$$a \in \left[1 - \frac{\pi}{2}, 1 + \frac{\pi}{2}\right]$$

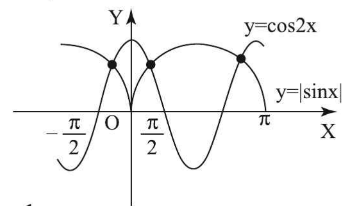
EXAMPLE 2.24 Find the complete set of values of x, satisfying the inequality $\cos 2x > |\sin x|$, where $x \in \left(-\frac{\pi}{2}, \pi\right)$.

SOLUTION $\cos 2x > |\sin x|$

If $\cos 2x = \sin x$ we get $2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \sin x = -1, \frac{1}{2}$

Clearly, $\sin x \neq -1$



$\Rightarrow \sin x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

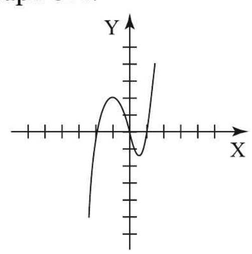
From the figure, the solution set is

$\left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$.

PRACTICE PROBLEMS [D]

1. Construct the graph of the following functions:
   (i) $y = |1 + 2x|$ (ii) $y = -|-4x + 2|$
2. Draw the graph of the following functions:
   (i) $y = \frac{1}{|x|}$ (ii) $y = |x^2 - x|$
   (iii) $y = |x^2 + x|$ (iv) $y = -|x^2 - 2x|$
3. Draw the graphs of the following functions:
   (i) $y = x^2 - 4|x| + 3$ (ii) $y = x^2 + 4|x| + 3$
4. Draw the graphs of the following functions:
   (i) $y = |x^2 - 3|x| + 2|$ (ii) $y = -|x^2 - |x| - 6|$
5. Draw the graph of the following functions:
   (i) $y = 1 + |\sin x|$ (ii) $y = |1 - |x^2 - 2||$
6. Draw the graph of the following functions:
   (i) $y = \sin|2x|$ (ii) $y = x^2 - 3|x|$
   (iii) $y = \log|x| - \log x^2$ (iv) $\log_2|x - 1| - 1$
7. Construct the graph of the following functions:
   (i) $y = \frac{1}{2 - |x|}$ (ii) $y = \frac{2}{|x - 1| - 1}$
   (iii) $y = \frac{|x - 1|}{|x| - 1}$.

8. Draw the graph of the function functions:
   (i) $|y| = |e^x|$ (ii) $|y| = |\log_{1/4}(x/4)|$
   (iii) $|y| = |1 - |x - 1||$
9. Construct the following curves:
   (i) $\dfrac{y}{x+1} = -1$ (ii) $|y| + x = -1$ (iii) $|x| + |y| = 2$ (iv) $|y - 3| = |x - 1|$.
10. The following graph shows the graph of f:



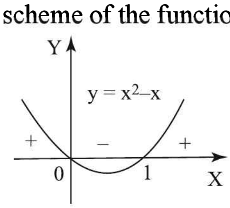
Draw the graph of the following functions:
   (i) $y = \dfrac{1}{2}f(x)$ (ii) $y = -|f(x)|$
   (iii) $y = |f(-x)|$ (iv) $y = f\left(-\dfrac{|x|}{2}\right)$.

2.8 | MISCELLANEOUS GRAPHS

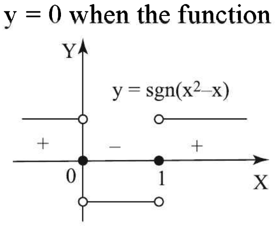
Graphs of functions involving signum function

EXAMPLE 2.1 Draw the graph of the function
(i) $y = \text{sgn}(x^2 - x)$ (ii) $y = \text{sgn}(\ln x)$

SOLUTION We first draw the graph of $y = x^2 - x$ to find the sign scheme of the function.



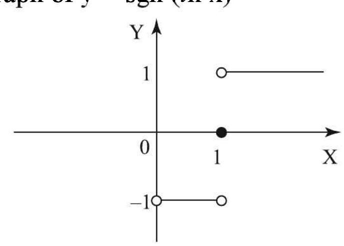
For $y = \text{sgn}(x^2 - x)$,
we draw $y = 1$ when the function is positive;
draw $y = -1$ when the function is negative;
and draw $y = 0$ when the function is zero.



Note

For clarity, we apply rings: solid ring to denote inclusion of point and hollow ring to denote exclusion of point. Using solid ring is optional, but helps to identify points on the graph.

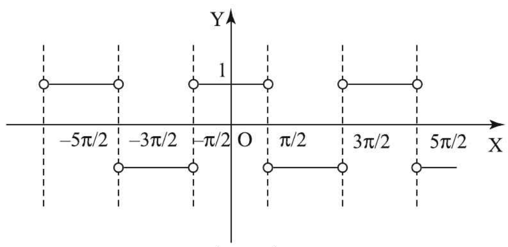
(ii) Graph of $y = \text{sgn}(\ln x)$



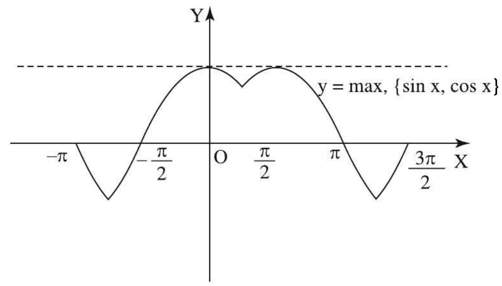
EXAMPLE 2.2 Draw the graph of

$$f(x) = \frac{|\cos x|}{\cos x}.$$

SOLUTION The value of the function is 1 or -1 according as $\cos x$ is positive or negative. $f(x)$ is undefined when $x = (2n + 1)\pi/2$, $n \in I$.



$y = \frac{|\cos x|}{\cos x}$



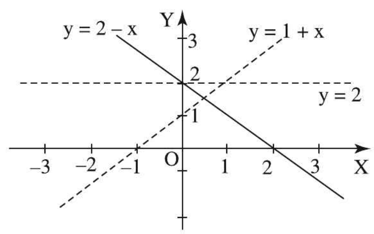
Graphs of max.- min. functions

EXAMPLE 2.3 Draw the graph of the function

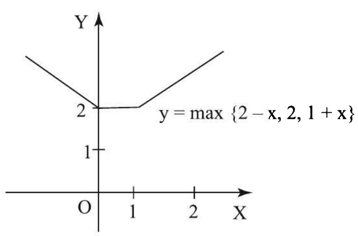
$y = \max\{2 - x, 2, 1 + x\}$

SOLUTION First we draw the graphs of

$y = 2 - x$, $y = 2$ and $y = 1 + x$.

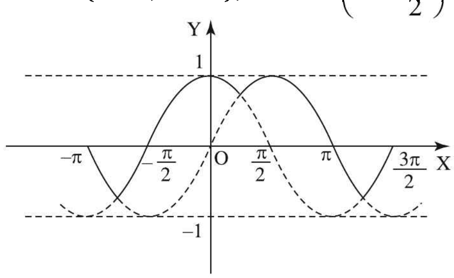


Then the graph of $y = \max\{2 - x, 2, 1 + x\}$ is selected a s the maximum value out of the three functions in the above graph:



EXAMPLE 2.4 Draw the graph of

$y = \max\{\sin x, \cos x\}$, for $x \in \left(-\pi, \frac{3\pi}{2}\right)$.



EXAMPLE 2.5 Draw

$y = \min.\{|x| - 1, |x - 1| - 1, |x - 2| - 1\}$.

SOLUTION We must plot the curves

$y = |x| - 1$
$y = |x - 1| - 1$
$y = |x - 2| - 1$

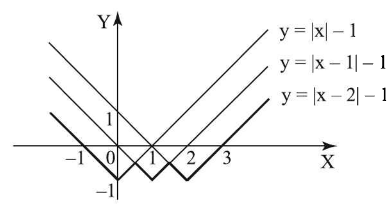
on the same reference frame.

The plot of $y + 1 = |x|$ is obtained by shifting the origin to $(0, -1)$

The plot of $y + 1 = |x - 1|$ is obtained by shifting the origin to $(1, -1)$.

The plot of $y + 1 = |x - 2|$ is obtained by shifting the origin to $(2, -1)$.

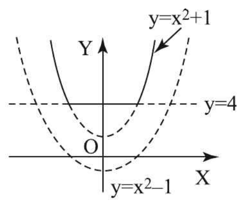
The plot of the three curves is shown below.



The darkened portion is the minimum of the three curves taken together and hence represents the required curve.

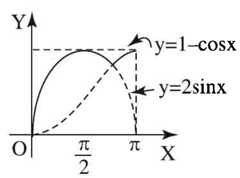
EXAMPLE 2.6 Let $f(x) = \text{maximum}\{4, x^2 + 1, x^2 - 1\}\ \forall x \in R$. Show that $f(x)$ is positive for all x.

SOLUTION The bold curve represents the graph of $y = f(x)$. Clearly $f(x)$ is above the x-axis at all points. Hence, $f(x)$ is positive for all x.



EXAMPLE 2.7 If f(x) = max. {2 sinx, 1 – cosx} $\forall$ x $\in$ [0, π], then find x at which f(x) attains the value of 1.

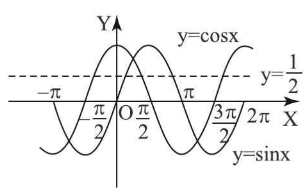
SOLUTION The bold curve represents the graph of f(x).



Clearly, f(x) = 1 at x = $\frac{\pi}{2}$, π.

EXAMPLE 2.8 Find the total number of solutions of the equation max. (sinx, cosx) = $\frac{1}{2}$ in x $\in$ (–2π, 5π).

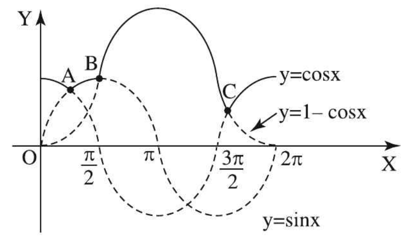
SOLUTION f(x) = max(sinx, cosx) is periodic with period 2π.



The graphs of y = f(x) and y = $\frac{1}{2}$, meet two times in [0, 2π]. Thus, there are 7 solutions in (–2π, 5π).

EXAMPLE 2.9 Let f(x) = max. {sinx, cosx, 1 – cosx}. Find the points belonging to (0, 2π) where f(x) = 1 – cos x.

SOLUTION The bold curve represents the graph of y = f(x).



For abscissa of point B, we solve
sin x = 1 – cosx

$\Rightarrow$ x = $\frac{\pi}{2}$. For abscissa of point C, we solve

cos x = 1 – cosx $\Rightarrow$ x = $\frac{5\pi}{3}$.

Hence the solution of f(x) = 1 – cos x is

$$x \in \left[\frac{\pi}{2}, \frac{5\pi}{3}\right].$$

EXAMPLE 2.10 Draw the graph of

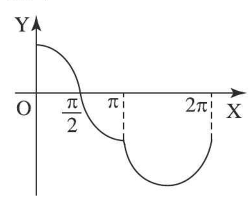
$$g(x) = \begin{cases} \text{minimum } \{f(t)\ ;\ 0 \le t \le x\}, & x \in [0, \pi] \\ \sin x - 1, & x > \pi \end{cases},$$

where f(x) = cosx.

SOLUTION Since cos x decreases in [0, π],

$$g(x) = \begin{cases} \cos x\ , & 0 \le x \le \pi \\ \sin x - 1\ , & x > \pi \end{cases}$$

The graph of g(x) is shown below.

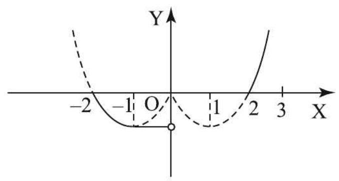


EXAMPLE 2.11 Let f(x) = x^2 – 2|x| and

$$g(x) = \begin{cases} \text{minimum } \{f(t) : -2 \le t \le x\}, & x \in [-2, 0) \\ \text{maximum } \{f(t) : 0 \le t \le x\}, & x \in [0, 3] \end{cases}.$$

Then find the solution set of the inequality g(x) > f(x).

SOLUTION The bold curve represents the graph of g(x). Clearly g(x) > f(x) when $x \in (0, 2)$.



Graphs of linear inequalities

The inequalities of the form $ax + by + c > 0$, $ax + by + c < 0$, $ax + by + c \geq 0$ and $ax + by + c \leq 0$ are called linear inequations in two variables x and y.

The solution of the linear inequality

$ax + by + c \leq 0$ (or, $ax + by + c \geq 0$)

is called a half plane. Geometrically speaking, the half plane consists of all of the points lying on the bounding line $ax + by + c = 0$ and on one side of the line.

First draw the line $ax + by + c = 0$. We select a point in one of the half planes and put it into the given inequality. If that point satisfies the given inequality, then all points in that half plane satisfy the inequality.

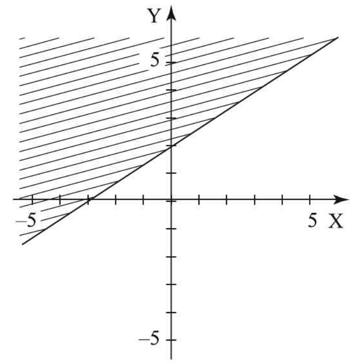
For strict inequality i.e., $ax + by + c > 0$ or $ax + by + c < 0$ draw the line dotted (.....) otherwise draw it thick(—). The shaded portion represents the solution set of the given inequality. The dotted line is not a part of the solution, but thick line is a part of the solution.

EXAMPLE 2.12 Draw the graph of the inequality $2x - 3y \leq - 6$.

SOLUTION First plot the line $2x - 3y = -6$.

Then choose an arbitrary point that doesn't belong to the line, say (0, 0), and test it to see if it belongs to the solution set.

Since (0, 0) does not satisfy the given inequality, the points on the other side of the line belong to the solution set as shown in the figure.



Graph : $2x - 3y \leq -6$

EXAMPLE 2.13 Sketch the solution set of each linear inequality:

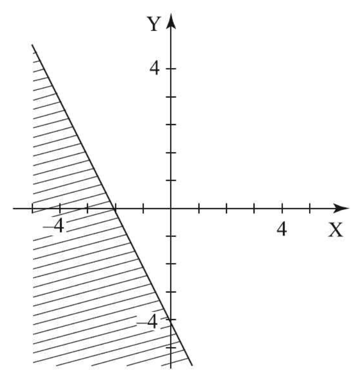
(i) $2x + y \leq -3$ (ii) $4x - 3y > -6$

(iii) $2x - 3y \leq 0$.

In each case, first plot the line bounding the half plane by finding atleast two points on the line ; then choose a point which doesn't belong to the line and test it in the given inequality.

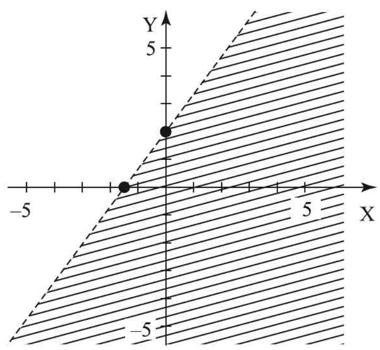
SOLUTION

(i) On the line $2x + y = -4$; if $x = 0$, $y = -4$; if $y = 0$, $x = -2$. Draw the line through (0, –4) and (–2, 0). Test, say, the origin (0, 0). Since (0, 0) does not satisfy $2x + y \leq -4$, i.e., $0 \neq -4$, shade the side of the line which does not contain the origin.



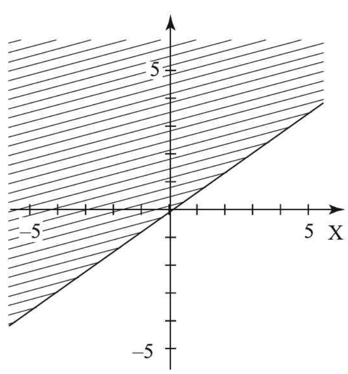
Graph : $2x + y \leq -4$

(ii) On the line $4x - 3y = -6$; if $x = 0$, $y = 2$; if $y = 0$, $x = -1.5$. Draw the line through (0, 2) and (–1.5, 0). Test the origin (0, 0). Since (0, 0) satisfies the given inequality $4x - 3y > -6$, shade the side of the line containing the origin.



Graph : $4x - 3y > -6$

(iii) On the $2x - 3y = 0$; if $x = 0$, $y = 0$; if $x = 3$, $y = 2$. Draw the line through (0, 0) and (3, 2). Since the origin belongs to the line, test the point (2, 0). Since (2, 0) does not satisfy the given inequality $2x - 3y \leq 0$, shade the other side of the line.



Graph : $2x - 3y \leq - 0$

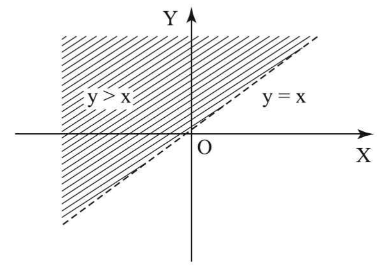
EXAMPLE 2.14 Draw the graphs of the solution set of the following inequalities:

(a) $y > x$ (b) $y > | x |$

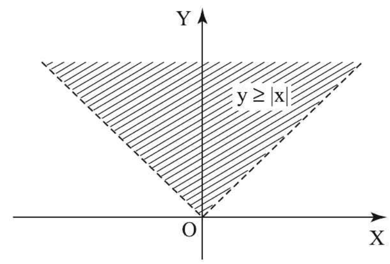
(c) $2x - y \geq 1$

SOLUTION

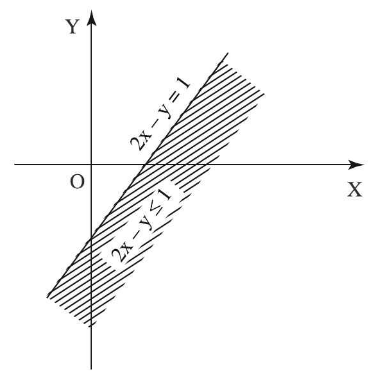
(a) $y > x$



(b) $y > | x |$



(c) $2x - y \geq 1$



EXAMPLE 2.15 Sketch the solution set of the following system of linear inequalities:

(i) $x + 2y \leq 4$
$x - y \geq 0$
$y \geq -1$

(ii) $x + 2y \leq 4$
$x - y \leq 0$
$y \geq -1$

SOLUTION In each case the bounding lines of the given half planes are $x + 2y = 4$, $x - y = 0$ and $y = -1$.

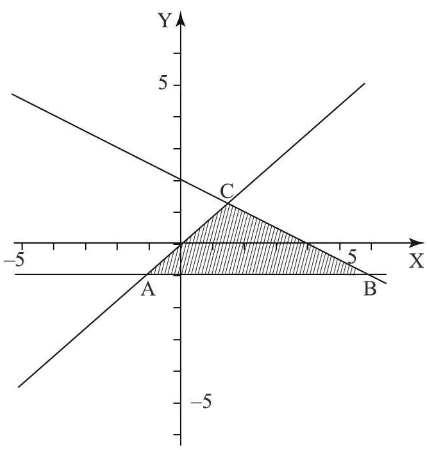
Furthermore,

A = $(-1, -1)$, is the solution of $\begin{cases} x - y = 0 \\ y = -1 \end{cases}$

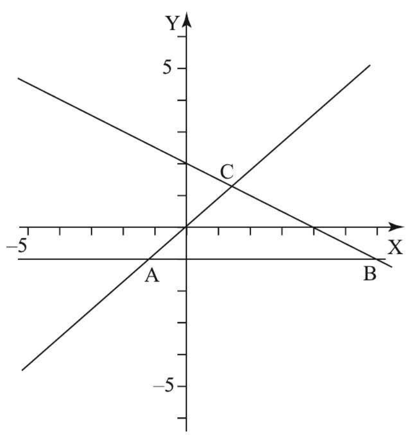
B = $(6, -1)$, is the solution of $\begin{cases} x + 2y = 4 \\ y = -1 \end{cases}$

C = $\left(\dfrac{4}{3}, \dfrac{4}{3}\right)$, is the solution of $\begin{cases} x + 2y = 4 \\ x - y = 0 \end{cases}$

(i)



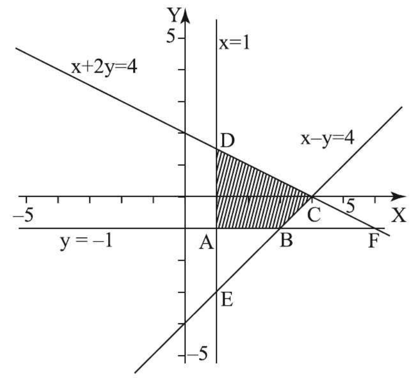
(ii)



EXAMPLE 2.16 Sketch the solution set of the following system of linear inequalities :

$x + 2y \le 4 \qquad x - y \le 4$
$x \ge 1 \qquad y \ge -1$

SOLUTION The convex polygon shaded in the adjoining diagram is the solution to the system of inequalities.



EXAMPLE 2.17 Plot $|x| + |y| = |x-3| + |y-2|$

SOLUTION There are nine cases:

Let $x \ge 3, y \le 2$

The given equation reduces to

$x + y = x - 3 + y - 2$

i.e. $0 = -5$ not possible.

Let $x \ge 3, y \le 0$

$x - y = x - 3 + 2 - 2$

i.e. $0 = -1$ not possible.

Let $0 \le x \le 3, y \le 2$

$x + y = 3 - x + y - 2$

i.e. $x = \dfrac{1}{2}$

Let $0 \le x \le 3, y \le 0$

$x + y = 3 - x + 2 - y$

i.e. $x = \dfrac{5}{2}$

Let $x \le 0, y \ge 2$

$-x + y = 3 - x + y - 2,$

i.e. $0 = 1$ not possible

Let $x \le 0, y \ge 0$

$-x - y = 3 - x + 2 - y$

i.e. $0 = 5$ not possible.

Let $x \ge 3, 0 \le y \le 2$

$x + y = x - 3 + 2 - y$

i.e. $y = \dfrac{1}{2}$

Unacceptable since it does not lie in $0 \le y \le 2$.

Let $0 \le x \le 3,\ 0 \le y \le 2$

$$x + y = 3 - x + 2 - y$$

i.e. $x + y = \frac{5}{2}$

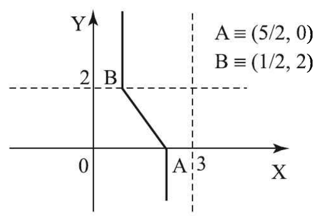
Let $x \le 0,\ 0 \le y \le 2$

$$-x + y = 3 - x + 2 - y$$

i.e. $y = \frac{5}{2}$

unacceptable since it does not lie in $0 \le y \le 2$.

Thus, the required curve is as shown alongside.



EXAMPLE 2.18 On the X-Y plane, indicate the points satisfying the inequality

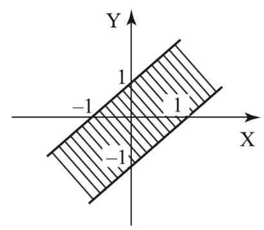
(i) $|x - y| \le 1$ (ii) $|x| + |y| \le 3$

(iii) $|y| \le 2^{-|x|} - \frac{1}{2}$

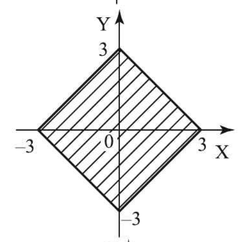
SOLUTION

(i) We first draw the bounding lines $x - y = 1$ and $x - y = -1$ and then test $(0, 0)$ in the given inequality $|x - y| \le 1$.

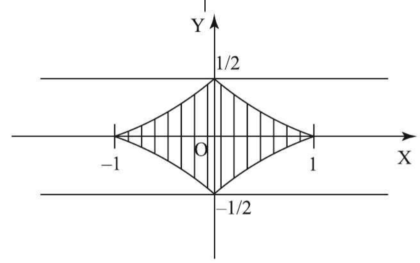
(i)



(ii)



(iii)



PRACTICE PROBLEMS [E]

1. Draw the graph of the following functions:

   (i) $y = \frac{|\sin x|}{\sin x}$ (ii) $y = \text{sgn}\,(x - |x|)$ (iii) $y = |\log_2 x| / \log_2 x$ (iv) $y = \frac{|x-1|}{x-1}(x^2+3)$

2. Draw the graph of the following functions:

   (i) $y = \max\{1 - x, 1 + x, 2\}$ (ii) $y = \min\{|x|, |x - 2|, 2 - |x - 1|\}$

   (iii) $y = \min\left\{e^x, \frac{3}{2}, 1 + e^{-x}\right\}$

3. Plot the solution of each pair of inequalities:

   (i) $2x - 3y \le 6$
   $2x + y \le 4$

   (ii) $2x + 5y \le 6$
   $x \ge 1$

4. Sketch the solution set of the following system of linear inequalities:

   (i) $x + 2y \le 4$
   $x - y \le 0$
   $y \le -1$

   (ii) $x + 2y \ge 4$
   $x - y \le 0$
   $y \le -1$

5. Indicate the points on the plane X-Y which satisfy the following equations:
   (i) $y + |y| - x - |x| = 0$ (ii) $|x + y| + |x - y| = 4$
   (iii) $|y|x = x$ (iv) $|x - y| + y = 0$

6. On the plane X-Y, indicate the points satisfying the inequality
   (i) $|x + y| \geq 2$ (ii) $|x| - |y| \geq 1$;
   (iii) $|x - 1| + |y + 1| \geq 2$ (iv) $|x + y| + |x - y| \leq 2$

2.9 | TRANSFORMATION OF GRAPHS BY GREATEST INTEGER AND FRACTIONAL PART FUNCTIONS

1. Drawing the graph of y = [f(x)], from the graph of y = f(x)

The graph of y = f(x) is transformed to y = [f(x)] by applying changes to the output of the function. Whatever be the function values, they will be changed to integral values following definition of greatest integer values.

Clearly, real values of f(x) are truncated to integer values in the interval of unity i.e. [–1, 0), [0, 1), [1.2) etc. along y-axis. We need to modify the graph of y = f(x) as :

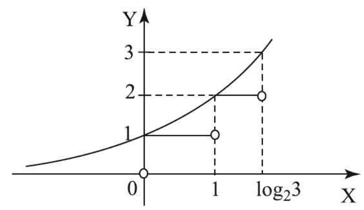
(i) Draw lines parallel to x-axis (horizontal lines) at integral values along y-axis to cover the graph of y = f(x).

(ii) Identify points of intersections of graph with parallel lines drawn in the earlier step. Draw lines parallel to y-axis (vertical lines) from the intersection points identified.

(iii) Take x-projection of curve from the point of intersection between two consecutive vertical lines such that it lies on horizontal line of lower value.

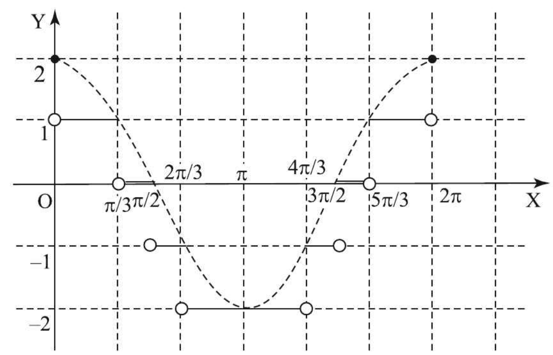
EXAMPLE 2.1 Draw the graph of $y = [2^x]$.

SOLUTION Following the above construction steps, graph of $y = [2^x]$ is drawn as follows:



EXAMPLE 2.2 Draw the graph of y = [2cos x].

SOLUTION Following the construction steps, graph of y = [2cos x] is drawn by transforming y = 2 cos x as shown here.

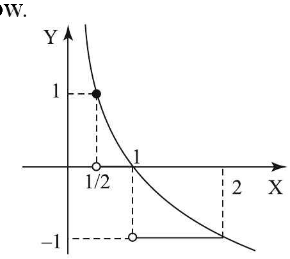


Note two individual solid circles. We should analyze their existence while constructing the graph.

EXAMPLE 2.3 Draw the graph of

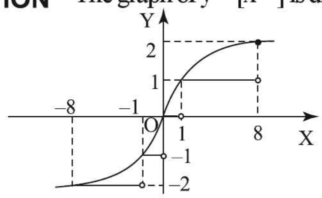
$y = [\log_{1/2}x]$, $1/2 \leq x \leq 2$.

SOLUTION The graph of $y = [\log_{1/2}x]$ is drawn below.



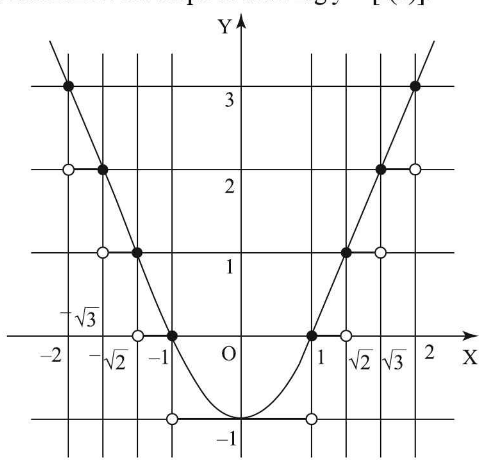
EXAMPLE 2.4 Draw the graph of $y = [x^{1/3}], -8 \le x \le 8$.

SOLUTION The graph of $y = [x^{1/3}]$ is drawn below.



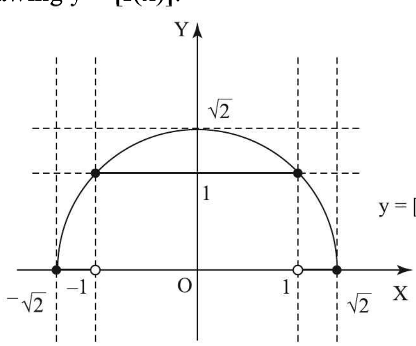
EXAMPLE 2.5 Draw the graph of $y = [x^2 - 1]$.

SOLUTION We first draw the parabola $y = x^2 - 1$ and then follow the steps of drawing $y = [f(x)]$.



EXAMPLE 2.6 Draw the graph of $y = \sqrt{2 - x^2}$

SOLUTION We first draw the semi-circle $y = \sqrt{2 - x^2}$ and then follow the steps of drawing $y = [f(x)]$.



2. Drawing the graph of y = f([x]), from the graph of y = f(x)

The graph of $y = f(x)$ is transformed to $y = f([x])$ by virtue of changes in the argument values due to operation on the independent variable. The independent variable of the function is subjected to greatest function operator. This changes the normal real value input to the function. The independent variable to the function is rendered to be integers depending on the value of x and interval it belongs to. A value like $x = -2.3$ is passed to the function as -3 in the interval $-3 \le x < -2$.

For example,

$$y = f([x]) = \begin{cases} f(-1) &, \quad -1 \le x < 0 \\ f(0) &, \quad 0 \le x < 1 \\ f(1) &, \quad 1 \le x < 2 \end{cases}$$

Clearly, real values of x are truncated to integer values in the interval of unity i.e. $[-1, 0)$, $[0, 1)$, $[1.2)$ etc. It means that values of the function $y=f([x])$ will remain same as that of its value corresponding to integral value of x till value of x changes to next interval. Knowing that truncation takes place for successive integral values of x, we divide graph of $y=f(x)$ to correspond to 1 unit segments of x-axis.

For this, we draw lines parallel to y-axis at integral points along x-axis. From intersection point of these lines and function's graph, we draw lines parallel to the x-axis for the whole interval which extends for a unit value. This ensures that function values remain same to that of function value for the lower integral value of x in a particular interval of one.

In short, for constructing the graph of $y = f([x])$, we need to modify the graph of $y = f(x)$ as:

(i) Draw lines parallel to y-axis (vertical lines) at integral values along x-axis to cover the graph of $y = f(x)$.

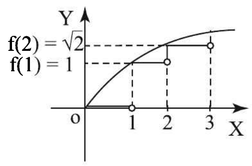
(ii) Identify points of intersections of graph with parallel lines drawn in the earlier step.

(iii) Draw lines of 1 unit parallel to x-axis from intersection points in the direction of positive x. The line ends at the next parallel line on right. Include intersection point but exclude other end of the line.

EXAMPLE 2.7 Draw the graph of $y = \sqrt{[x]}$.

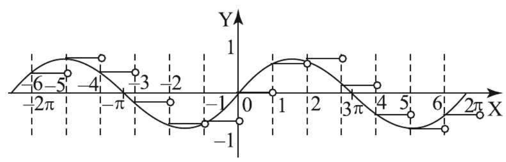
SOLUTION Here $f(x) = \sqrt{x}$, and we have to draw $y = f([x])$.

Following the above construction steps, graph of $y = \sqrt{[x]}$ is drawn as shown here.

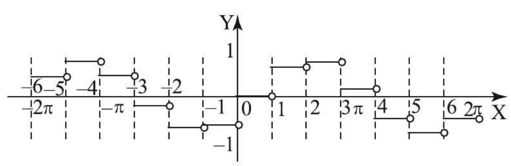


EXAMPLE 2.8 Draw the graph of $y = \sin[x]$

SOLUTION Following the above construction steps, graph of $y = \sin [x]$ is drawn as shown here.

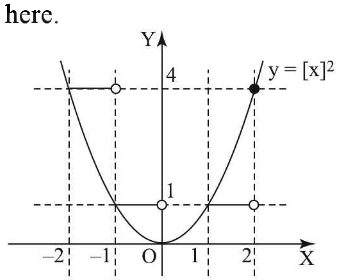


The graph of $y = \sin[x]$:



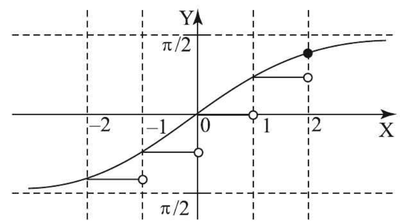
EXAMPLE 2.9 Draw the graph of $y = [x]^2$, $-2 \le x \le 2$.

SOLUTION The graph of $y = [x]^2$ is drawn as shown here.



EXAMPLE 2.10 Draw graph of $\tan^{-1}[x]$, $x \in [-2, 2]$.

SOLUTION The graph of $y = \tan^{-1}[x]$ is drawn as shown here.

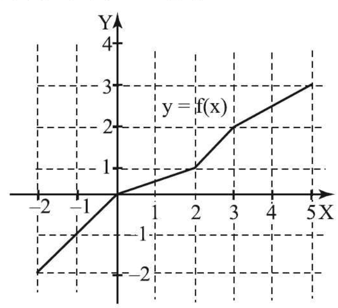


See that the function value corresponding to $x = 2$ is not included in the preceding interval on the graph. As such, we need to put a solid circle at $x = 2$.

Further, we need to remove the original graph of $y = \tan^{-1}x$ (this step is not shown in the figure above).

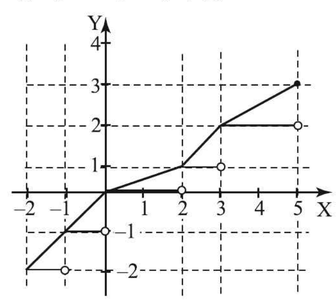
EXAMPLE 2.11 Consider the graph of $y = f(x)$ given below. Draw the graphs of the following functions:

(i) $y = [f(x)]$ (ii) $y = f([x])$

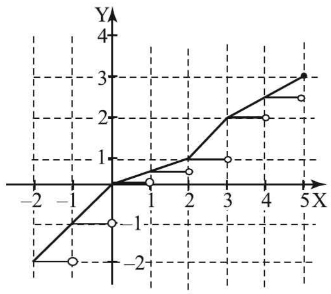


SOLUTION

(i) The graph of $y = [f(x)]$ is shown below:



(ii) The graph of $y = f([x])$ is shown below:



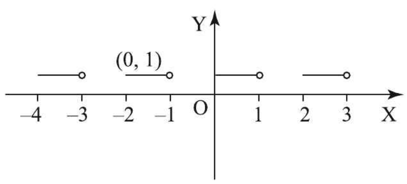
EXAMPLE 2.12 Draw the graph of

$f(x) = \text{sgn}(\cot^{-1} x) + \tan \frac{\pi}{2}[x]$.

SOLUTION Clearly the domain of f(x) is

$x \in \bigcup_{n \in I} \{[2n,\ 2n+1)\}$

Further $f(x) = 1,\ \forall x \in D_f$



We see that f(x) is periodic with period 2.

3. Drawing the graph of y = {f(x)}, from the graph of y = f(x)

The graph of y = f(x) is transformed in y = {f(x)} by applying changes to the output of the function. Whatever be the function values, they will be changed to fraction values following the definition of fraction part function. The values of y will lie in the interval [0,1).

Mathematically, {f(x)} = f(x) – [f(x)]

Clearly, {f(x)} depends on f(x), but lies in the interval [0,1).

For constructing the graph of y={f(x)}, we need to modify the graph of y=f(x) as :

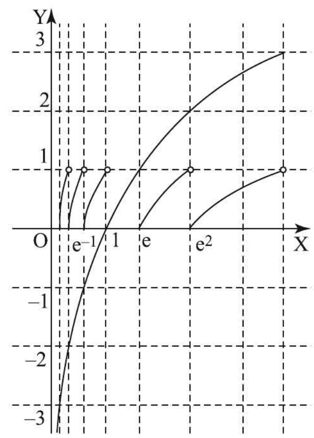
(i) Draw lines parallel to x-axis (horizontal lines) at integral values along y-axis to cover the graph of y = f(x).

(ii) Identify segments of graph between two consecutive vertical intervals. Transfer these segments to y-interval [0,1).

(iii) Include end point corresponding to y = 0 and exclude end point corresponding to y = 1.

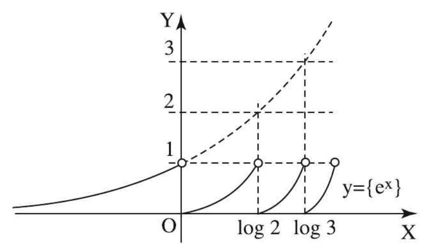
EXAMPLE 2.13 Draw the graph of y = {ln x}.

SOLUTION Following the construction steps, graph of y = {ln x} is drawn as shown here.



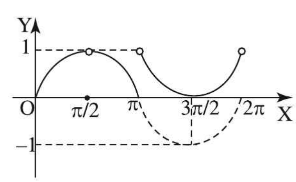
EXAMPLE 2.14 Draw the graph of $y = \{e^x\}$.

SOLUTION



EXAMPLE 2.15 Draw the graph of y = {sinx}.

SOLUTION y = {sinx} = sinx – [sinx]

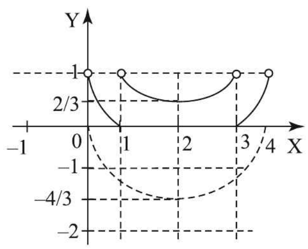


EXAMPLE 2.16 Draw the graph of

$$y = \left\{\frac{x^2 - 4x}{3}\right\}.$$

SOLUTION

$$y = \left\{\frac{x^2 - 4x}{3}\right\} = \frac{x^2 - 4x}{3} - \left[\frac{x^2 - 4x}{3}\right]$$



4. Drawing the graph of y = f({x}), from the graph of y = f(x)

The graph of $y = f(x)$ is transformed to $y = f(\{x\})$ by virtue of changes in the argument values. The independent variable is subjected to fractional part function. This changes the normal real value input to function. Instead of real numbers, independent variable to function is rendered to be fractions irrespective of values of x. A value like $x = -2.3$ is passed to the function as 0.7 in the interval [0,1). Clearly, real values of x are truncated to fraction values. It means that the same set of values of the function $y = f(\{x\})$ corresponding to interval of x defined by [0,1) will repeat in other intervals along x-axis.

The fractional part function is a periodic function with a period of 1. Taking advantage of this fact, we obtain graph of $y = f(\{x\})$ by repeating part of graph for x in [0,1) to other intervals along x-axis. Clearly, the transformed function $y = f(\{x\})$ is periodic with a period of 1. For constructing the graph of $y = f(\{x\})$, we need to modify the graph of $y = f(x)$ as:

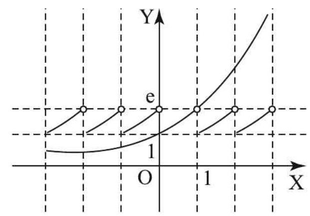
(i) Draw the vertical lines $x = 0$ and $x = 1$.

(ii) Identify part of the graph for values of x in [0,1). Include end point corresponding to $x = 0$ and exclude end point corresponding to $x = 1$.

(iii) Repeat the part of the graph identified in step (ii) for other intervals of x.

EXAMPLE 2.17 Draw the graph of $y = e^{\{x\}}$

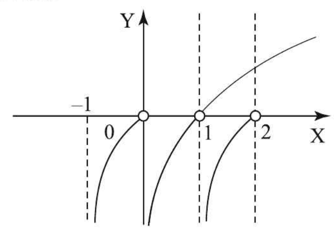
SOLUTION Following the construction steps, graph of $y = e^{\{x\}}$ is drawn by transforming $y = e^x$ as shown here.



We repeat the part of the graph identified between 0 and 1 to other intervals of x.

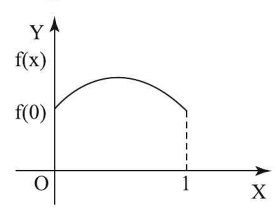
EXAMPLE 2.18 Draw the graph of $y = \ln\{x\}$.

SOLUTION



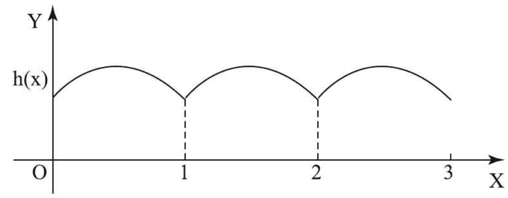
EXAMPLE 2.19 Let $f(x) = x - x^2 + 1$, then draw the graph of the function $y = f(\{x\})$ and show that it does not have any break.

SOLUTION The graph of f in the interval [0, 1] is shown in the figure.



The graph of $y = f(\{x\})$ is periodic with period 1. We see that f is continuous with $f(0) = f(1)$

$\Rightarrow f(\{x\})$ is continuous for all real x.



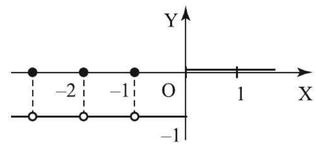
EXAMPLE 2.20 Draw the function

f (x) = [|x|] − |[x]|,

where [x] denotes the greatest integer function.

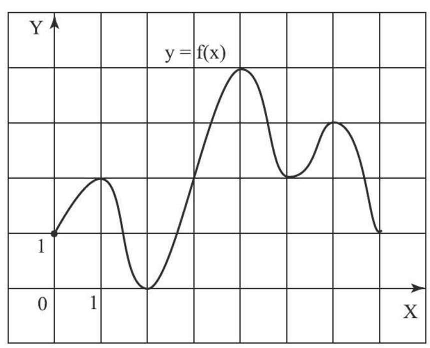
SOLUTION

$$f(x) = [|x|] - |[x]| = \begin{cases} 0 & x = -1 \\ -1 & -1 < x < 0 \\ 0 & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases}$$

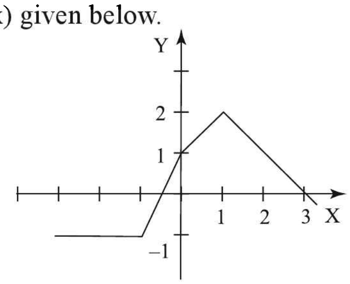


PRACTICE PROBLEMS [F]

1. Draw the graph of following functions where [.] denotes greatest integer function
   (i) $y = [2x] + 1$ (ii) $y = x^{[x]}$, $1 \le x \le 3$
2. Draw the graph of y = [2 sinx].
3. Draw the graph of f(x) = [4 sinx] in [π, 2π].
4. Draw the graph of the following functions:
   (i) $y = 2\{x\}^2 - 3\{x\} + 2$ (ii) y = [sin x + cos x]
5. Draw the graph of $f(x) = \tan\left(\frac{\pi}{2} + \pi\{x\}\right)$.
6. Consider the graph of y = f(x) given below.

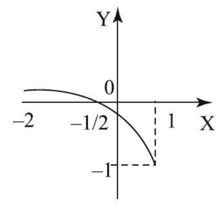


   Draw the graph of y = f([x]).
7. Consider the graph of y = f(x) given below.



   Draw the graph of y = f([x]).

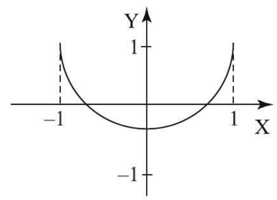
8. Consider the graph of $y = f(x)$ given below.



Draw the graphs of the following functions

(i) $y = [f(x)]$, (ii) $y = [f([x])]$.

9. Consider the graph of $y = f(x)$ given below.



Draw the graphs of the following functions

(i) $y = f(-\{x\})$ (ii) $y = \{|f(x)|\}$.

10. Draw the graphs of the following functions:

(i) $y = \sin \frac{\pi}{2}[x]$ (ii) $y = \cos^{-1}\{2x\}$

11. Let $[x]$ denote the greatest integer in x. Then, find the number of solutions of the equation, $x^2 - 3x + [x] = 0$, in the interval $[0, 3]$.

12. Find the complete solution set of the inequality $[\sin x] \geq \frac{x}{2}$.

13. Find the number of hollow rings used in the graph of $y = [\cos x], -\frac{x}{2} \leq x \leq \frac{7\pi}{2}$.

14. Find the area of the region represented by $[x + y]^2 = 1$ and lying in the first quadrant.

2.10. | GRAPHS OF FUNCTIONS BY OPERATIONS ON FUNCTIONS

Addition of Graphs

The addition of two functions f and g

$$y = f(x) + g(x)$$

is defined for those values of x which lie in the domain of both the functions. For construction of the graph, we follow the following steps:

(i) We draw the graph of f and g on the same axes.

(ii) At any x lying in the common domain, we draw vertical arrows representing the values of the two functions.

(iii) We now draw a vertical arrow at the same x with a height equal to the sum of the heights (with sign) of the above two arrows.

(iv) We repeat the steps (ii) and (iii) at several values of x and join the tips of the resultant arrows by a smooth curve.

EXAMPLE 2.1 Construct the graph of the following functions:

(i) $y = x + \dfrac{1}{x}$ (ii) $y = 2^x + 2^{-x}$

(iii) $y = x + \sin x$.

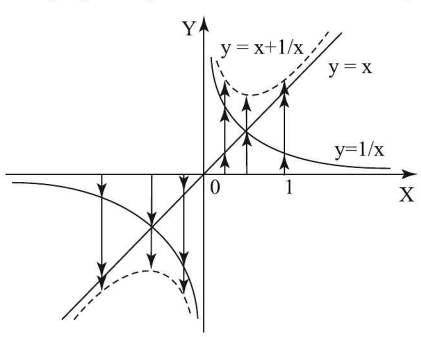
SOLUTION

(i) The graph of $y = x + (1/x)$ is constructed by combining two graphs, $y_1 = x$ and $y_2 = 1/x$. In other words, for each admissible value of the argument (that is, for every $x \neq 0$) the corresponding ordinate y is built up as an algebraic sum of the ordinates y_1 and y_2 corresponding to the same value of the argument.

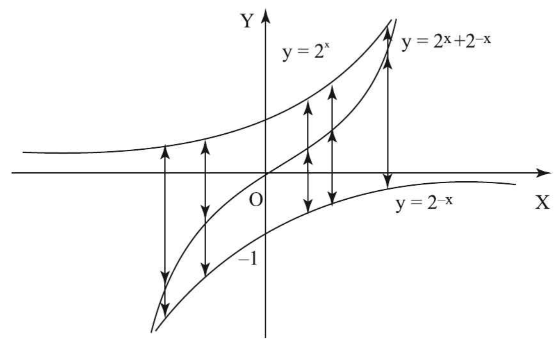
It is easy to figure out the shape of the graph of the function on the positive x-axis : for each value $x > 0$, the corresponding ordinate of the straight line $y_1 = x$ has to be increased by an ordinate of the hyperbola $y_2 = 1/x$ corresponding to the same value of x. It is quite obvious that for a positive x tending to zero, the expression $x + (1/x)$ tends to ∞, and for x tending to ∞, the desired graph approaches the line $y_1 = x$, since the summand $1/x$ becomes smaller and smaller.

It is easy in this case to determine the smallest value of the function y (recall that so far we are only considering positive values of x): indeed, when $x > 0$ the inequality $x + (1/x) \geq 2$ holds true, which is to say the smallest value is equal to 2 and is reached when $x = 1$.

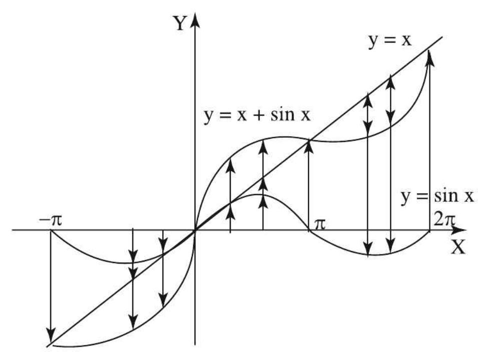
Construction of the graph is similar on the negative x-axis as well. Incidentally, we could take advantage of the fact that the function y is odd and, hence, its graph is symmetric about the origin.



(ii) We follow the steps as outlined above for addition of graphs of $y = 2^x$ and $y = 2^{-x}$. The graph of $y = 2^x + 2^{-x}$ is shown below.



(iii) We combine the graphs of $y = x$ and $y = \sin x$, to obtain the graph of $y = x + \sin x$, which is shown below.



EXAMPLE 2.2 Construct the graph of the following functions:

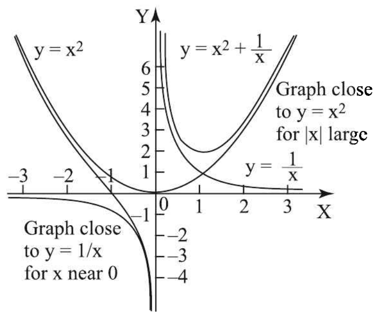
(i) $y = x^2 + \dfrac{1}{x}$

(ii) $y = \dfrac{x^2 - 1}{x}$

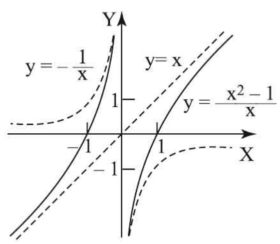
(iii) $y = \operatorname{cosec} x - \dfrac{1}{x^2}, \ 0 < x < \pi$

SOLUTION

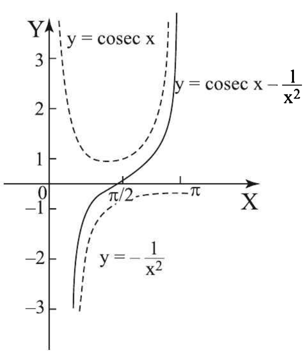
(i) The graph of $y = x^2 + \frac{1}{x}$ is shown below:



(ii) The graph of $y = \frac{x^2 - 1}{x}$ is shown below:



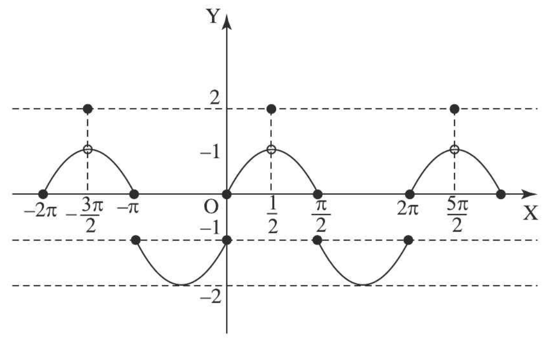
(ii) The graph of $y = \operatorname{cosec} x - \frac{1}{x^2}$, $0 < x < \pi$ is shown below:



EXAMPLE 2.3 Draw the graph of $y = [\sin x] + \sin x$, where [.] denotes the greatest integer function.

SOLUTION Since $-1 \le \sin x \le 1$, $[\sin x] = -1, 0, 1$

We have $[\sin x] + \sin x = 2$ when $x = 2n\pi + \frac{\pi}{2}$, $n \in I$ and in general, $-2 \le [\sin x] + \sin x \le 2$



Multiplication of graphs

The multiplication of two functions f and g

$y = f(x) \,.\, g(x)$

is defined for those values of x which lie in the domain of both the functions.

In general, it is difficult to predict the graph of the product of two functions. But, it can be done if one of the functions is a trigonometric function like sin x, cos x, $\sin^2 x$, etc.

That is, we wish to draw functions like

$y = f(x) \sin x$,

$y = f(x) \cos x$, $y = f(x) \sin^2 x$, etc.

EXAMPLE 2.4 Construct the graph of the function $y = x \sin x$.

SOLUTION Taking advantage of the fact that the formula is a product, we apply the technique of multiplication of graphs. The required graph will be constructed by multiplying two graphs $y_1 = x$ and $y_2 = \sin x$. In other words, for each value of the argument, the corresponding ordinate y is constructed as a product of the ordinates y_1 and y_2 which correspond to the same value of the argument. We first construct the graph of the function y for nonnegative values of the argument. For each value of x we multiply the value of the corresponding ordinate of the straight line $y_1 = x$ and the value of the ordinate of the sine curve $y_2 = \sin x$, and are thus able to construct a smooth curve that gives an approximate idea of the behaviour of the graph of the function on the nonnegative x-axis.

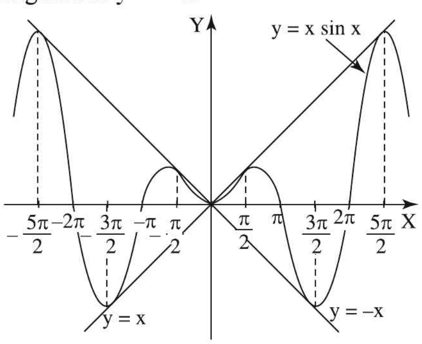
The aspect of the curve can be improved somewhat by plotting a few characteristic points. First of all, it is clear that y = 0 for those values

of x for which sin x = 0, and so the graph of the function y crosses the positive x-axis at the points $x = k\pi$, k = 0, 2, ... Furthermore, for x > 0 the obvious inequality

$-x \le x \sin x \le x$ holds true.

This means that for positive values of the argument the graph of the function y does not extend above the straight line y = x or below the straight line y = – x. In this case, the points of the graph of the function y that correspond to the values of x > 0 for which sin x = 1, i.e., to the values

$x = (\pi/2) + 2k\pi$, k = 0, 1, 2,..., lie on the straight line y = x, and the points corresponding to the values of x > 0 for which sin x = – 1, i.e., to the values $x = (3\pi/2) + 2k\pi$, k = 0, 1, 2,..., lie on the straight line y = – x.

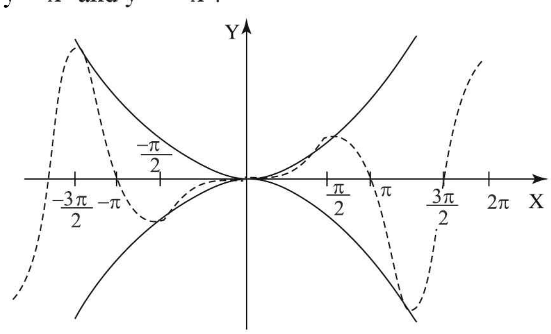


It is very easy to construct the graph of the function y on the negative x-axis since the function y is even and so its graph is symmetric about the y-axis.

EXAMPLE 2.5 Draw the graph of $y = x^2 \sin x$.

SOLUTION We $-x^2 \le x^2 \sin x \le x^2$

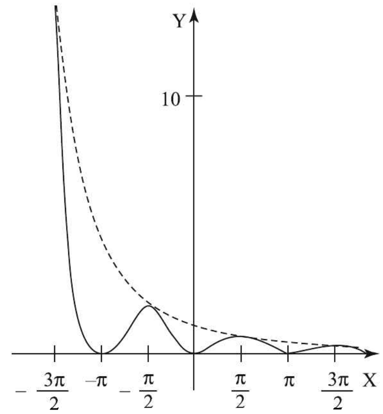
Thus, the graph lies between two auxiliary graphs $y = x^2$ and $y = -x^2$.



EXAMPLE 2.6 Draw the graph of $y = (1.3)^{-2x} \sin^2 x$.

SOLUTION

We have $0 \le (1.3)^{-2x} \sin^2 x \le (1.3)^{-2x}$



Reciprocal of Graph

Given the graph of y = f(x) we now wish to draw the graph of $y = \frac{1}{f(x)}$.

For construction of the graph, we follow the following steps:

(i) We draw the graph of f.

(ii) We note that the sign scheme of $\frac{1}{f(x)}$ is same as that of f(x). At the points where f(x) is zero, the graph of $\frac{1}{f(x)}$ goes to infinity with due consideration of sign.

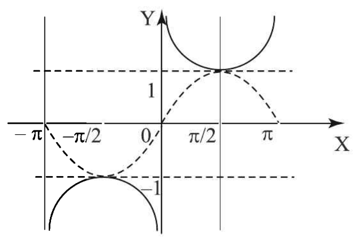
(iii) At the points where f(x) is 1 or – 1, the graph of $\frac{1}{f(x)}$ is also 1 or – 1. That is, the points of intersection of the graph of y = f(x) with the lines y = ± 1 also lie on the graph of $y = \frac{1}{f(x)}$.

(iv) When f(x) increases, $\frac{1}{f(x)}$ decreases and vice-versa.

(v) When f(x) approaches infinity, then $\frac{1}{f(x)}$ approaches zero.

EXAMPLE 2.7 Construct the graph of the function $y = \text{cosec } x$ using the graph of $y = \sin x$.

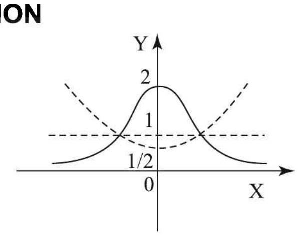
SOLUTION $y = \sin x$, $y = \text{cosec } x$



EXAMPLE 2.8 Construct the graph of the function

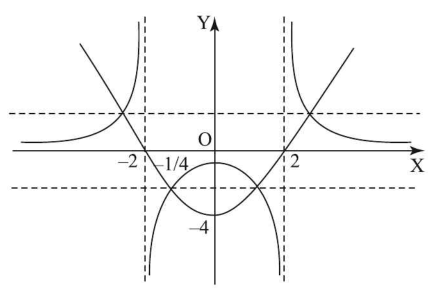
$$y = \frac{1}{x^2 + 1/2}.$$

SOLUTION



EXAMPLE 2.9 Construct the graph of the function $y = \dfrac{1}{x^2 - 4}$.

SOLUTION



Composition of graphs

EXAMPLE 2.10 Sketch the graph of the function $y = 2^{1/x}$.

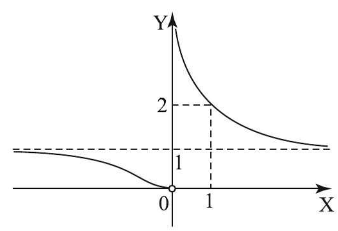
SOLUTION Here we have to construct the graph of a composition of two functions.

The domain of the function y consists of all real numbers except $x = 0$.

Since for $x > 0$ the exponent $1/x > 0$, it follows that $y > 1$ for all positive values of the argument. Note that $y = 2$ when $x = 1$. If x increases without bound, then the expression $1/x$ decreases to zero monotonically, remaining positive, and so $2^{1/x}$ decreases to unity monotonically, remaining greater than unity.

When x is positive and tends to zero, the exponent $1/x$ increases without bound and, hence, $2^{1/x}$ also increases without bound. This enables us to sketch a graph of the function y when $x > 0$.

It is easy to demonstrate that the inequality $0 < y < 1$ is true on the negative x-axis. Using similar reasoning, we construct the graph of the function y for $x < 0$ as well. Note that the origin does not belong to the graph.



EXAMPLE 2.11 Construct the graph of the function

$f(x) = 2^{\cos x}$.

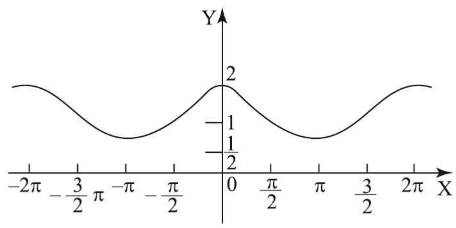
SOLUTION The function $f(x) = 2^{\cos x}$ is defined on the entire real axis. The function cosx being even, the function f(x) is also even.

The function cos x is periodic with a period 2π, therefore, the function f(x) is also periodic with the same period. Consequently, it is sufficient to construct the graph of the function on the interval $[0, 2\pi]$.

On the interval $[0, \pi]$ the function cosx decreases from 1 to –1 and, therefore, the function f(x) decreases from 2 to 1/2.

On the interval $[\pi, 2\pi]$ the function cosx increases from – 1 to 1 and the function f(x) increases from 1/2 to 2. At the points $x = 2\pi n$, $n \in I$, the function f(x) possesses the maximum value

equal to 2, and at the points $x = 2\pi n + \pi$, $n \in I$, the function f(x) possesses the minimum value equal to 1/2. The graph of the function $f(x) = 2\cos x$ is shown below.



EXAMPLE 2.12 Construct the graph of the function

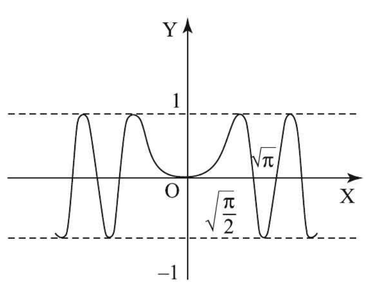
$y = \sin x^2$.

SOLUTION Letusfirstconsiderthenonnegative values of the argument and partition the semi-axis $x \geq 0$ into intervals over which the function y increases or decreases.

If x^2 increases from 0 to $\pi/2$ (which is to say that x increases from 0 to $\sqrt{\pi/2}$), then $\sin x^2$ increases from 0 to 1. If x^2 increases from $\pi/2$ to $3\pi/2$ (i.e., x increases from $\sqrt{\pi/2}$ to $\sqrt{3\pi/2}$), then $\sin x^2$ decreases from 1 to – 1; if x^2 increases from $3\pi/2$ to $5\pi/2$ (i.e. x increases from $\sqrt{3\pi/2}$ to $\sqrt{5\pi/2}$), then $\sin x^2$ increases from – 1 to 1, and so on.

The graph of the function y therefore is of wavelike nature with an amplitude of 1. It is easy to obtain the x-intercepts of this graph : all we have to do is solve the equation $\sin x^2 = 0$. It is clear that the nonnegative roots of this equation are the numbers $x = \sqrt{k\pi}$, $k \in N$.

On the negative x-axis, the graph is drawn at once since the function y is even.



EXAMPLE 2.13 Draw the graph of the function $y = \sin (2^x)$.

SOLUTION We have $-1 \leq \sin (2^x) \leq 1$.

The function $y = \sin (2^x)$ is non periodic.

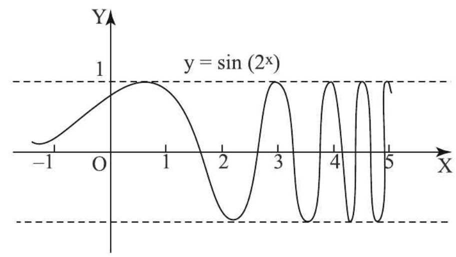
The graph cuts the x-axis when

$\sin (2^x) = 0 \Rightarrow 2^x = n\pi$, $n \in N$

$\therefore$ $x = \log_2 (n\pi)$, $n \in N$

i.e., $x = \log_2 \pi, \log_2 2\pi, \log_2 3\pi, ...$

The graphs of $y = \sin (2^x)$ is shown below.

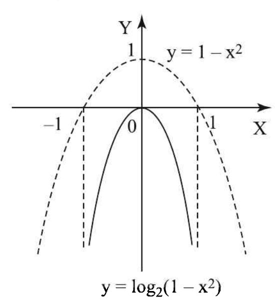


EXAMPLE 2.14 Construct the graph of the function $y = \log_2 (1 - x^2)$.

SOLUTION First draw the graph of the inner function $y_1 = 1 - x^2$. This is a parabola which is shown in the figure by the dashed curve. We then construct the graph of the logarithm of this function.

For $x = 0$ we have $y = \log_2 1 = 0$. If x is increased from 0 to 1, then, as may be seen from the graph of the inner function, $1 - x^2$ decreases from 1 to 0 and so $\log_2 (1 - x^2)$ decreases from 0 to $-\infty$. The function is even.

Similar graph is drawn in (–1, 0). For the remaining values of x, that is, for $x \leq -1$ and $x \geq 1$, we have $1 - x^2 \leq 0$, so that $\log_2 (1 - x^2)$ is meaningless. The graph of the function $y = \log_2 (1 - x^2)$ is shown below.



EXAMPLE 2.15 Construct the graph of the function

$y = \log_{\sin x} 1/2$.

SOLUTION The domain of this function is the collection of all values of x for which, simultaneously, $\sin x > 0$ and $\sin x \neq 1$, that is, the set

$$2k\pi < x < \frac{\pi}{2} + 2k\pi,\ \frac{\pi}{2} + 2k\pi < x < (2k+1)\pi,\ k \in I.$$

The function y is clearly periodic with period 2π. And so we can confine ourselves to an interval of length 2π, say, the interval $0 \le x \le 2\pi$. But the function is meaningful (over this interval) only for $0 < x < \pi/2$, and $\pi/2 < x < \pi$. It is precisely on these intervals that we first of all have to construct the graph (then we can simply extend it over the entire domain because of its periodicity).

The function y can, in its domain, be rewritten as

$$y = \frac{1}{\log_{1/2} \sin x}$$

We first of all construct the graph of the function

$y_1 = \log_{1/2} \sin x$.

It will only interest us over the interval $0 < x < \pi$. Taking the piece of the sine curve

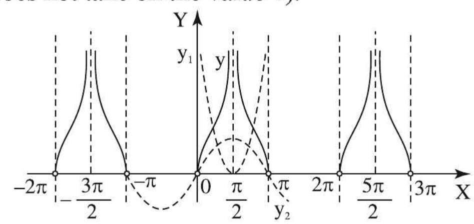
$y_2 = \sin x$

corresponding to this interval, we can use composition of graphs to obtain the graph of the composite function y_1 (the graphs y_1 and y_2 are depicted by dashed lines).

We now consider the interval $0 < x < \pi/2$. Since for any value of x in this interval, the corresponding value of the function y is the reciprocal of the value of y_1 corresponding to the same value of the argument, it is easy to obtain a rough sketch of the graph of y for $0 < x < \pi/2$ (the solid line in the figure). Note that the origin does not belong to the graph.

It is easy to prove that the function y monotonically increases when x varies from 0 to $\pi/2$. if x increases from 0 to $\pi/2$, then sin x increases monotonically from 0 to 1, and then $\log_{1/2} \sin x$ decreases monotonically from ∞ to 0; and hence, the value of y increases monotonically from 0 to ∞.

Let us stress that if x approaches $\pi/2$, remaining all the time less than this value, then the value of the function y_1, tends to zero, remaining all the time positive, and therefore the value of y increases without bound. But if x approaches zero and remains positive, then the value of y_1 increases without bound and so the value of y tends to zero (although it does not take on the value 0).



The construction is similar for the graph of the function at hand when $\pi/2 < x < \pi$.

Graph of Inverse of a Function

If $f : A \to B$ is a bijective function, then the function, $f^{-1} : B \to A$ which associates each element $b \in B$ and $a \in A$ such that $f(a) = b$ is called the inverse of the function f.

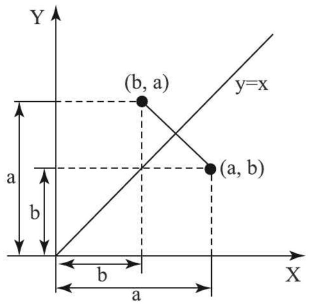
$y = f(x) \Leftrightarrow f^{-1}(y) = x,\ \forall\, x \in A, y \in B$.

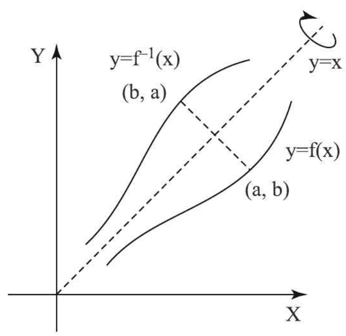
When a formula for f^{-1} is obtained by solving the equation $y = f(x)$ for x as a function of y, the resulting formula has y as the independent variable. It is perferable to have x as the independent variable for f^{-1} We solve $y = f(x)$ for x as a function of y, and then replace y by x in the final formula for f^{-1} in which case the final equation will be $y = f^{-1}(x)$.

Our objective is to explore the relationship between the graphs of f and f^{-1}. For this purpose, it will be desirable to use x as the independent variable for both functions, which means that we will be comparing the graphs of $y = f(x)$ and

$y = f^{-1}(x)$.

If (a, b) is a point on the graph $y = f(x)$, then $b = f(a)$. This is equivalent to the statement that $a = f^{-1}(b)$, which means that (b, a) is a point on the graph of $y = f^{-1}(x)$. In short, reversing the coordinates of a point on the graph of f produces a point on the graph of f^{-1}. However, the geometric effect of reversing the coordinates of a point is to reflect that point about the line $y = x$ (Figure), and hence the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about this line.





EXAMPLE 2.16 A function is given as:

$f : R \rightarrow R$ by $f(x) = x^3 + 1$.

Construct the graph of the inverse function.

SOLUTION Here, we assume that the given function is bijective.

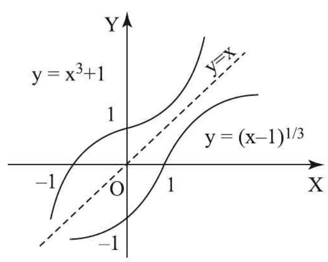
From $\quad y = x^3 + 1$

$\qquad x = (y - 1)^{1/3}$

Changing notations, $f^{-1}(x) = (x - 1)^{1/3}$.

The inverse function, therefore, is given as:

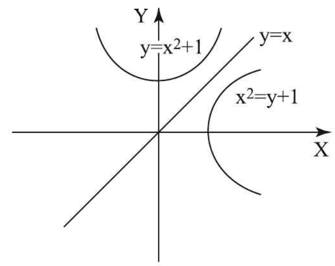
$f^{-1} : R \rightarrow R$ where $f^{-1}(x) = (x - 1)^{1/3}$.



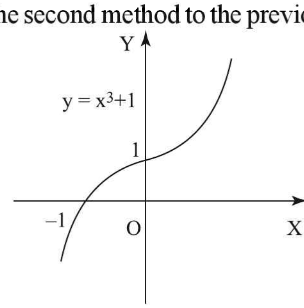
Note

From the graph of $y = f(x)$, we obtain the graph of $x = f(y)$ either by reflecting the graph of $y = f(x)$ w.r.t. to the line $y = x$, or by first rotating the graph of $y = f(x)$ by 90° anti-clockwise, and then reflecting the resulting graph about the vertical axis.

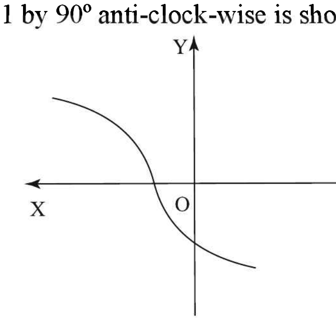
For example, the graph of $x^2 = y + 1$ can be obtained from the graph of $y^2 = x + 1$.



We apply the second method to the previous question.

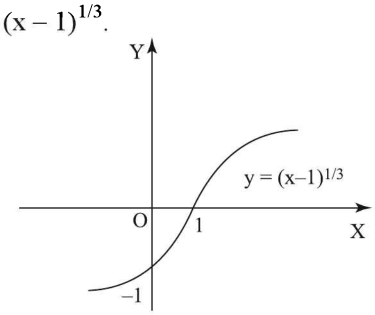


The graph obtained by rotating the graph of $y = x^3 + 1$ by 90° anti-clock-wise is shown below:



We now reflect the above graph about the vertical axis to obtain the graph of $x = y_3 + 1$

i.e. $y = (x - 1)^{1/3}$.



PRACTICE PROBLEMS [G]

1. Construct the graph of the following functions:

   (i) $y = |x + 2| + |x - 3|$ (ii) $y = |2x + 1| - |2x - 2|$ (iii) $y = x + \dfrac{x}{|x|}$.

2. Construct the graph of the following functions:

   (i) $y = \dfrac{x^4 + 1}{x^2}$ (ii) $y = \sec x + \dfrac{1}{x},\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

   (iii) $y = \tan x + \dfrac{1}{x^2},\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

3. Draw the graph of $y = \sqrt{x}\,\sin(1/x)$.

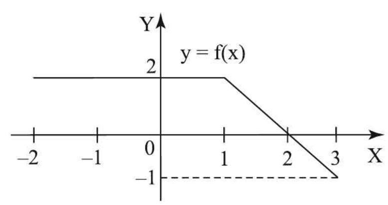
4. Draw the graph of the function $y = \dfrac{\sin x}{x}$.

5. Draw the graph of the following functions:

   (i) $y = x \sin \dfrac{1}{x}$ (ii) $y = 2^x \cdot \cos x$ (iii) $y = |x| \sin^2 x$.

6. Draw the graph of the function $y = \dfrac{1}{x^2 - 1}$.

7. Use the graph of f to sketch a graph of $y = \dfrac{1}{f(x)}$ on the interval [–2, 3].



8. Construct the graphs of the following functions:

   (i) $y = \log_2(2 - x)^2$ (ii) $y = \dfrac{|\ln x|}{\ln x}$

   (iii) $y = e^{|\ln x|}$ (iv) $|y| = \log_2(-x)$.

9. Construct the graphs of the following functions :

   (i) $y = \log^2(x^2 - 2x)$ (ii) $y = \log_2\left|\dfrac{x}{x-1}\right|$

   (iii) $y = \log_2 \sin x$.

10. Construct the graphs of the following functions :

    (i) $y = \ln(x^2 + 1)$ (ii) $y = 2^{\sin x}$

    (iii) $y = \sin(\sin^{-1}(\log_{1/2} x))$ (iv) $y = x^{\log_x 2}$.

11. Construct the graphs of the following functions :

    (i) $y = \sin^{-1} x^2$ (ii) $y = -\cos^{-1}(-x^2)$

    (iii) $y = \tan^{-1}(x^2 - 1)$ (iv) $y = \cot^{-1}(4 - x^2)$.

12. Draw the graph of the function

$$f(x) = \tan^{-1}\left(\frac{1-x^2}{1+x^2}\right).$$

2.11 | GRAPH USING DIFFERENTIAL CALCULUS

Monotonic Functions

A monotonic function is one whose successive values are increasing, decreasing or constant.

In general, a function may or may not maintain its order of change in its domain. However, we can always identify monotonic behaviour in an appropriately chosen subset of domain unless it is a point function.

Consider the graph of sine function. As a whole, the function is not monotonic as the order of the function is not preserved over the domain of the function, which is R. However, if we consider an interval, say, between 0 and $\pi/2$, then we find that function keeps increasing with the increasing independent variable. Therefore, sine function is monotonic in this interval.

From the point of view of monotonic behaviour, we classify functions in the following categories :

(i) Strictly increasing : Function values change as independent variable varies in accordance with following condition:
If $x_1 < x_2$ then $f(x_1) < f(x_2)$, for all $x_1, x_2 \in$ domain.

(ii) Non-decreasing or increasing:
If $x_1 < x_2$ then $f(x_1) \le f(x_2)$, for all $x_1, x_2 \in$ domain.

(iii) Strictly decreasing:
If $x_1 < x_2$ then $f(x_1) > f(x_2)$, for all $x_1, x_2 \in$ domain.

(iv) Non-increasing or decreasing :
If $x_1 < x_2$ then $f(x_1) \ge f(x_2)$, for all $x_1, x_2 \in$ domain.

According to the definition, a constant function is an increasing, decreasing or both kinds of function.

An increasing or non-decreasing function actually captures the notion of an overall increasing function, which is intermittently constant and thereby distinguishes this class from strictly increasing order.

Derivative and Nature of Function

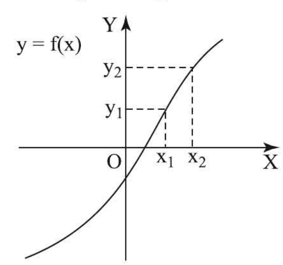
We shall learn subsequently that the first derivative of a function is defined in terms of ratio of the differences between two successive values of the function and that of the independent variable, where the difference in independent variable is small. We have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Depending on the monotonic nature of the function, the relative values of f(x) and f(x+h) are different and so is the sign of the first derivative.

Strictly Increasing Function

The value of the function increases as the value of the independent variable increases. In other words, the preceding values are less than successive values that follow. Mathematically,
if $x_1 < x_2$, then $f(x_1) < f(x_2)$.



As $f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ domain, the difference $f(x + h) - f(x)$ is positive for small h. This implies that the first derivative of the function is

positive. Note that the first derivative can be equal to zero for few points in the interval in which it is strictly increasing.

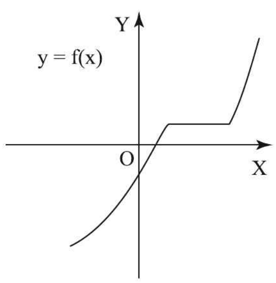
Thus, for a strictly increasing function, $f'(x) \geq 0$, where equality sign holds for distinct points only and not on a continuous sub-interval of the domain.

The order of a function provides an easy technique to determine the range of a continuous function, corresponding to a given domain interval. For example, if domain of a continuously increasing function, $f(x)$, is $[x_1, x_2]$, then the least value of the function is $f(x_1)$ and the greatest value of the function is $f(x_2)$. Hence, the range of the function is $[f(x_1), f(x_2)]$.

Non-decreasing Function or Increasing

The successive value of function increases or remains constant as the value of the independent variable increases. In other words, the preceding values are less than or equal to successive values that follow. Mathematically,

if $x_1 < x_2$ then $f(x_1) \leq f(x_2)$



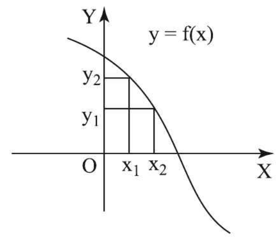
As $f(x_1) \leq f(x_2)$ all $x_1, x_2 \in$ domain, the difference $f(x + h) - f(x)$ is non-negative for small h. This implies that the first derivative of the function is non-negative. Note that the first derivative can be equal to zero for few points or sub-intervals in which it is increasing.

Thus, for non-decreasing function $f'(x) \geq 0$ where equality sign holds for few points or a continuous sub-interval of the domain.

Strictly Decreasing Function

The successive value of function decreases as the value of the independent variable increases. In other words, the preceding values are greater than successive values that follow. Mathematically,

if $x_1 < x_2$ then $f(x_1) > f(x_2)$



As $f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ domain, the difference $f(x + h) - f(x)$ is negative for small h. This implies that the first derivative of the function is negative. Note that the first derivative can be equal to zero for few points in the interval in which it is strictly decreasing.

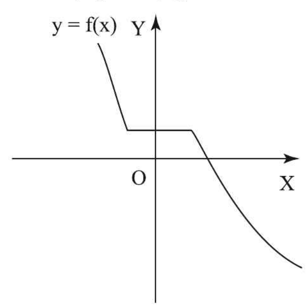
Thus, for strictly decreasing function, $f'(x) \leq 0$, where equality sign holds for distinct points only and not on a continous sub-interval of the domain.

If the domain of a continuously decreasing function $f(x)$, is $[x_1, x_2]$, then the least value of the function is $f(x_2)$ and the greatest value of the function is $f(x_1)$. Hence, the range of the function is $[f(x_2), f(x_1)]$.

Non-increasing Function or Decreasing

The successive value of function decreases or remains constant as the value of the independent variable increases. In other words, the preceding values are greater than or equal to successive values that follow. Mathematically,

if $x_1 < x_2$ then $f(x_1) \geq f(x_2)$



As $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ domain, the difference $f(x + h) - f(x)$ is non-positive for small h. This implies that the first derivative of function is

non-positive. Note that the first derivative can be equal to zero at points or in sub-intervals in which it is decreasing.

Thus, for a non-decreasing function, $f'(x) \le 0$, where equality sign holds for few points or a continuous sub-interval of the domain.

Intervals of Monotonicity

In the discussion about monotonic functions, we observed that the order of change in the function values is related to the sign of the derivative of the function. The task of finding increasing and decreasing intervals is, therefore, about finding sign of derivative of function in different intervals and determining points or intervals where derivative turns zero or does not exist.

The steps for determining intervals of monotonicity are as follows :

(i) Determine the derivative of the function $f'(x)$ and find the critical points i.e. points where the derivative is zero or does not exist.

(ii) Determine the sign of $f'(x)$ in different intervals formed by the critical points.

(iii) Determine monotonic nature of function in accordance with following categorization:

$f'(x) \ge 0$: equality holding for points only $\Rightarrow$ strictly increasing interval

$f'(x) \ge 0$: equality holding for sub-intervals $\Rightarrow$ non-decreasing or increasing interval

$f'(x) \le 0$: equality holding for points only $\Rightarrow$ strictly decreasing interval

$f'(x) \le 0$: equality holding for subintervals $\Rightarrow$ non-increasing or decreasing interval.

In order to illustrate the above steps, we consider a function, $f(x) = x^2 - x$

Its first derivative is $f'(x) = 2x - 1$

Here, $2x - 1 = 0$ gives the critical point as 1/2. The first derivative, $f'(x)$, is positive for $x > 1/2$ and negative for $x < 1/2$. The signs of derivative obey strict inequalities. It means that the function is strictly increasing for $x > 1/2$ and strictly decreasing for $x < 1/2$.

EXAMPLE 2.1 Determine the intervals of monotonicity of the function

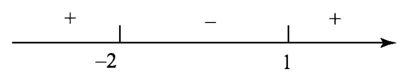
$f(x) = 2x^3 + 3x^2 - 12x + 1$.

SOLUTION The first derivative of the given function is $f'(x) = 6x^2 + 6x - 12$.

Now, the roots of the quadratic equation $6x^2 + 6x - 12 = 0$ are $x = 1, -2$

$\Rightarrow\ f'(x) = 6(x - 1)(x + 2) = 0$

Here, coefficient of x^2 is positive. Hence, the sign scheme is:



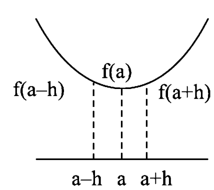
The function f(x) is strictly increasing in the interval $(-\infty, -2) \cup (1, \infty)$ and strictly decreasing in the interval $(-2, 1)$.

Minimum and Maximum Values

In general, a function may have multiple minimum and maximum values in the domain of function. These minimum and maximum values are "local" minimum and maximum, which belongs to finite sub-intervals within the domain of function. The least minimum and greatest maximum in the domain of function are "global" minimum and maximum respectively in the entire domain of the function. Clearly, the least and greatest values are one of the local minimum and maximum values.

Local Minimum and Maximum

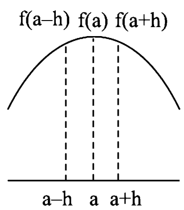
The idea of local minimum and maximum is clearly understood from graphical representation. A function has a local minimum at a point $x = a$, if the function values in the immediate neighbourhood on either side of point are less than the value at the point. Mathematically, $f(a) < f(a + h)$ and $f(a) < f(a - h)$ for small h.



A function has a local maximum at a point

$x = a$, if the function values in the immediate neighbourhood on either side of point are greater than the value at the point. Mathematically,

$f(a) > f(a + h)$ and $f(a) > f(a - h)$ for small h.



Extreme value or extremum is either a minimum or maximum value. A function, f(x), has a extremum at $x = a$, if it has either a minimum or maximum value at that point.

Minimum and maximum of function cannot occur at points where function is not defined, because there is no function value corresponding to undefined points.

A function can have minimum and maximum at points where it is discontinuous. Consider fraction part function in the finite domain. The function is not continuous at $x = 1$, but minimum occurs at this point (recall its graph).

A function can have minimum and maximum at points where it is continuous but not differentiable. In other words, maximum and minimum can occur at corners. For example, the modulus function $|x|$ has minimum at the corner point $x = 0$ (recall its graph).

We see that minimum and maximum of function can occur at the following points:

(a) Points on the graph of function, where derivative of function is zero.

(b) Points where function is continuous but not differentiable. Consider, for example, the corner of modulus function graph at $x = 0$.

(c) Points where function is discontinuous (note that the function is discontinuous but not undefined).

We can summarize that critical points are those points where (i) derivative of function does not exist or (ii) derivative of function is equal to zero.

The first statement covers the cases described at (b) and (c) above. The second statement covers the case described at (a). We should note that critical points are points where minimum or maximum "can" exist, not that they will exist.

The First Derivative Test

Suppose that $x = a$ is a critical point of a continuous function $y = f(x)$.

(i) If $f'(x)$ changes from positive to negative at $x = a$, then f has a local maximum at $x = a$.

(ii) If $f'(x)$ changes from negative to positive at $x = a$, then f has a local minimum at $x = a$.

(iii) If $f'(x)$ does not change sign at $x = a$ (that is, if $f'(x)$ is positive on both sides of $x = a$ or negative on both sides), then f has no local maximum or minimum at $x = a$.

Method to determine local maximum or minimum of continuous functions using the first derivative. We follow the following steps:

(i) Determine the first derivative.

(ii) Find sign scheme of first derivative using the critical points.

(iii) If the function is decreasing to the left and increasing to the right of a critical point in the sign scheme, then the function attains a point of local minimum.

(iv) If the function is increasing to the left and decreasing to the right of a critical point in the sign scheme, then the function attains a point of local maximum.

EXAMPLE 2.2 Find the points of local maxima or minima for $f(x) = \sin 2x - x$, $x \in (0, \pi)$.

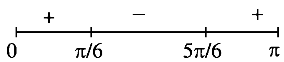
SOLUTION

$$f(x) = \sin 2x - x$$

$$f'(x) = 2\cos 2x - 1$$

$$f'(x) = 0 \Rightarrow \cos 2x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Sign scheme of $f'(x)$



Since the sign scheme of first derivative changes from positive to negative about $x = \pi/6$, we have a local maximum at $x = \pi/6$.

And it changes from negative to positive about $x = 5\pi/6$, hence we have a local minimum at $x = 5\pi/6$.

EXAMPLE 2.3 Find the local maximum and minimum values of the function $y = \dfrac{x^2 - 7x + 6}{x - 10}$.

SOLUTION This function is not defined for $x = 10$. The function is continuous except at this point.

$$\Rightarrow y' = \frac{(x-10)(2x-7) - (x^2 - 7x + 6)}{(x-10)^2}$$

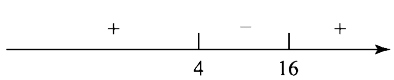
$$= \frac{2x^2 - 27x + 70 - x^2 + 7x - 6}{(x-10)^2}$$

$$\Rightarrow y' = \frac{x^2 - 20x + 64}{(x-10)^2}$$

Now denominator is a positive number for all x. Thus, the sign scheme of the first derivative is same as that of the numerator. In order to draw sign scheme, we need to factorize the numerator.

$x^2 - 20x + 64 = (x - 4)(x - 16)$

Hence, critical points are 4 and 16. The sign scheme is as shown in the figure. At $x = 4$, the function is increasing to its left and decreasing to its right. It means $f(x)$ has a local maximum at $x = 4$.

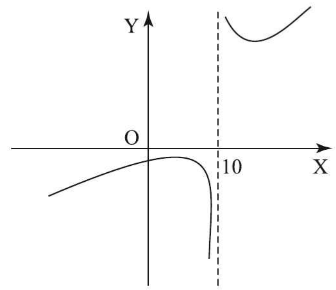


Maximum value $= f(4) = 1$.

At $x = 16$, the function is decreasing to its left and increasing to its right. It means $f(x)$ has a local minimum at $x = 16$.

Minimum value $= f(16) = 25$.

The graph of the function is shown below:



Test for Local Maximum/Minimum when $f(x)$ is not differentiable at $x = a$.

Case 1: When $f(x)$ is continuous at $x = a$ and $f'(a^-)$ and $f'(a^+)$ exist and are non-zero, then $f(x)$ has a local maximum or minimum at $x = a$ if $f'(a^-)$ and $f'(a^+)$ are of opposite signs.

If $f'(a^-) > 0$ and $f'(a^+) < 0$, then $x = a$ is a point of local maximum . If $f'(a^-) < 0$ and $f'(a^+) > 0$, then $x = a$ is a point of local minimum.

Case 2: When $f(x)$ is continuous and one or both of $f'(a^-)$ and $f'(a^+)$ are either zero or does not exist, then we should consider the signs of $f'(a - h)$ and $f'(a + h)$, where h is a small positive arbitrary number.

If $f'(a - h) > 0$ and $f'(a + h) < 0$, then $x = a$ is a point of local maximum. If $f'(a - h) < 0$ and $f'(a + h) > 0$, then $x = a$ is a point of local minimum.

Case 3: If $f(x)$ is discontinuous at $x = a$, then we should find about the existence of local maxima/minima using the basic definition of local maxima/minima i.e. compare the values of $f(x)$ at the neighbouring points of $x = a$.

EXAMPLE 2.4 Find the minimum value of $f(x) = |x^3|$.

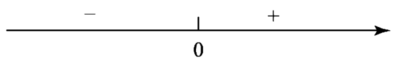
SOLUTION $f(x) = |x^3| = \begin{cases} x^3 & ; x \geq 0 \\ -x^3 & ; x < 0 \end{cases}$

For $x > 0$, $f'(x) = 3x^2 > 0$

Since the first derivative is positive, the given function is increasing for $x > 0$.

For $x < 0$, $f'(x) = -3x^2 < 0$.

Since the first derivative is negative, the given function is decreasing for $x < 0$. The sign scheme of the derivative is shown here:



At $x = 0$, the function is decreasing to its left and increasing to its right. It means that the function has a local minimum at $x = 0$ and the minimum value is 0.

EXAMPLE 2.5 Draw the graph of the function

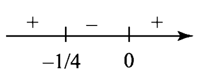
$f(x) = x + \sqrt{|x|}$.

SOLUTION

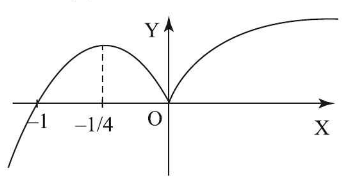
$$f(x) = \begin{cases} x + \sqrt{x} & \text{if } x \geq 0 \\ x + \sqrt{-x} & \text{if } x < 0 \end{cases}.$$

$$f'(x) = \begin{cases} 1 + \dfrac{1}{2\sqrt{x}} & \text{if } x > 0 \\ 1 - \dfrac{1}{2\sqrt{-x}} & \text{if } x < 0 \end{cases}$$

There are two critical points : $x = -1/4, 0$. The derivative fails at $x = 0$.



The graph of $f(x)$ is shown below:



The Second Derivative Test

Let $x = a$ be a stationary point of a function f (i.e. $f'(a) = 0$). The function f has a local maximum at a if $f''(a)$ is negative, and a local minimum if $f''(a)$ is positive.

EXAMPLE 2.6 Determine the local minimum and maximum values of the function

$$f(x) = \frac{x^3}{3} - x.$$

SOLUTION Differentiating with respect x, we have :

$f'(x) = x^2 - 1 = (x - 1)(x + 1)$

The roots of the corresponding equation are –1 and 1. Now, differentiating with respect to x again,

$f''(x) = 2x$

Putting, $x = -1$, $f''(-1) = -2 < 0$. Hence, function has maximum value at $x = -1$.

$$\text{Maximum value} = \frac{(-1)^3}{3} - (-1) = -\frac{1}{3} + 1 = \frac{2}{3}.$$

EXAMPLE 2.7 Determine the local minimum and maximum values of function

$f(x) = 2x^3 - 9x^2 + 12x - 11.$

SOLUTION Differentiating with respect x, we have

$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$
$= 6(x - 1)(x - 2)$

The roots are 1 and 2. Now, differentiating with respect to x again,

$f''(x) = 12x - 18$

Putting, $x = 1$, $f''(x) = -6 < 0$. Hence, the function has a local maximum at $x = 1$.

The local maximum value

$= 2x^3 - 9x^2 + 12x - 10$
$= 2.1^2 + 12.1 - 11 = -6$

Putting, $x = 2$, $f''(x) = 6 > 0$. Hence, the function has a minimum value at $x = 2$.

The local minimum value $= 2x^3 - 9x^2 + 12x - 11$
$= 2.2^3 - 9.2^2 + 12.2 - 11 = -7.$

EXAMPLE 2.8 Determine the local minimum and maximum values of function

$f(x) = x^5 - 5x^4 + 5x^3 - 5.$

SOLUTION Differentiating with respect x, we have

$f'(x) = 5x^4 - 20x^3 + 15x^2$

Equating to zero, we have:

$5x^4 - 20x^3 + 15x^2 = 0$

$\Rightarrow x^4 - 4x^3 + 3x^2 = 0$

$\Rightarrow x^2(x^2 - 4x + 3) = 0$

$\Rightarrow x^2(x - 1)(x - 3) = 0$

The roots are 0, 1 and 3.

Now, differentiating with respect to x again,

$f''(x) = 20x^3 - 60x^2 + 30x$

Putting $x = 0$, $f''(x) = 0$. We differentiate further to investigate this point.

$f'''(x) = 60x^2 - 120x + 30$

Putting, $x = 0$, $f'''(x) = 30 > 0$. Hence, the function has neither minimum nor maximum value at $x = 0$.

Putting, $x = 1$, $f''(x) = -10 < 0$. Hence, the function has a local maximum value at $x = 1$, the maximum value being $1^5 - 5.1^4 + 5.1^3 - 5 = -4$,

Putting, $x = 3$, $f''(x) = 90 > 0$. Hence, the function has a local minimum value at $x = 3$, the minimum value being

$$3^5 - 5.3^4 + 5.3^3 - 5 = 243 - 5.81 + 5.27 - 5 = -32.$$

Global Minimum and Maximum

Global minimum is also known as "least value" or "absolute minimum". A function has one global minimum in the domain [a, b]. Global minimum, $f(\alpha)$, is less than or equal to all function values in the domain. Thus,

$f(\alpha) \le f(x)$ for all $x \in [a, b]$.

If the domain interval is open like (a, b), then global minimum, $f(\alpha)$, also needs to be less than or equal to function value, which is infinitesimally close to boundary values. Hence,

$f(\alpha) \le f(x)$ for all $x \in (a, b)$

$f(\alpha) \le \lim_{x \to a^+} f(x)$

$f(\alpha) \le \lim_{x \to b^-} f(x)$

Similarly, the global maximum is also known as "greatest value" or "absolute maximum". A function has one global maximum in the domain [a, b]. Global maximum, $f(\beta)$, is greater than or equal to all function values in the domain. Thus,

$f(\beta) \ge f(x)$ for all $x \in [a, b]$

If the domain interval is open like (a, b), then global maximum, $f(\beta)$, also needs to be greater than or equal to the function value, which is infinitesimally close to the boundary values. Hence,

$f(\beta) \ge f(x)$ for all $x \in (a, b)$

$f(\beta) \ge \lim_{x \to a^+} f(x)$

$f(\beta) \ge \lim_{x \to b^-} f(x)$

Determining Minimum and Maximum Values

In order to find the global maximum and minimum of a continuous function f(x) in [a, b]:

Find out all the critical points of f(x) in (a, b). Let $c_1, c_2, \ldots, c_n$ be the different critical points.

Find the value of the function at these critical points. Let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at the critical points.

Let $M = \max \{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

and $m = \min \{f(a), f(c), f(c_2), \ldots, f(c_n), f(b)\}$

Then M is the global maximum (greatest value) of f(x) in [a, b] and m is the global minimum (least value) of f(x) in [a, b].

Extreme Value Theorem

The extreme value theorem of continuous function guarantees existence of minimum and maximum values in a closed interval. Mathematically, if f(x) is a continuous function in the closed interval [a,b], then there exists $f(\alpha) \le f(x)$ and $f(\beta) \ge f(x)$ such that $f(\alpha)$ is global minimum and $f(\beta)$ is global maximum of function.

Range of Function

If a function is continuous, and the global minimum value is "m" and the global maximum value is "M", in the domain of function, then the range of the function is [m, M].

If a function is not continuous or if a function can not assume certain values, then we need to suitably analyze the function and find the range.

EXAMPLE 2.9 Find the greatest and the least value of the function $f(x) = 3x - x^3$ on the closed interval [–2, 3].

SOLUTION We find the derivative:

$f'(x) = 3 - 3x^2$; $3 - 3x^2 = 0$,
i.e. $x = \pm 1$ are stationary points.

Next we determine the values of the function at these points : $f(1) = 2$, $f(-1) = -2$. Then we calculate the values of the given function at the end points of the closed interval:

$f(-2) = 2$, $f(3) = -18$. From the four values we have obtained, we choose the greatest and the least values.

Thus, the greatest value of the function on the given interval is equal to 2 and the least to –18.

EXAMPLE 2.10 Find the greatest and least values of

$$f(x) = 3x^4 - 8x^3 - 18x^2 + 1$$

SOLUTION We have

$f(x) = 3x^4 - 8x^3 - 18x^2 + 1$ and

$f'(x) = 12x^3 - 24x^2 - 36x = 12x(x + 1)(x - 3)$.

The critical points are $x = -1, 0, 3$ [$\because f'(1) = 0$ and $f'(0)$ = does not exist] and the end points are infinite.

Now, $f(-1) = -6$, $(f(23) = -134$, $f(0) = 1$,
and $f(\pm\infty) \to \infty$.

Hence, the least value of the function is – 134 whereas the greatest value does not exist.

EXAMPLE 2.11 Graph the function

$$y = x - \cos 2x.$$

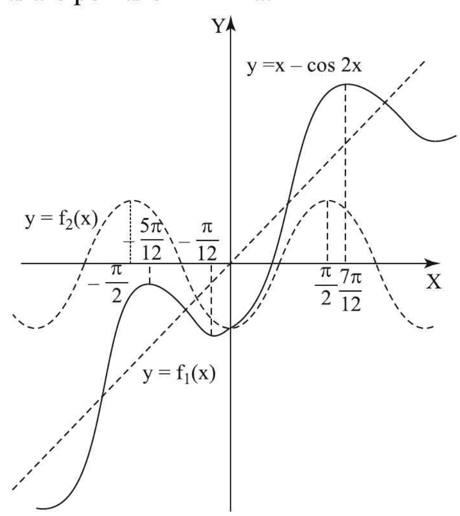
SOLUTION To construct the graph of the given function, we apply the method of "adding the graphs" of two functions. Let us construct the graphs of the functions $f_1(x) = x$ and $f_2(x) = -\cos 2x$, using dashed lines (see figure).

Now, the ordinate of any point of the graph of the function $x - \cos 2x$ is equal to the sum of the ordinates of the points on the graphs of auxiliary functions (for an arbitrary value of x). By adding the ordinates of points, it is possible to construct a sufficient number of points belonging to the graph of the function $x - \cos 2x$, and then to join them with a smooth curve.

$y' = 1 + 2 \sin 2x$. The derivative vanishes at the points where $1 + 2 \sin 2x = 0$, that is, at the points $x = (-1)^{n+1} \dfrac{\pi}{12} + \dfrac{\pi n}{2}$ $(n \in I)$.

Note that if $n = 2m + 1$, $m \in I$, then $x = \dfrac{7\pi}{12} + \pi m$. At these points, the derivative changes sign from plus to minus, and therefore $x = \dfrac{7\pi}{12} + \pi m$ $(m \in I)$ are points of maxima.

If $n = 2m$, $m \in I$, then $x = -\dfrac{\pi}{12} + \pi m$ $(m \in I)$, and, when passing through these points, the derivative changes sign from minus to plus, therefore these points are points of minima.



Concavity

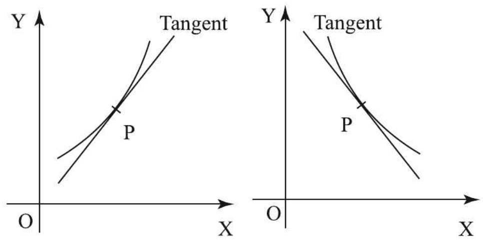
An arc of a curve is said to be concave up or down if it lies entirely on one side of the tangent drawn through any point of the arc. It is of course supposed here that tangent can be drawn at each point of the arc.

(a) Concave upwards

A curve is said to be concave up at a point P when in the immediate neighbourhood of P it lies wholly above the tangent at P.

Mathematically, $\dfrac{dy}{dx}$ increases as x increases.

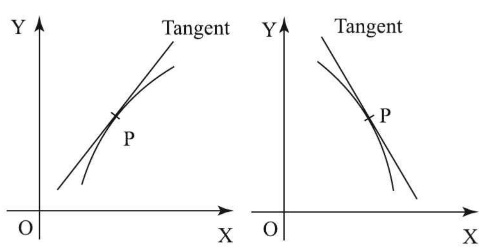
$$\Rightarrow \frac{d^2y}{dx^2} > 0$$



(b) Convex upwards or concave downwards

A curve is said to be concave down when in the immediate neighbourhood of P it lies wholly below the tangent at P.

$\frac{dy}{dx}$ decreases as x increases. $\Rightarrow \frac{d^2y}{dx^2} < 0$

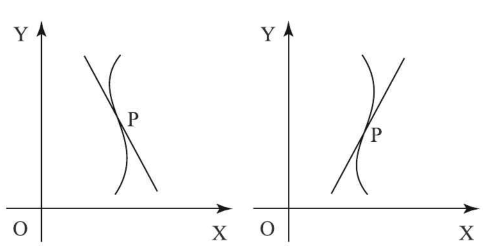


Thus, for a curve $y = f(x)$, if the second derivative $f''(x)$ is everywhere positive within an interval, the arc of the curve $y = f(x)$ corresponding to that interval is concave up. If the second derivative $f''(x)$ is everywhere negative in an interval, the corresponding arc of the curve $y = f(x)$ is concave down.

Point of Inflection

A point of a curve separating its concave up arc from a concave down arc is termed as point of inflection.

At a point of inflection, the tangent intersects the curve. In the vicinity of such a point the curve lies on both sides of its tangent drawn through that point.



Mathematically,

(i) $\frac{d^2y}{dx^2} = 0$ or does not exist at the point.

(ii) $\frac{d^2y}{dx^2}$ changes its sign about the point.

The concavity of the curve $y = f(x)$ can change only at points where $f''(x) = 0$ or $f''(x)$ does not exist. Let $x = c$ be such a point and the signs of $f''(c - h)$ and $f''(c + h)$ be opposite, then the point $x = c$ is called a point of inflection.

EXAMPLE 2.12 Find the point of inflection in the curve $y = x\,e^{-x}$

SOLUTION

$y = xe^{-x}$

$y' = e^{-x} - x\,e^{-x} = (1\,^{-x})e^{-x} > 0$ for $x < 1$

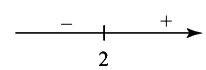
Hence the curve is increasing for $x < 1$ and decreasing for $x > 1$.

$y'' = -e^{-x} - [e^{-x} - xe^{-x}] = e^{-x}[-1 - 1 + x]$

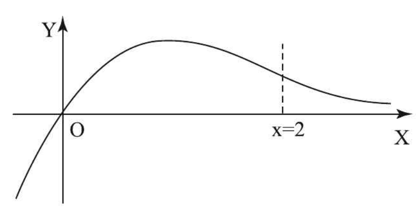
$= (x - 2)e^{-x}$

$y'' = 0 \Rightarrow x = 2.$

Sign scheme of y''



The curve is concave down for $x < 2$ and concave up for $x > 2$. Since the sign of y'' changes about the point $x = 2$, we have a point of inflection at this point.



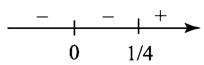
EXAMPLE 2.13 Draw the function

$f(x) = x^{1/3}(x - 1).$

SOLUTION $f(x) = x^{1/3}(x - 1)$

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot\frac{1}{x^{2/3}} = \frac{1}{3x^{2/3}}[4x - 1]$$

Sign scheme of $f'(x)$

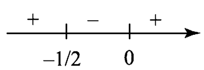


Hence, f is increasing for $x > \frac{1}{4}$ and decreasing for $x < \frac{1}{4}$.

Now $\quad f''(x) = \frac{4}{9} \cdot \frac{1}{x^{2/3}} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{x^{5/3}}$

$$= \frac{2}{9x^{2/3}}\left[2 + \frac{1}{x}\right] = \frac{2}{9x^{2/3}}\left[\frac{2x+1}{x}\right]$$

Sign scheme of $f''(x)$

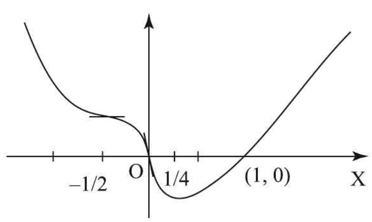


$\therefore$ f (x) is concave up for $x < -\frac{1}{2}$ and $x > 0$.

It is concave down for $-\frac{1}{2} < x < 0$.

Further $x = -\frac{1}{2}$ and $x = 0$ are points of inflection.

The graph of f (x) is shown below:



EXAMPLE 2.14 Draw the graph of $f(x) = \ln(1 - \ln x)$.

SOLUTION The domain of $f(x) = \ln(1 - \ln x)$ is $(0, e)$.

Also, $f(1) = 0$.

$f'(x) = -\frac{1}{(1 - \ln x)} \cdot \frac{1}{x} < 0$

$\Rightarrow$ decreasing $\forall$ x in its domain.

Now $\lim\limits_{x \to e^-} f(x) = -\infty$; $\lim\limits_{x \to 0^+} f(x) = \infty$

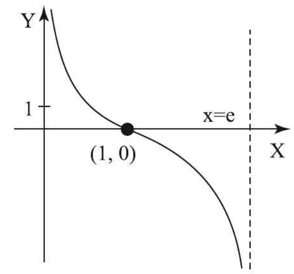
Thus, $x = 0$ and $x = e$ are two vertical asynmptotes

$f''(x) = \frac{-\ln x}{x^2(1 - \ln x)^2}$

f(x) is concave up for $0 < x < 1$ and concave down for $1 < x < e$.

The concavity changes at $x = 1$, which is a point of inflection.

The graph is shown below:



Graph of Polynomial Functions

Let us first understand the behaviour of a polynomial function at infinity.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_0$

We write f(x) in the form

$$f(x) = x^n\left(a_n + \frac{a_{n-1}}{x} + \ldots. + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right).$$

Then we conclude that the behaviour of f(x) as $x \to \pm\infty$ is much the same as that of its leading term $a_n x^n$, because all the terms that have powers of x in the denominator approach zero as $x \to \pm\infty$.

In particular, if $a_n > 0$, then

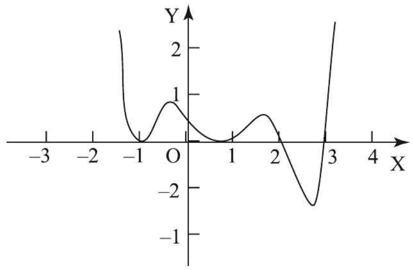
$$\lim_{x \to \infty} f(x) = \infty \qquad \text{...(1)}$$

meaning that f(x) increases without bound as $x \to \infty$. Also

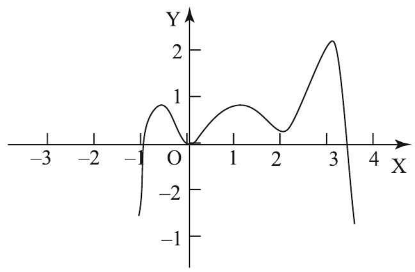
$$\lim_{x \to -\infty} f(x) = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases} \qquad \text{...(2)}$$

If $a_n < 0$, simply reverse the signs on the right-hand sides in equations (1) and (2). It follows that the graph of any (nonconstant) polynomial function exhibits one of the four "behaviours as $x \to \pm\infty$" that are illustrated in figure.

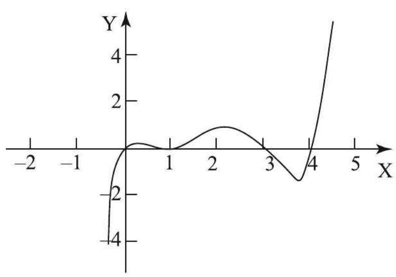
(a) Northwest-northeast if n is even and $a_n > 0$



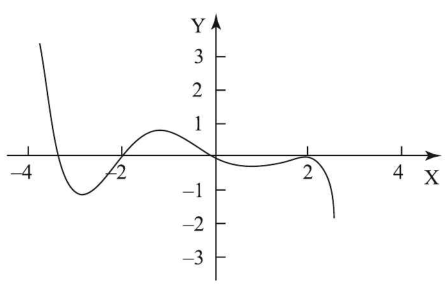
(b) Southwest-southeast if n is even and $a_n < 0$



(c) Southwest-northeast if n is odd and $a_n > 0$



(d) Northwest-southeast if n is odd and af′ < 0



EXAMPLE 2.15 Determine the intervals of monotonicity of the function $f(x) = 3x^4 - x^3$. Find the point of inflection and draw the graph of f(x).

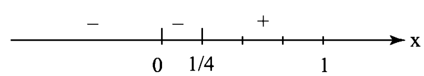
SOLUTION Its first derivative is:

$$f'(x) = 12x^3 - 3x^2 = 3x^2(4x - 1)$$

Here, critical points are 0,1/4. We need to write given function in terms of factors as:

$$f'(x) = 3x^2(4x - 1)$$

Since zero is repeated even times, the sign of the derivative does not change at x = 0. The sign scheme is shown in the figure.



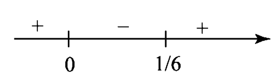
The first derivative, f′(x), is positive for x > 1/4 and negative for x < 1/4 (zero at x = 0 only).

It means that the function is strictly increasing for x > 1/4 and strictly decreasing for x < 1/4.

$$f''(x) = 36x^2 - 6x = 6x(6x - 1)$$

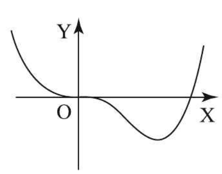
$$f''(x) = 0 \Rightarrow x = 0, \frac{1}{6}.$$

Again examining sign of f″(x)



Thus $x = 0, \frac{1}{6}$ are the inflection points

Hence the graph of f(x) is



PRACTICE PROBLEMS [H]

1. Draw the graphs of the following functions :
   (i) $f(x) = 12x^3 + 24x^2 + 11x - 1$ (ii) $f(x) = x^3 - 8x^2 + 20x - 13$
   (iii) $f(x) = 3x^2 - 2x^3$
2. In the graph of $f(x) = x(x + 1)(x - 1)$
   (i) for which values of x is $f(x) = 0$
   (ii) where does the graph cross the x axis ?
   (iii) where does the graph cross the y axis ?
3. Draw the graphs of the following functions:
   (i) $y = (x + 1)(x - 1)(x - 2)$ (ii) $y = (x - 1)(x - 3)^2(x - 4)$
   (iii) $y = (x - 1)(x - 3)^2$ (iv) $y = -x^2(x - 4)$
4. Find the greatest and the least value of the function $f(x) = x^3 - 3x^2 + 2$ on the interval $[-1, 3]$ and construct its graph on that interval.
5. Given a function $y = (6x^2 - x^4)/9$. Investigate and make a rough drawing of the graph.
6. Draw the graph of the following functions :
   (i) $y = 3x^4 - 4x^3$ (ii) $y = \frac{2}{5}x - \frac{1}{2}x^3 + \frac{1}{10}x^5$
   (iii) $y = (x + 2)^{2/3} - (x - 2)^{2/3}$ (iv) $y = (x + 1)^{2/3} + (x - 1)^{2/3}$
7. Draw the graph of the function $y = e^{-x^2}$.
8. Draw the graph of the function $y = 2 \sin x + \cos 2x$
9. Consider the function $f(x) = x \cos x - \sin x$. Prove that (i) f is strictly decreasing at $x = 0$, (ii) f has a point of inflection at $x = 0$, (iii) f has a local maxima at $x = -\pi$. Also draw the graph of the function.

3.12 | GRAPH OF RATIONAL FUNCTIONS

A rational function of x is defined as the ratio of two polynomials of x, say P(x) and Q(x) where $Q(x) \neq 0$ i.e. f(x) is a rational function of x if

$$f(x) = \frac{P(x)}{Q(x)};\ Q(x) \neq 0,$$

Following are some examples of rational functions of x.

$$f(x) = \frac{2x + 1}{x^2 + x + 1};$$

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 5x + 6};\ x \neq 2, x \neq 3$$

$$f(x) = \frac{x^4 + 2x^3 + 3x + 1}{(x - 1)^2};\quad x \neq 1$$

The following properties of rational functions help in drawing their graphs :

(i) Singularity
(ii) Holes
(iii) Asymptotes : vertical, horizontal and slant

Singularities

Singularities are values of x for which denominator of rational function is zero. The function is not defined for such values and as such these values are excluded from the domain set of the function.

Factorising numerator and denominator of rational function helps to identify singularities of rational function. Singularities correspond to x values resulting from equating linear factors in denominator to zero. The important thing to note here is that singularity occurs when the denominator of a rational function turns zero no matter whether linear factor in the denominator cancels out with the linear factor in numerator or not. To understand this point, let us consider few rational functions :

$$f(x) = \frac{(x-1)(x+2)}{(x-1)(x+1)}$$

$$g(x) = \frac{(x-1)^2(x+2)}{(x-1)(x+1)}$$

$$h(x) = \frac{(x-1)(x+2)}{(x-1)^2(x+1)}$$

We can see that h(x) contains a linear factor $(x-1)$ in the denominator after cancellation of like linear factors. On the other hand, functions f(x) and g(x) do not contain $(x-1)$ in the denominator after cancellation of like linear factors. The function g(x), however, contains $(x-1)$ in the numerator after cancellation. Notwithstanding these possibilities, the denominator of the rational function turns zero at $x = 1$. As such, the point specified by $x = 1$ is a singularity for all three function forms shown above.

We shall see that either a hole or vertical asymptote occurs at the point of singularity. It depends on how do the linear factors in the denominator relate to linear factors in the numerator.

Holes

Hole exists at a singularity when corresponding linear factor of the denominator cancel out completely or when linear factor remains in the numerator after cancellation. Hole is a point on the graph where the function value is not defined.

We determine the x-coordinate by equating the linear factor in the denominator to zero. Its y-coordinate is obtained by plugging x-value in the reduced (after cancellation of common factors in numerator and denominator) function form. As pointed out, there are two situations with respect to position of a hole:

(i) If linear factor cancels completely, then hole lies anywhere but not on the x-axis.

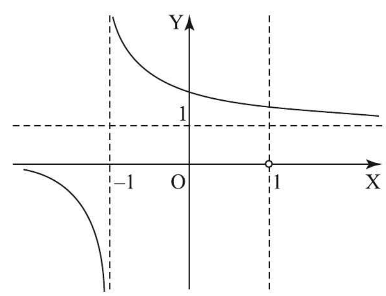
$$f(x) = \frac{(x-1)(x+2)}{(x-1)(x+1)}$$

Here, singularities occur at $x = -1$ and 1. The linear factor $(x-1)$ is present in both numerator and denominator and as such cancels out completely. Therefore, there is a hole at $x = 1$. On the other hand, no cancellation is involved at $x = -1$. There exists a vertical asymptote at $x = -1$. We shall learn about vertical asymptote subsequently.

Now, the y-coordinate of hole at $x = 1$ is :

$$f(x) = \frac{(x+2)}{(x+1)} = \frac{3}{2} = 1.5$$

The graph of the function f(x), is shown in the figure below :



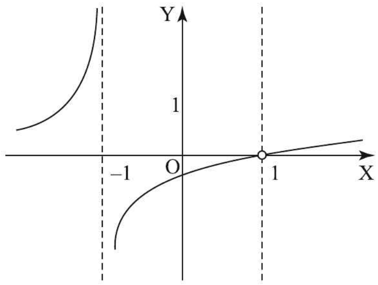
(ii) If the linear factor remains in the numerator after cancellation, then hole lies on the x-axis. The graph tends to intercept x-axis. As such, hole exists at the x-axis.

$$g(x) = \frac{(x-1)^2(x+2)}{(x-1)(x+1)}$$

Here, singularities occur at $x = -1$ and 1. There is a vertical asymptote at $x = -1$, but a hole at $x = 1$. The y-coordinate of hole is:

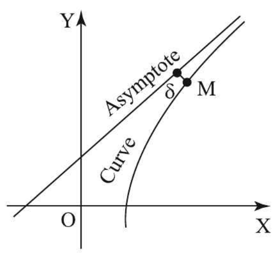
$$g(x) = \frac{(x-1)(x+2)}{(x+1)} = \frac{0.3}{2} = 0$$

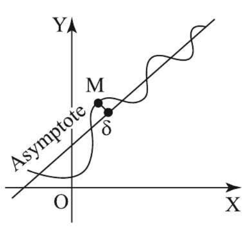
Thus, the hole lies on the x-axis.



Asymptotes

A straight line is called an asymptote to a curve, if the distance δ from the variable point M of the curve to this straight line approaches zero as the point M tends to infinity.





Vertical Asymptote

Vertical asymptote is a vertical line to which graph of the function comes closer and closer. Vertical asymptotes correspond to very large y-values, where difference between x-value and asymptote is infinitesimally small. An equation of vertical asymptote has the form, x = a.

The line x = a is a vertical asymptote if at least one of the following statements is true :

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

Vertical asymptotes occur at singularity when linear factor in the denominator remains after cancellation or otherwise. Let us investigate three functions given earlier for existence of vertical asymptote.

$$f(x) = \frac{(x-1)(x+2)}{(x-1)(x+1)}$$

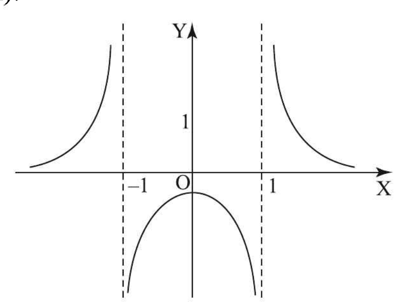
$$g(x) = \frac{(x-1)^2(x+2)}{(x-1)(x+1)}$$

$$h(x) = \frac{(x-1)(x+2)}{(x-1)^2(x+1)}$$

For function, f(x) singularities exists at x = 1 and –1. The linear factor (x + 1), does not cancel out. Thus, a vertical asymptote exists at x = –1. See the graph drawn earlier for f(x).

For function, g(x), also singularities exists at x = 1 and –1. The linear factor (x + 1), does not cancel out. Thus, a vertical asymptote exists at x = –1. See the graph drawn earlier for g(x).

For function, h(x), also singularities exists at x = 1 and –1. Here, a vertical asymptote exists at x = 1 as the linear factor (x – 1) remains in the denominator. The linear factor (x + 1), however, does not cancel out. Thus, a vertical asymptote also exists at x= –1. See the graph shown here for h(x).



The function value assumes large values close to singularity where asymptote exists. The values are directed either in the same or opposite directions. It depends on the reduced function. If the reduced function has linear factor raised

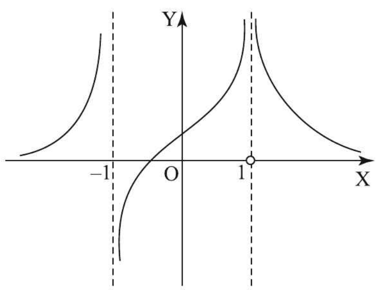
to even power, then values asymptotes in the same direction. On the other hand, if the reduced function has linear factor raised to odd power, then values asymptotes in opposite directions. Let us consider function as defined here,

$$f(x) = \frac{(x-1)(x+2)}{(x-1)^3(x+1)}$$

After simplification, the function reduces to :

$$f(x) = \frac{(x+2)}{(x-1)^2(x+1)}$$

Clearly, (x – 1) is raised to even power 2. The graph asymptotes towards large positive values i.e. in the same direction from either side of the asymptote. On the other hand, the linear factor (x + 1) is raised to 1 i.e. odd power. Hence function value asymptotes in opposite directions.

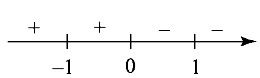


EXAMPLE 2.1 Draw the graph of $y = \frac{x^2}{x^2-1}$.

SOLUTION The function $y = \frac{x^2}{x^2-1}$ is not defined at x = ± 1.

$$y = 1 + \frac{dy}{dx} \Rightarrow y' = -\frac{2x}{(x^2-1)^2}$$

Sign scheme of y ′



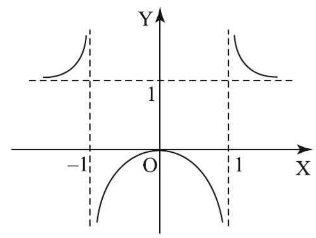
$$\frac{dy}{dx} = 0 \Rightarrow x = 0 \text{ (point of local maxima)}$$

As $x \to 1^+, y \to \infty$; $x \to 1^-, y \to -\infty$

$x \to -1^+, y \to -\infty$; $x \to -1^-, y \to \infty$

Thus, the lines x = ±1 are two vertical asymptotes.

The graph of $y = \frac{x^2}{x^2-1}$ is as shown:



EXAMPLE 2.2

Let $f(x) = \frac{x^3 - 2x^2 - 9x + 18}{x^2 - 4}$. Then show that f is increasing every where in its domain and the graph has one hole and one vertical asymptote.

SOLUTION

$$f(x) = \frac{x^2(x-2) - 9(x-2)}{(x-2)(x+2)}$$

$$= \frac{(x^2-9)(x-2)}{(x-2)(x+2)} = \frac{x^2-9}{x+2} \quad (x \neq 2)$$

$$f(x) = \frac{x^2-9}{x+2}, x \neq 2.$$

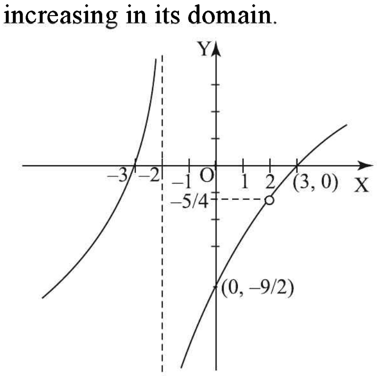
Further, as $x \to -2^+, y \to -\infty$

$x \to -2^-, y \to +\infty$

⇒ The graph has a hole at x = 2 and a vertical asymptote at x = – 2.

$$f'(x) = \frac{2x(x+2) - (x^2-9)}{(x+2)^2} = \frac{x^2+4x+9}{(x+2)^2} > 0$$

⇒ f is increasing in its domain.



EXAMPLE 2.3 Find the number of roots of the equation

$$3x = \frac{1}{(x+1)^3}.$$

SOLUTION

Let $f(x) = \frac{1}{(x+1)^3} - 3x$

$$f'(x) = -\frac{3}{(x+1)^4} - 3 < 0$$

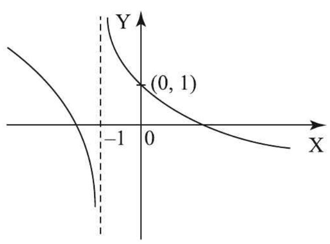
Note that f has a discontinuity at $x = -1$.

Hence f (x) is strictly decreasing in two separate intervals $(-\infty, -1)$ and $(-1, \infty)$.

Also as $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$

Hence the function has one positive and one negative root.

The graph of y = f(x) is as shown



Horizontal Asymptote

Horizontal asymptote is a horizontal line including the x–axis to which the graph of function comes closer and closer. The difference between y–value and asymptote is infinitesimally small for large values of x. An equation of horizontal asymptote has the form, y = c.

If either $\lim_{x\to\infty} f(x) = c$ or $\lim_{x\to-\infty} f(x) = c$, then the line y = c is a horizontal asymptote of the curve y = f(x).

Existence of horizontal asymptote in case of rational functions depends on the degree of the polynomial in the numerator (n) and degree of polynomial in the denominator (m). There are following three cases:

(i) If n > m, then there is no horizontal asymptote. However, if n = m + 1, then there exists slant asymptote.

(ii) If n < m, then x–axis is horizontal asymptote.

(iii) If n = m, then horizontal asymptote exists. In this case, the equation of horizontal asymptote is:

$$y = \frac{\text{Coefficient of highest power term in numerator}}{\text{Coefficient of highest power term in denominator}}$$

EXAMPLE 2.4 Construct the graph of the function $f(x) = \frac{x^2-1}{x^2+1}$.

SOLUTION The function f(x) is defined for all x. It is even since f(x) = f(–x) for any $x \in R$.

The function f(x) is not periodic.

Since $f(x) = 1 - \frac{2}{x^2+1}$,

$f(x) \to 1$ as $x \to \infty$ and as $x \to -\infty$.

Hence, the straight line y = 1 is a horizontal asymptote to the graph of the function as $x \to \infty$.

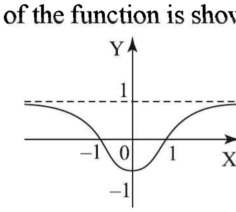
Further, the graph of the function f(x) lies below the line y = 1.

Since the function $\frac{2}{x^2+1}$ decreases on the interval $(0, \infty)$, the function $f(x) = 1 - \frac{2}{x^2+1}$ increases on that interval. On the interval $(-\infty, 0)$ the function $\frac{2}{x^2+1}$ increases and, therefore, the function $f(x) = 1 - \frac{2}{x^2+1}$ decreases on the interval $(-\infty, 0)$.

The graph of the function f(x) cuts the x-axis at x = 1 and x = – 1 and the y-axis at y = – 1.

The function $\frac{2}{x^2+1}$ assumes the greatest value at x = 0 and, consequently, at x = 0, the function f(x) assumes the least value equal to – 1.

The graph of the function is shown in the figure.



EXAMPLE 2.5 Draw the graph of

$$f(x) = \frac{x^2 - 6x + 4}{x^2 + 2x + 4}.$$

Also find the range of f(x).

SOLUTION

$$f(x) = \frac{x^2 - 6x + 4}{x^2 + 2x + 4} = 1 - \frac{8x}{x^2 + 2x + 4}$$

$$f'(x) = -8\left[\frac{(x^2 + 2x + 4) - x(2x + 2)}{(x^2 + 2x + 4)^2}\right]$$

$$= -8\left[\frac{-x^2 + 4}{(x^2 + 2x + 4)^2}\right] = \frac{8(x^2 - 4)}{(x^2 + 2x + 4)^2}$$

$f'(x) = 0 \quad \Rightarrow \quad x = 2$ or -2

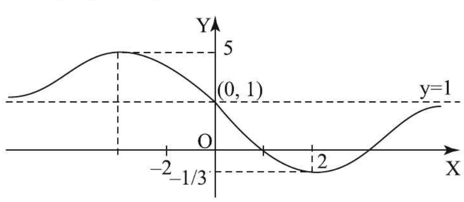
$$f(2) = \frac{4 - 12 + 4}{4 + 4 + 4} = \frac{-4}{12} = \frac{-1}{3}$$

$$f(-2) = \frac{4 + 12 + 4}{4 - 4 + 4} = 5$$

We have $\lim\limits_{x \to \pm\infty} \dfrac{x^2 - 6x + 4}{x^2 + 2x + 4} = 1$

Hence, y = 1 is a horizontal asymptote.

The graph of y = f (x) is as shown



We can see from the graph that the range of f(x) is $\left[-\frac{1}{3}, 5\right]$.

Oblique (Slant) Asymptotes

If there are limits

$$\lim_{x \to \infty} \frac{f(x)}{x} = m_1 \text{ and } \lim_{x \to \infty} [f(x) - m_1 x] = c_1,$$

then the straight line $y = m_1 x + c_1$ will be an asymptote (a right inclined asymptote or, when $m_1 = 0$, a right horizontal asymptote).

If there are limits

$$\lim_{x \to -\infty} \frac{f(x)}{x} = m_2 \text{ and } \lim_{x \to -\infty} [f(x) - m_2 x] = c_2,$$

then the straight line $y = m_2 x + c_2$ is an asymptote (a left inclined asymptote or, when $m_2 = 0$, a left horizontal asymptote).

A rational function has a slant or oblique asymptote when order of numerator (n) is greater than order of denominator (m).

The equation of slant asymptote is obtained by dividing numerator polynomial by denominator polynomial. The quotient of division is equation of asymptote. Clearly, asymptote is a straight line. As such, quotient should be a linear expression. The requirement that asymptote is a straight line implies that the order of numerator polynomial is higher than order of denominator polynomial by 1 i.e. n = m + 1.

In the nutshell, slant asymptote exists when n = m + 1. The slant asymptote is obtained by dividing numerator and denominator. We neglect remainder. The equation of the slant asymptote is given by quotient equated to y.

EXAMPLE 2.6 Construct the graph of the function $y = \dfrac{x^3 + 4}{x^2}$.

SOLUTION

(i) The domain of the function is $(-\infty, 0) \cup (0, \infty)$

(ii) The function is neither even nor odd.

(iii) Next we find the points of intersection of the graph and x-axis: we have $\dfrac{x^3 + 4}{x^2} = 0$; $x = -\sqrt[3]{4}$.

(iv) The point of discontinuity is x = 0, with $\lim\limits_{x \to 0} y = \infty$. Consequently, x = 0 (the y-axis) is a vertical asymptote of the graph.

Let us now find the slant asymptotes:

$$m = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^3 + 4}{x^3} = 1.$$

$$c = \lim_{x\to\infty}[f(x) - mx] = \lim_{x\to\infty}\left(\frac{x^3+4}{x^2} - x\right)$$

$$= \lim_{x\to-\infty}\frac{4}{x^2} = 0$$

The slant asymptote is specified by the equation $y = mx + c$ i.e. $y = x$.

(v) Let us find the extrema of the function and the intervals of monotonicity.

We have $y' = 1 - x/x^3 = (x^3 - 8)/x^3$
$y' = 0$ for $x = 2$.

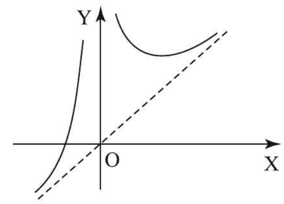
The points $x = 0$ and $x = 2$ divide the number axis into the intervals $(-\infty, 0)$ $(0, 2)$ and $(2, \infty)$, with $y' > 0$ on the intervals $(-\infty, 0)$ and $(2, \infty)$ (the function increases) and $y' < 0$ on the interval $(0, 2)$(the function decreases).

Next we find $y'' = 24/x^4$; $y''(2) > 0$.

Hence, $x = 2$ is the point of minimum: $y_{min} = 3$.

(vi) Now we find the intervals of concavity of the curve and its points of inflection. Since $y'' > 0$, the graph of the function is concave up everywhere. The curve has no point of inflection.

Using the above analysis, we construct the graph of the function.



PRACTICE PROBLEMS [I]

1. Draw the graph of the functions:

   (i) $f(x) = \dfrac{1}{(x-1)(x+2)}$

   (ii) $f(x) = \begin{cases} \dfrac{5x^2 - 10x - 15}{x^2 - x - 6}, & \text{if } x \neq 3 \\ 4, & \text{if } x = 3 \end{cases}$

2. Given a function $y = x^2/(x - 2)$. Investigate its behaviour and make a rough drawing of the graph.

3. Investigate the behaviour of the function $y = (x^3 - 4)/(x - 1)^3$ and construct its graph. How many roots does the equation $(x^3 - 4)/(x - 1)^3 = c$ possess?

4. Draw the graph of the function

   $f(x) = \dfrac{\tan\left(x + \frac{\pi}{6}\right)}{\tan x}$ and mark the points of extrema.

5. Draw the curve $y = \dfrac{x}{x^2+1}$. Also find the range of values of m for which the line $y = mx$ and the given curve enclose a region.

6. Draw the graph of the function

   $f(x) = \begin{cases} \dfrac{|x-1|}{x^2+1} & \text{if } x > -1 \\ x^2 & \text{if } x \leq -1 \end{cases}$.

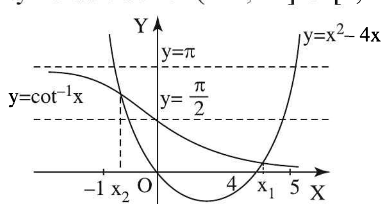
   and mark all the critical point(s) on the graph.

TARGET PROBLEMS for JEE ADVANCED

PROBLEM 2.1 Show that the inequality $x^2 - 4x > \cot^{-1}x$ holds true for $(-\infty, -1] \cup [5, \infty)$.

SOLUTION The solution of the inequality $x^2 - 4x > \cot^{-1}x$ is $x < x_2$ or $x > x_1$, as shown in the figure.

Clearly, $5 > x_1$ and $-1 < x_2$. Hence, the inequality holds true for $(-\infty, -1] \cup [5, \infty)$.



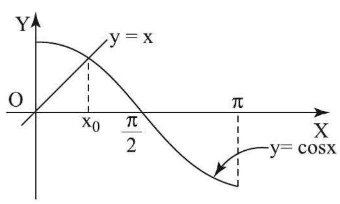
PROBLEM 2.2 If x_0 is the solution of the equation $\cos x = x$ then find $\text{sgn}(\cos x_0 - \pi/4)$.

SOLUTION In the figure, we see that $x_0 < \pi/4$

since $\cos \pi/4 < \pi/4$.

Thus, $\cos x_0 - \pi/4 = x_0 - \pi/4 < 0$.

$\therefore$ $\text{sgn}(\cos x_0 - \pi/4) = -1$.



PROBLEM 2.3 Draw graph of function defined as:

$$y = \frac{1}{|x|+1}$$

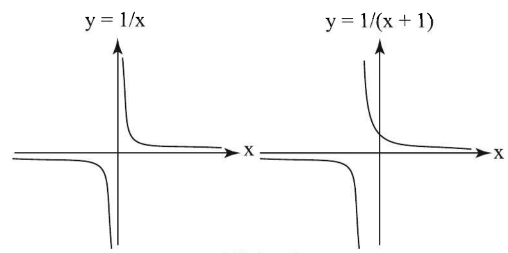
SOLUTION It is clear that we can obtain the given function by applying modulus operator to the independent variable of the function:

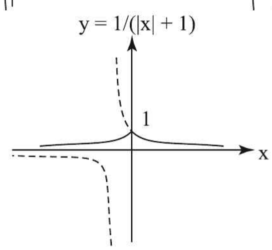
$$y = \frac{1}{x+1}$$

This function, in turn, can be obtained by applying shifting modification to the argument of the function given as:

$$y = \frac{1}{x}$$

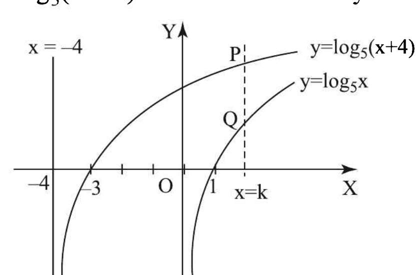
We, therefore, first draw $f(x) = 1/x$. Then we draw $g(x) = f(x + 1) = 1/(x + 1)$ by shifting the graph left by 1 unit. Finally, we draw $h(x) = g(|x|) = 1/(|x| + 1)$ by removing left half of the graph and taking mirror image of right half of the graph in y-axis.





PROBLEM 2.4 A line $x = k$ intersects the graph of $y = \log_5 x$ and the graph of $y = \log_5(x + 4)$. The distance between the points of intersection is 0.5. Given $k = a + \sqrt{b}$, where a and b are integers, find the value of $(a + b)$.

SOLUTION Obviously the curve $y = \log_5(x + 4)$ is above the curve $y = \log_5 x$.



Hence for $x = k$, $\log_5(k + 4) - \log_5 k = \frac{1}{2}$.

$$\Rightarrow \log_5\left(\frac{k+4}{k}\right) = \frac{1}{2}$$

$\Rightarrow \dfrac{k+4}{k} = \sqrt{5} \quad \Rightarrow \quad 1 + \dfrac{4}{k} = \sqrt{5}$

$\Rightarrow k = \dfrac{4}{\sqrt{5}-1} = \sqrt{5} + 1 \quad \Rightarrow \quad a = 1; b = 5$

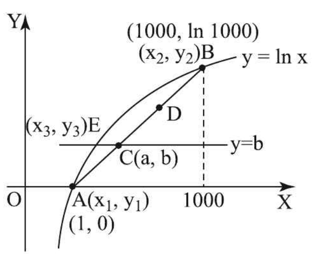
$\therefore$ a + b = 6.

PROBLEM 2.5 Two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are chosen on the graph of $f(x) = \ln x$ with $0 < x_1 < x_2$. The points C and D trisect the line segment AB with AC < CB. Through C a horizontal line is drawn to cut the curve at $E(x_3, y_3)$. If $x_1 = 1$ and $x_2 = 1000$ then find the value of x_3.

SOLUTION Using section formula

$$a = \frac{2\cdot 1 + 1\cdot 1000}{3} = 334$$

$$b = \frac{2\cdot 0 + 1\cdot \ln 1000}{3} \Rightarrow b = \frac{\ln 1000}{3} \qquad \text{...(1)}$$



Now line y = b intersects the curve $y = \ln x$

$\therefore$ $b = \ln x$ ...(2)

From (1) and (2)

$$\frac{\ln 1000}{3} = \ln x \quad \Rightarrow \quad \ln (1000)^{1/3} = \ln x$$

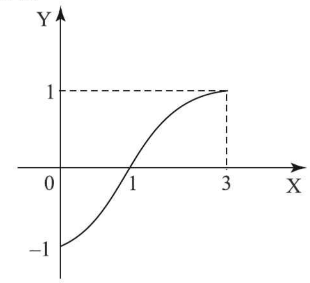
$\therefore$ $x = (1000)^{1/3} = 10.$

PROBLEM 2.6 Suppose that the domain of the function f (x) is set D and the range is the set R, where D and R are the subsets of real numbers. Consider the functions: f (2x), f (x + 2), 2 f (x), $f\left(\dfrac{x}{2}\right)$, $\dfrac{f(x)}{2} - 2$. If m is the number of functions listed above that must have the same domain as f and n is the number of functions that must have the same range as f (x), then find the m and n.

SOLUTION 2f(x) and $\dfrac{f(x)}{2} - 2$ have the same domain as f(x) and f(2x), f(x + 2), $\dfrac{f(x)}{2}$ have the range as f(x)

$\Rightarrow$ m = 2, n = 3.

PROBLEM 2.7 The graph of y = f(x) is shown below.



Draw the graph of the following functions:

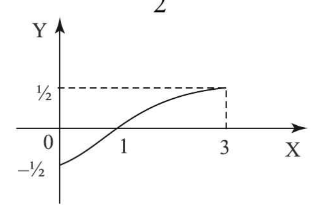
(a) $y = \dfrac{1}{2} f(x)$ (b) y = f(2x)

(c) $y = \dfrac{|f(x)|}{f(x)}$ (d) y = [f(x)] G.I.F.

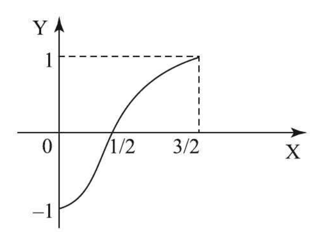
(e) y = f(–x + 1) (f) x = f(y).

SOLUTION

(a) The graph of $y = \dfrac{1}{2} f(x)$ is shown below.

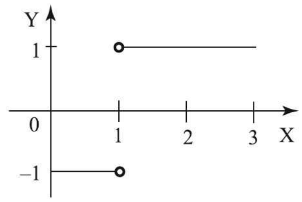


(b) The function y = f(x), x ∈ [0, 3] transforms to $y = f(2x), x \in \left[0, \dfrac{3}{2}\right]$ whose graph is shown below.



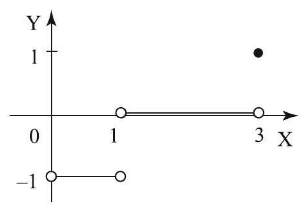
(c) We have $\dfrac{|f(x)|}{f(x)} = \begin{cases} -1 & 0 \le x < 1 \\ 1 & 1 < x \le 3 \end{cases}$

whose graph is shown below.



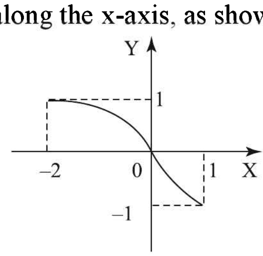
(d) We have $[f(x)] = \begin{cases} 1 & x = 3 \\ 0 & 1 \le x < 3 \\ -1 & 0 \le x < 1 \end{cases}$

The graph of y = [f(x)] is shown below.

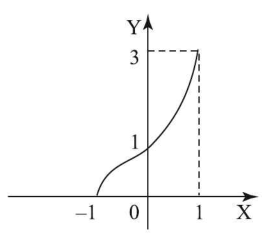


(e) We have f(–x + 1) = f{–(x – 1)}

Its graph is drawn by taking the mirror image of the graph of f(x) in the x-axis and then translating it by 1 unit along the x-axis, as shown below.

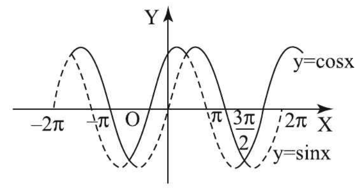


(f) The graph of x = f(y) is drawn by taking the mirror image of the graph of y = f(x) in the line y = x, as shown below.



PROBLEM 2.8 Solve max {sinx, cosx} < 0, $\forall\, x \in (-2\pi, 2\pi)$

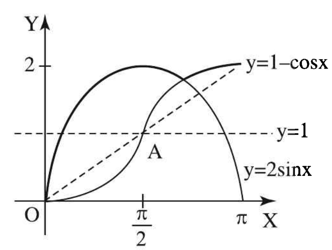
SOLUTION The bold curve represents the graph of y = f(x).



Clearly f(x) < 0 for $x \in \left(-\pi, -\dfrac{\pi}{2}\right) \cup \left(\pi, \dfrac{3\pi}{2}\right)$.

PROBLEM 2.9 Let f(x) = min{2sinx, 1 – cosx, 1}. Find all positive values of m so that the line y = mx intersects the graph of y = f(x) atleast twice in [0, π].

SOLUTION The bold curve represents the graph of f(x).



The line y = mx with the greatest slope intersecting the graph of f(x) atleast twice is OA. Its slope is $\dfrac{2}{\pi}$.

PROBLEM 2.10

Given f(t) = |t – 1| – |t| + |t + 1|, t ∈ R.

(a) Plot the curve y = f(t).

(b) Define g(x) = max {f(t) : x + 1 ≤ t ≤ x + 2}, x ∈ R in piecewise form.

SOLUTION

(a) We have

f(t) = t – 1 – t + t + 1, t ≥ 1

$= 1 - t - t + t + 1, \quad 0 \le t < 1$

$= 1 - t + t + t + 1, \quad -1 \le t < 0$

$= 1 - t + t - t - 1, \quad t < -1$

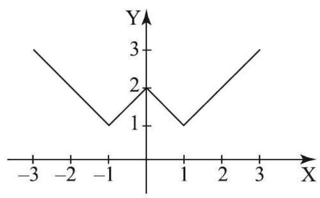
i.e. $f(t) = -t, \quad t < -1$

$= t + 2, \quad -1 \le t < 0$

$= -t + 2, \quad 0 \le t < 1$

$= t, \quad t \ge 1$

The plot of y = f(t), is shown below.



(b) We have $g(x) = \max \{f(t) : x + 1 \le t \le x + 2\}$

As an example, g(1) = max value of f(t) in the interval

$2 \le t \le 3$

= f(3) using the above plot.

Case I Let $x < -3 \Rightarrow t < -1$.

Consider an interval [x + 1, x + 2] (which is of unit length) in the region $t < -1$.

The maximum value of f(x) on this interval is at x + 1. Hence, we have

$g(x) = f(x + 1), \quad x < -1$

$= -(x + 1)$

Case II Let $-3 \le x < -2.5 \Rightarrow -2 \le t < -0.5$

Consider an interval [x + 1, x + 2] in the region $-2 \le t < -0.5$. From the graph of f(t), we can see that the maximum value of f(t) is at x + 1.

Hence, we have

$g(x) = f(x + 1), -2 \le x + 1 < -1.5$

$= -(x + 1)$.

Case III Let $-2.5 < x \le -2 \Rightarrow -1.5 < t \le 0$

Consider an interval [x + 1, x + 2] in the region $-1.5 < t \le 0$. From the graph of f(t), we can see that the maximum value of f(t) is at x + 2.

Hence, we have

$g(x) = f(x + 2), -0.5 \le x + 2 < 0$

$= (x + 2) + 2 = x + 4$.

Case IV Let $-2 < x \le -1.5 \Rightarrow -1 < t \le -0.5$

Consider an interval [x + 1, x + 2] in the region $-1 < t \le 0.5$. From the graph of f(t), we can see that the maximum value of f(t) is at x + 2.

Hence, we have

$g(x) = f(x + 2), 0 < x + 2 \le 0.5$

$= -(x + 2) + 2 = -x$.

Case V Let $-1.5 < x \le -1 \Rightarrow -0.5 < t \le 1$

Proceeding as above, we have

$g(x) = f(x + 2), 0.5 < x + 2 \le 1$

$= -(x + 2) + 2 = -x$.

Case VI Let $-1 < x \le -0.5 \Rightarrow -0 < t \le 1.5$

Proceeding as above, we have

$g(x) = f(x + 1), 0 < x + 1 \le 0.5$

$= -(x + 1) + 1 = -x + 1$.

Case VII Let $-0.5 < x \Rightarrow -0.5 < t$

Proceeding as above, we have

$g(x) = f(x + 2), x + 2 > 1.5$

$= x + 2$.

Hence, we have

$g(x) = -x - 1, x < -2.5$

$= x + 4, -2.5 < x \le -2$

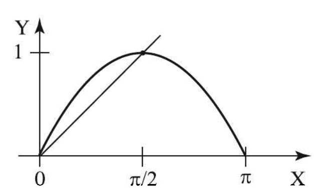
$= -x, -2 < x \le -1$

$= -x + 1, -1 < x \le -0.5$

$= x + 2, -0.5 < x$.

PROBLEM 2.11 Give a graphical argument justifying the Jordan's inequality $\frac{2}{\pi} x \le \sin x$ for $0 \le x \le \frac{\pi}{2}$.

SOLUTION The equation of the straight line joining (0,0) and $\left(\frac{\pi}{2}, 1\right)$ is $y = \frac{2}{\pi}x$. From the graphs we see that, the graph of $y = \frac{2}{\pi}x$ lies below that of y = sin x in the interval $\left[0, \frac{\pi}{2}\right]$.



PROBLEM 2.12 Find the number of real roots of the following equations :

(i) $x^2 \tan x = 1$, $x \in [0, 3\pi/2)$

(ii) $2\cos x = |\sin x|$, $x \in [0, 2\pi]$

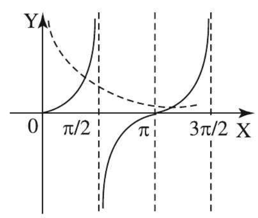
(iii) $7^{|x|}(|5 - |x|| = 1$.

SOLUTION

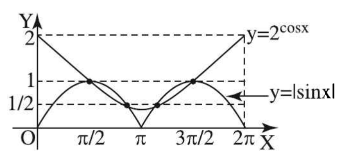
(i) $\tan x = \dfrac{1}{x^2}$

We draw the curves $y = \tan x$ and $y = \dfrac{1}{x^2}$.

From the graph, it is clear that the equation has two real roots.



(ii) See the graphs of $y = 2^{\cos x}$ and $y = |\sin x|$. The two curves meet at four points for $x \in [0, 2\pi]$.



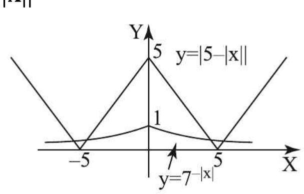
So, the equation $2^{\cos x} = |\sin x|$ has four solutions.

(iii) $7^{|x|} (|5 - |x||) = 1$.

$\Rightarrow |5 - |x|| = 7^{-|x|}$

We draw the graphs of $y = 7^{-|x|}$ and

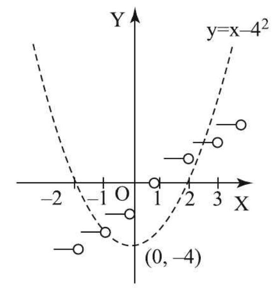
$y = |5 - |x||$



From the graph, the number of roots is 4.

PROBLEM 2.13 Find the sum of square of roots of $x^2 - 4 - [x] = 0$ is (where [.] denotes the greatest integer function).

SOLUTION The given equation is $x^2 - 4 = [x]$. Then, the solutions of the equation are values of x where graph $y = x^2 - 4$ and $y = [x]$ intersect.



From the graph, it is seen that graph intersects when

$x^2 - 4 = 2$ and $x^2 - 4 = -2$

$\Rightarrow x^2 = 6 \quad \text{or} \quad x^2 = 2$

$\Rightarrow x = \sqrt{6} \quad \text{or} \quad -\sqrt{2}$

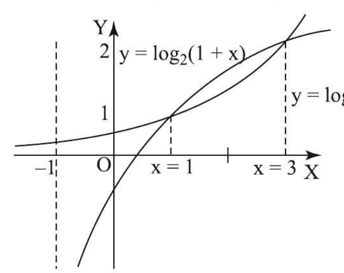
Hence, the sum of square of roots is 8.

PROBLEM 2.14 Solve the equation

$\log_2(1+ x) = \log_3(1 + 2^x)$.

SOLUTION We draw the graphs of the functions $y = \log_2(1+ x)$ and $y = \log_3(1 + 2^x)$.

We can see that the curves intersect two times. We find the roots by trial. They are $x = 1, 3$.



PROBLEM 2.15 The curve $y = x - \dfrac{2}{x^2}$ undergoes the following successive transformations:

(i) a translation one unit to the right,

(ii) a reflection about the y-axis,

(iii) a translation one unit down.

What is the equation of the resulting curve?

SOLUTION Put $a(x) = x - \frac{2}{x^2}$. After a translation one unit right we get a

$$(x-1) = x - 1 - \frac{2}{(x-1)^2} = b(x), \text{ say. After a}$$

reflection about the y axis we get

$$b(-x) = -x - 1 - \frac{2}{(-x-1)^2}$$

$$= -x - 1 - \frac{2}{(x+1)^2} = c(x), \text{ say.}$$

After a translation one unit down we get

$$c(x) - 1 = -x - 1 - \frac{2}{(x+1)^2} - 1$$

$$= -x - 2 - \frac{2}{(x+1)^2}$$

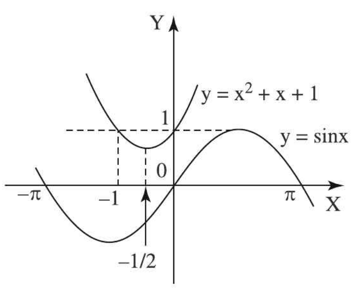
Hence the resulting curve is

$$y = -x - 2\ \frac{2}{(x+1)^2}.$$

PROBLEM 2.16 Solve the equation

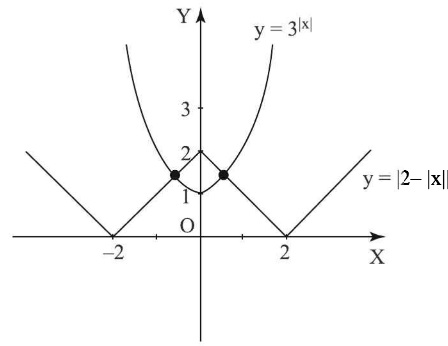
$\sin x = x^2 + x + 1$.

SOLUTION Let us first draw the graphs of $y = \sin x$ and $y = x^2 + x + 1$. It is clear from the graphs that the curves do not intersect and so the equation has no solution.



PROBLEM 2.17 Find the number of solutions of the equation $3^{|x|} = |2 - |x||$.

SOLUTION It is clear from the graph there are two points of intersection. Hence the number of solutions is two.



PROBLEM 2.18 Find the number of solutions of

$\sin x = \frac{x}{10}$.

SOLUTION

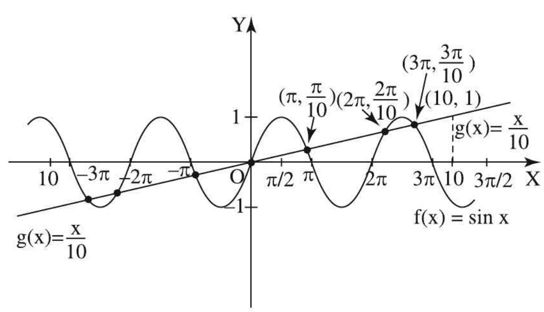
Here, let $f(x) = \sin x$ and $g(x) = \frac{x}{10}$

also we know; $-1 \le \sin x \le 1$

$-1 \le \frac{x}{10} \le 1$

$\Rightarrow\ -10 \le x \le 10$

Thus, we sketch both curves when $x \in [-10, 10]$

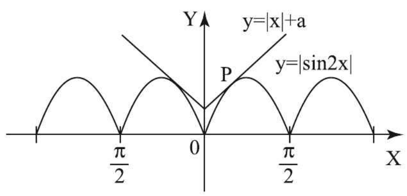


From above figure $f(x) = \sin x$ and $g(x) = \frac{x}{10}$ intersect at 7 points.

PROBLEM 2.19 Find the complete set of values of 'a' for which the equation

$|\sin 2x| - |x| - a = 0$ does not have any real solution.

SOLUTION Let us consider the limiting case when the graphs of $y = |\sin 2x|$ and $y = |x| + a$, touch each other.



If P is the point of contact, then at point P, the slope of the curves are equal. Hence, $2\cos 2x = 1$.

$\Rightarrow\ x = \dfrac{\pi}{6}$. Thus, $P \equiv \left(\dfrac{\pi}{6}, \dfrac{\sqrt{3}}{2}\right)$.

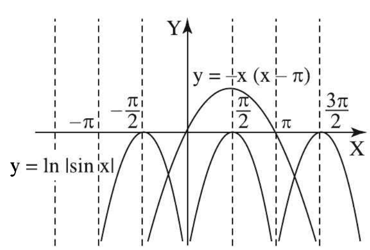
Now, since the point P also lies of the line $y = x + a$, we have $\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{6} + a \Rightarrow a = \dfrac{3\sqrt{3} - \pi}{6}$.

If $a > \dfrac{3\sqrt{3} - \pi}{6}$, then the given equation has no solution. Thus, the required set values of a is $\left(\dfrac{3\sqrt{3} - \pi}{6}, \infty\right)$.

PROBLEM 2.20 Find the total number of solutions of $ln\,|\sin x| = -x^2 + \pi x$ in $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$.

SOLUTION $ln\,|\sin x| = -x\,(x - \pi)$

The graphs of $y = ln|\sin x|$ and $y = -x(x - \pi)$ meet exactly two times in $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$.



PROBLEM 2.21 Sketch the graph of

(i) $y = xe^{-x}$ (ii) $y = x\,ln\,x$.

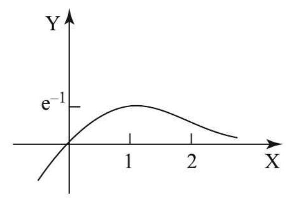
SOLUTION $y = xe^{-x}$

$\lim\limits_{x \to -\infty} y = 0$. So, the x axis is a horizontal asymptote.

Also $\lim\limits_{x \to -\infty} xe^{-x} = -\infty$.

$y' = e^{-x}(1 - x)$ and $y'' = e^{-x}(x - 2)$.

Then $x = 1$ is a critical point. By the second derivative test, there is a relative maximum at $(1, 1/e)$ since $y'' < 0$ at $x = 1$.

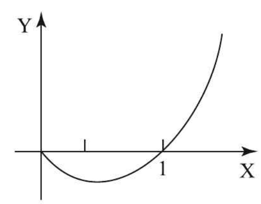


(ii) $y = x \ln x$

Clearly $\lim\limits_{x \to -\infty} y = \infty$. There is no horizontal asymptote.

$\lim\limits_{x \to 0^+} y = 0$.

$y' = 1 + \ln x$. Thus, $x = 1/e$ is a critical point.



PROBLEM 2.22 Draw the graph of $f(x) = x^2 e^{-|x|}$ and find the value of k so that the curve $y = kx^2$ will intersect $y = e^{|x|}$ at exactly two points.

SOLUTION We have

$$f(x) = x^2 e^{-|x|} = \begin{cases} x^2 e^{-x}, & x \ge 0 \\ x^2 e^{x}, & x < 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} x e^{-x}(2 - x), & x \ge 0 \\ x e^{x}(x + 2), & x < 0 \end{cases}$$

$f(x)$ is increasing in $(-\infty, -2) \cup (0, 2)$

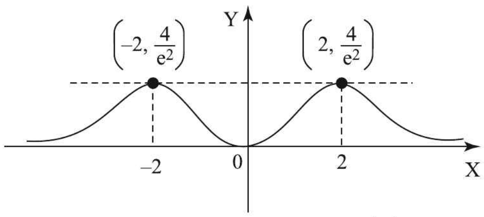
and $f(x)$ is decreasing in $(-2, 0) \cup (2, \infty)$.

Also $f(\pm 2) = \frac{4}{e^2}$, $f(0) = 0$

$\lim_{x \to \pm\infty} f(x) = 0$

Hence y = 0 is a horizontal asymptote to the curve.

$\Rightarrow$ The graph of f (x) is as shown below.



Now $y = kx^2$ will intersect $y = e^{|x|}$

when $kx^2 = e^{|x|} \Rightarrow x^2 e^{-|x|} = \frac{1}{k}$

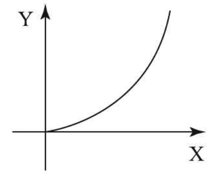
So, $y = x^2 e^{-|x|}$ will be cut by line $y = \frac{1}{k}$ at exactly two points if $\frac{1}{k} = \frac{4}{e^2} \Rightarrow k = \frac{e^2}{4}$.

Things to REMEMBER

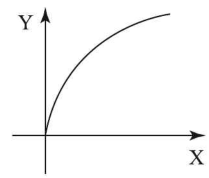
1. Symmetry of graph of F(x, y) = 0
   (i) The graph of F(x, y) = 0 is symmetric about the y-axis if on replacing x by – x, the equation of the curve does not change.
   (ii) The graph of F(x, y) = 0 is symmetric about the x-axis if on replacing y by – y, the equation of the curve does not change.
   (iii) The graph of F(x, y) = 0 is symmetric about the origin if on replacing x by – x and y by – y, the equation of the curve does not change.
   (iv) The graph of F(x, y) = 0 is symmetric about the line y = x if on interchanging x and y, the equation of the curve does not change.
   (v) The graph of F(x, y) = 0 is symmetric about the line y = –x if on replacing x by – y and y by – x, the equation of the curve does not change.

2. When we draw the graph of $y = x^p$ in the first quadrant we have three kinds of graph:

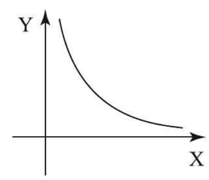
   (i) $y = x^p$ where $p > 1$



   (ii) $y = x^p$ where $0 < p < 1$



   (iii) $y = x^p$ where $p < 0$



3. Transformation of graphs (assume a > 0)
   (i) To draw the graph of y = f(x) + a, we take the graph of y = f(x) and move it a units up.
   (ii) To draw the graph of y = f(x – a), we take the graph of y = f(x) and shifted to the right by a units.
   (iii) The graph of y = a f(x) is obtained by stretching the graph of y = f(x), a times along y axis (more precisely, by stretching for a > 1 and by compressing for 0 < a < 1).
   (iv) The graph of function y = – f(x) is obtained from the graph of the function y = f(x) by reflection of the graph of y = f(x) about the x–axis.
   (v) The graph of y = f(ax) is obtained by shrinking the graph of y = f(x), a times along y-axis (more precisely, by shrinking for a > 1 and by expanding for 0 < a < 1).
   (vi) The graph of function y = f(–x) is obtained from the graph of the function y = f(x) by reflection of the graph of y = f(x) about the y–axis.

4. Transformation of graphs by modulus function
   (i) To construct the graph of $y = |f(x)|$, we take the mirror image of the lower half of the graph of $y = f(x)$, about the x-axis and remove the lower half of the graph.
   (ii) To construct the graph of $y = f(|x|)$, first we remove the left half of the graph of $y = f(x)$, and then take the mirror image of right half of the graph in the y-axis.
   (iii) The graph of $|y| = |f(x)|$ is the union of the graphs of $y = f(x)$ and $y = -f(x)$. We first draw the graph of $y = f(x)$ and then take the image of the graph of $f(x)$ about the x–axis considering it as a plane mirror.
   (iv) The graph of function $|y| = f(x)$ is obtained from the graph of the function $y = f(x)$ as follows : first we delete its portion located below the x–axis and then take the image of the positive part of $f(x)$ about the x–axis considering it as a plane mirror.
5. The graph of $y = f(x)$ is transformed to $y = [f(x)]$ as follows:
   (i) Draw lines parallel to x-axis (horizontal lines) at integral values along y-axis to cover the graph of $y = f(x)$.
   (ii) Identify points of intersections of graph with parallel lines drawn in the earlier step. Draw lines parallel to y-axis (vertical lines) from the intersection points identified.
   (iii) Take x-projection of curve from the point of intersection between two consecutive vertical lines such that it lies on horizontal line of lower value.
6. The graph of $y = f(x)$ is transformed to $y = f([x])$ as follows :
   (i) Draw lines parallel to y-axis (vertical lines) at integral values along x-axis to cover the graph of $y = f(x)$.
   (ii) Identify points of intersections of graph with parallel lines drawn in the earlier step.
   (iii) Draw lines of 1 unit parallel to x-axis from intersection points in the direction of positive x. The line ends at the next parallel line on right. Include intersection point but exclude other end of the line.
7. The graph of $y = f(x)$ is transformed to $y = \{f(x)\}$ as follows :
   (i) Draw lines parallel to x-axis (horizontal lines) at integral values along y-axis to cover the graph of $y = f(x)$.
   (ii) Identify segments of graph between two consecutive vertical intervals. Transfer these segments to y-interval given by $[0,1)$.
   (iii) Include end point corresponding to $y = 0$ and exclude end point corresponding to $y = 1$.
8. The graph of $y= f(x)$ is transformed to $y = f(\{x\})$ as follows:
   (i) Draw the vertical lines $x = 0$ and $x = 1$.
   (ii) Identify part of the graph for values of x in $[0,1)$. Include end point corresponding to $x = 0$ and exclude end point corresponding to $x = 1$.
   (iii) Repeat the part of the graph identified in step (ii) for other intervals of x.
9. From the graph of $y = f(x)$, we obtain the graph of $x = f(y)$ either by reflecting the graph of $y = f(x)$ w.r.t. to the line $y = x$, or by first rotating the graph of $y = f(x)$ by 90° anti-clock-wise, and then reflecting the resulting graph about the vertical axis.
10. The steps for determining intervals of monotonicity are as follows:
   (i) Determine the derivative of the function $f'(x)$ and find the critical points i.e. points where the derivative is zero or does not exist.
   (ii) Determine the sign of $f'(x)$ in different intervals formed by the critical points.
   (iii) Determine monotonic nature of function in accordance with following categorization :
   $f'(x) \geq 0$: equality holding for points only $\Rightarrow$ strictly increasing interval
   $f'(x) \geq 0$: equality holding for sub-intervals $\Rightarrow$ non-decreasing or increasing interval
   $f'(x) \leq 0$: equality holding for points only $\Rightarrow$ strictly decreasing interval
   $f'(x) \leq 0$: equality holding for subintervals $\Rightarrow$ non-increasing or decreasing interval.
11. The First Derivative Test : Suppose that $x = a$ is a critical point of a continuous function $y = f(x)$.

(i) If $f'(x)$ changes from positive to negative at $x = a$, then f has a local maximum at $x = a$.

(ii) If $f'(x)$ changes from negative to positive at $x = a$, then f has a local minimum at $x = a$.

(iii) If $f'(x)$ does not change sign at $x = a$ (that is, if $f'(x)$ is positive on both sides of $x = a$ or negative on both sides), then f has no local maximum or minimum at $x = a$.

12. The Second Derivative Test : Let $x = a$ be a stationary point of a function f (i.e. $f'(a) = 0$). The function f has a local maximum at a if $f''(a)$ is negative, and a local minimum if $f''(a)$ is positive.

13. In order to find the global maximum and minimum of a continuous function f(x) in [a, b] :
Find out all the critical points of f(x) in (a, b).
Let $c_1, c_2, \ldots, c_n$ be the different critical points.

Find the value of the function at these critical points.

Let $M = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$
and $m = \min\{f(a), f(c), f(c_2), \ldots, f(c_n), f(b)\}$
Then M is the global maximum (greatest value) of f(x) in [a, b] and m is the global minimum (least value) of f(x) in [a, b].

14. For a curve $y = f(x)$, if the second derivative $f''(x)$ is everywhere positive within an interval, the arc of the curve $y = f(x)$ corresponding to that interval is concave up. If the second derivative $f''(x)$ is everywhere negative in an interval, the corresponding arc of the curve $y = f(x)$ is concave down.

15. Let $x = c$ be a point where $f''(c) = 0$ or $f''(c)$ does not exist and the signs of $f''(c-h)$ and $f''(c+h)$ be opposite, then the point $x = c$ is called a point of inflection.

16. Asymptotes:

(i) If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$.

(ii) The line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$\lim\limits_{x\to a^+} f(x) = \infty$ $\quad$ $\lim\limits_{x\to a^-} f(x) = \infty$

$\lim\limits_{x\to a^+} f(x) = -\infty$ $\quad$ $\lim\limits_{x\to a^-} f(x) = -\infty$

(iii) If there are limits

$$\lim_{x\to\infty} \frac{f(x)}{x} = m_1$$

and $\lim\limits_{x\to\infty} [f(x) - m_1x] = c_1$,

then the straight line $y = m_1x + c_1$ will be an asymptote (a right inclined asymptote or, when $m_1 = 0$, a right horizontal asymptote).

If there are limits

$$\lim_{x\to-\infty} \frac{f(x)}{x} = m_2$$

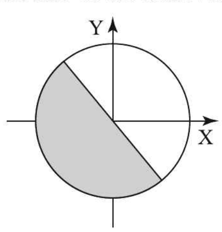
and $\lim\limits_{x\to-\infty} [f(x) - m_2x] = c_2$,

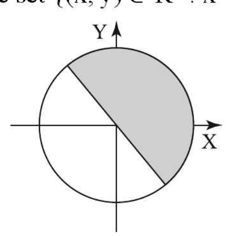
then the straight line $y = m_2x + c_2$ is an asymptote (a left inclined asymptote or, when $m_2 = 0$, a left horizontal asymptote).

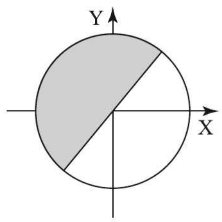
OBJECTIVE EXERCISES

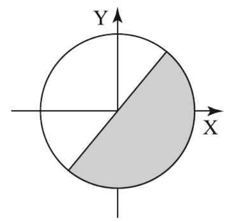
SINGLE CORRECT ANSWER TYPE

1. Which one of the following graphs best represents the set $\{(x, y) \in R^2 : x^2 + y^2 \le 16, y \ge -x\}$?

(a) 

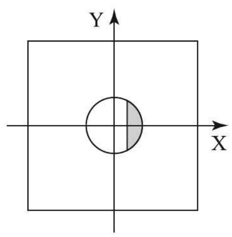
(b) 

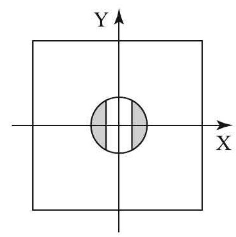
(c)


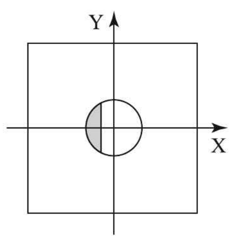
(d)


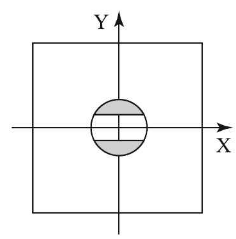
(A) a (B) b (C) c (D) d

2. Which of the following graphs represents the set $\{(x, y) \in R^2 : x^2 + y^2 \le 4, |x| \ge 1\}$?

(a)


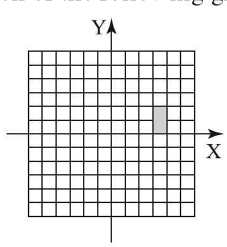
(b)


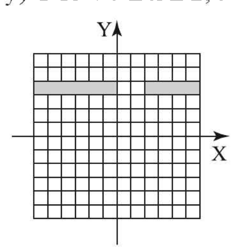
(c)


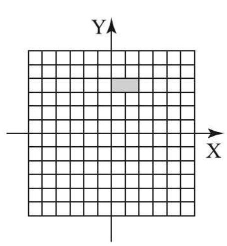
(d)


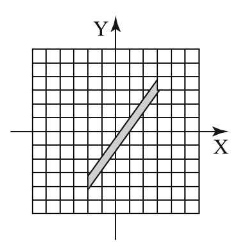
(A) a (B) b (C) c (D) d

3. Which of the following graphs represents the set $\{(x, y) \in R^2 : 0 \le x \le 2, 3 \le y \le 4\}$?

(a)


(b)


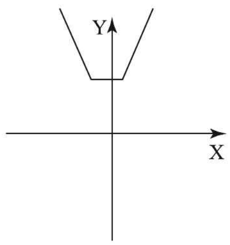
(c)


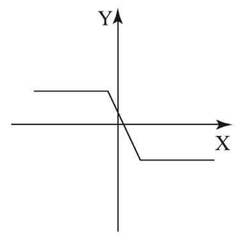
(d)


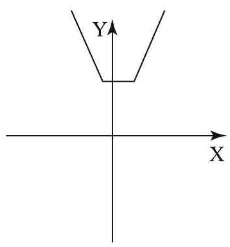
(A) a (B) b (C) c (D) d

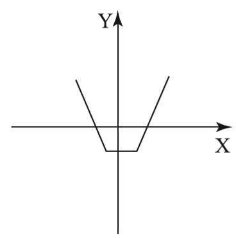
4. Which graph most resembles the curve

$y = |x - 2| + |x + 1|$?

(a) 

(b) 

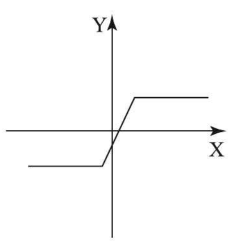
(c) 

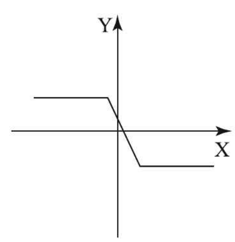
(d) 

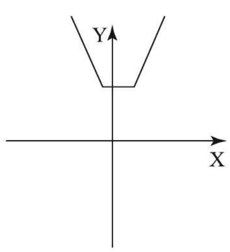
(A) a (B) b (C) c (D) d

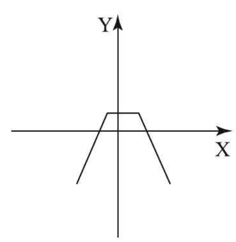
5. Which graph most resembles the curve

$y = |x - 2| - |x + 1|$?

(a) 

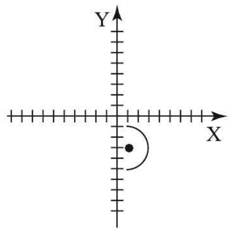
(b) 

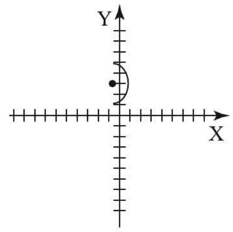
(c) 

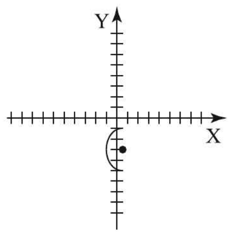
(d) 

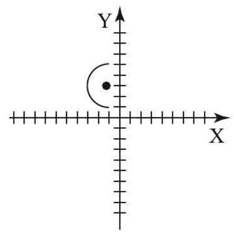
(A) a (B) b (C) c (D) d

6. Which figure represents the semicircle with equation $x = 1 - \sqrt{-y^2 - 6y - 5}$?

(a)


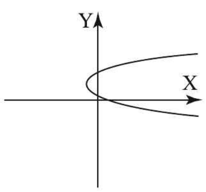
(b)


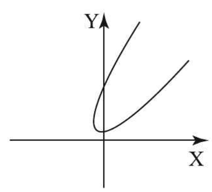
(c)


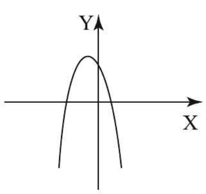
(d)


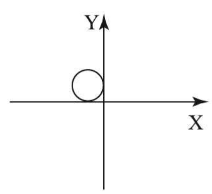
(A) a (B) b (C) c (D) d

7. Which one of the following represents a function

(a)


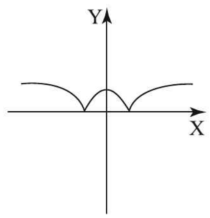
(b)


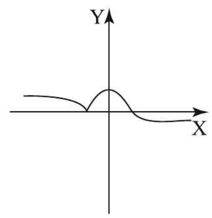
(c)


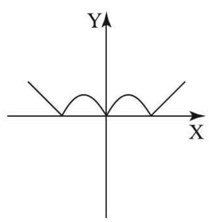
(d)


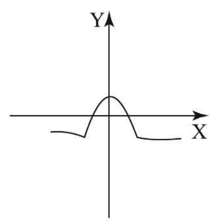
(A) a (B) b (C) c (D) d

8. Which one most resembles the graph of $y = f(x) = \begin{cases} \dfrac{1}{x} + 1 & \text{if } x \in (-\infty, -1] \\ 1 - x^2 & \text{if } x \in (-1, 1) \\ \dfrac{1}{x} - 1 & [1, \infty) \end{cases}$?

(a) 

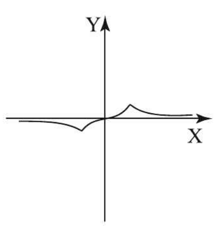
(b) 

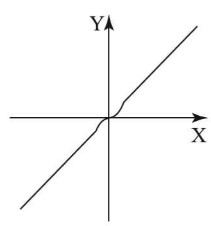
(c) 

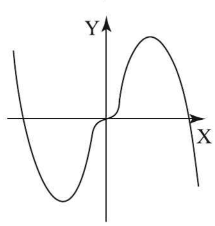
(d) 

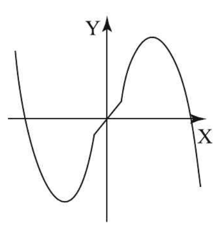
(A) a (B) b (C) c (D) d

9. Which one most resembles the graph of $y = f(x) = \begin{cases} (x+3)^2 - 5 & \text{if } x \in (-\infty, -1] \\ x^3 & \text{if } x \in (-1, 1) \\ 5-(x-3)^2 & [1, \infty) \end{cases}$?

(a) 

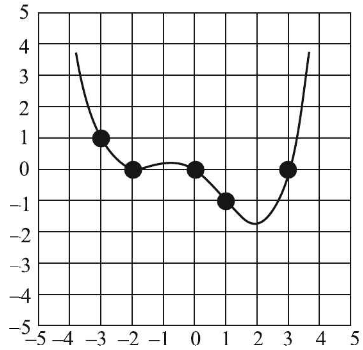
(b) 

(c) 

(d) 

(A) a (B) b (C) c (D) d

10. The polynomial p whose graph is shown below has degree 4. You may assume that the points marked below with a dot through which the polynomial passes have integer coordinates. Find the equation.



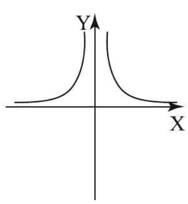
(A) $p(x) = x(x+2)^2(x-3)$

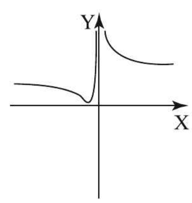
(B) $p(x) = -\dfrac{x(x+2)^2(x-3)}{18}$

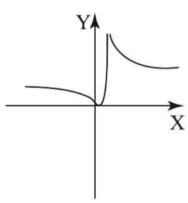
(C) $p(x) = \dfrac{x(x+2)^2(x-3)}{12}$

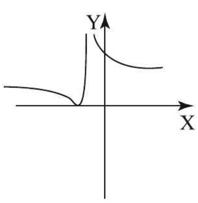
(D) $p(x) = \dfrac{x(x+2)^2(x-3)}{182}$

11. Which graph most resembles the curve $y = \left|\dfrac{1}{x-1} + 2\right|$?

(a)


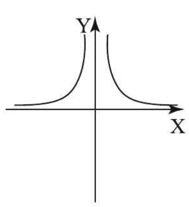
(b)


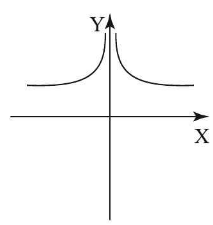
(c)


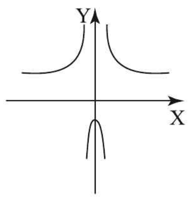
(d)


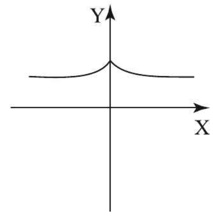
(A) a (B) b (C) c (D) d

12. Which graph most resembles the curve $y = \dfrac{1}{|x|-1} + 2$?

(a)


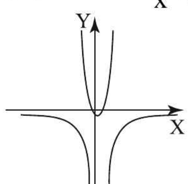
(b)


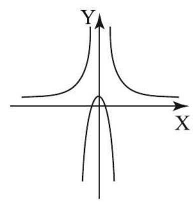
(c) 

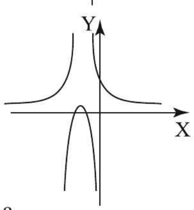
(d) 

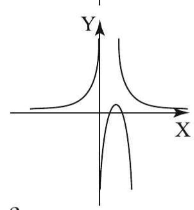
(A) a (B) b (C) c (D) d

13. The graph of $y = \dfrac{x^2 + x}{x^2 + x - 2}$ is

(a) 

(b) 

(c) 

(d) 

(A) a (B) b (C) c (D) d

14. Consider the rational function f, with f(x) $f(x) = \dfrac{(x+1)^2\,(x-2)}{(x-1)\,(x+2)^2}$. Which of the following is true ?

(A) f has zeroes at $x = -1$ and $x = 2$, and poles at $x = 1$ and $x = -2$.
(B) f has zeroes at $x = 1$ and $x = -2$, and poles at $x = -1$ and $x = 2$.
(C) f has zeroes at $x = 1$ and $x = 2$, and poles at $x = -1$ and $x = -2$.
(D) f has no zeroes and no poles.

15. The curve $y = \dfrac{x-1}{x+1}$ experiences the following successive transformations:

(i) a reflection about the y-axis, (ii) a translation 1/2 unit down,
(iii) a reflection about the x-axis.

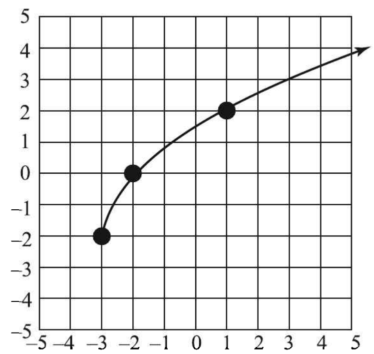
Find the equation of the resulting curve.

(A) $y = \dfrac{x+3}{2(1-x)}$ (B) $y = \dfrac{x}{2-x}$

(C) $y = \dfrac{2}{x-1}$ (D) $y = -\dfrac{x+3}{2(x-1)}$

16. What is the equation of the resulting curve after the curve $y = x|x + 1|$ has been successively translated one unit right and reflected about the y-axis ?
(A) $y = (x - 1)|x|$ (B) $y = -(x + 1)\,|x|$
(C) $y = -x|x|$ (D) $y = -x|x| - 1$

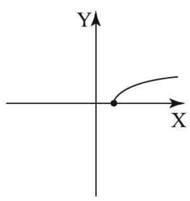
17. The graph in figure below belongs to a curve with equation of the form $y = A\sqrt{x+3} - 2$. Find A.

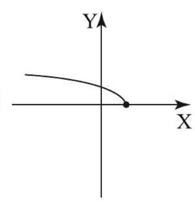


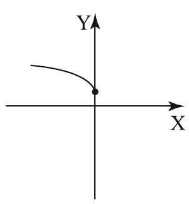
(A) $A = \frac{1}{2}$ (B) $A = 1$

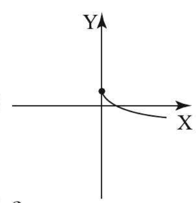
(C) $A = -2$ (D) $A = 2$

18. Which graph most resembles the curve $y = \sqrt{1-x}$?

(a) 

(b) 

(c) 

(d) 

(A) a (B) b (C) c (D) d

19. Which of the following pairs of equations have the same graph?

(A) $x^2 + y^2 = 1$ and $\sqrt{(x^2 + y^2)} = 1$ (B) $y = \sqrt{2x}$ and $\sqrt{y} = \sqrt{2x}$

(C) $x^2 + y^2 = 25$ and $y = \sqrt{(25 - x^2)}$ (D) none of these

20. The number of solutions of the equation $[x]^2 + [x + 1] = 3$ is : where $[x]$ dentotes greatest integer function.

(A) Two (B) Four (C) Infinite (D) None of these

21. Number of solutions of the equation $\log_{10}|x| = \sin x$ are

(A) 6 (B) 7 (C) 5 (D) 8

22. The minimum vertical distance between the graphs of $y = 2 + \sin x$ and $y = \cos x$ is
(A) 2 (B) 1 (C) 3 (D) 4

23. The range of a for which the equation $e^{-\cos x} + a = e^{\cos x}$ possess infinite solution
(A) $(-\infty, \infty)$ (B) $(-1, 1)$ (C) $(0, e - 1/e]$ (D) $(0, e + 1/e]$

24. The number of roots of $|\sin |x|| = x + |x|$ in $[-2\pi, 2\pi]$ is -
(A) 2 (B) 3 (C) 4 (D) 6

25. Number of solutions of the equation $|x - 1| + |x + 1| \le 1$ are
(A) 1 (B) 2 (C) infinite (D) 0

26. Let $f(x) = \max \{\tan x, \cot x\}$. Then number of roots of the equation $f(x) = \dfrac{1}{\sqrt{3}}$ in $(0, 2\pi)$ is
(A) 2 (B) 4 (C) 0 (D) None

27. Number of solutions of $3^{|x|} = |2 - |x||$ is
(A) 0 (B) 2 (C) 4 (D) infinite

28. How many roots does the following equation posses , $x^2 - 2x - \log_2 |1 - x| = 3$?
(A) 2 (B) 3 (C) 4 (D) 5

29. The equation $e^{2x} + e^x - e = 0$ has
(A) no real root in the interval [0, 1] (B) at least one real root in [0, 1]
(C) two real roots in [0, 1] (D) none of these

30. Area of the common region represented by the inequalities $x - y + 5 \ge 0$, $x - y - 5 \le 0$, $0 \le y \le 5$, is
(A) 25 sq. units (B) 50 sq. units (C) 75 sq. units (D) 100 sq. units

31. The number of times the graphs of $y = |1 - 2x|$ and $y = 1 - |x|$ intersect is
(A) once (B) twice (C) thrice (D) None

32. For the equation $|x^2 - 2x - 3| = b$ which statement or statements are true
(A) for $b < 0$ there are no solutions (B) for $b = 0$ there are three solutions
(C) for $0 < b < 1$ there are four solutions (D) for $b = 1$ there are two solutions

33. Area enclosed by the curve,
$|x + y - 1| + |2x + y + 1| = 1$ is
(A) 2 sq. units (B) 1 sq. units (C) 4 sq. units (D) none of these

34. Number of solutions of $\sin x/2 + 2\pi x = x^2 + \pi^2 + 1$ is
(A) 0 (B) 1 (C) 2 (D) none

35. The number of ordered pairs of integers x and y which satisfy $|x| + |y| < 21$ is
(A) 210 (B) 420 (C) 821 (D) 841

36. The graph of the curve $y = \dfrac{1}{x}$ and $y = px$ intersects at
(A) one point for any value of p (B) two points for any values of p
(C) two points for $p \ge 0$ (D) None of these

37. Number of roots of the equation $3^{|x|} . |2 - |x|| = 1$ is
(A) 0 (B) 2 (C) 4 (D) 8

38. Equation $|x - 4| + |x + 4| = ax + 8$ has
(A) exactly one solution if $a \in (-\infty, -2] \cup [2, \infty)$
(B) exactly one solution if $a \in (-2, 2)$

(C) exactly two solutions if $a \in (-2, 0) \cup (0, 2)$
(D) exactly two solutions if $a = 0$

39. The value of $b > 0$ for which the region bounded by both the x-axis and $y = -|2x| + b$ has an area of 72, is
(A) 12 (B) 36 (C) $6\sqrt{2}$ (D) 144

40. The equation $||x - 2| + a| = 4$ can have four distinct real solutions for x if a belongs to the interval
(A) $(-\infty, -4)$ (B) $(-\infty, 0)$ (C) $[4, \infty)$ (D) None of these

41. The solution set of the equation,
$(x + 1) \log_3^2 x + 4 x \log_3 x - 16 = 0$ is:
(A) an empty set (B) a singleton
(C) a set consisting of exactly two elements (D) a set consisting of more than two elements

42. If $\sin^{-1}[x] > \cos^{-1} \{x\}$, then the values of x satisfying the inequality is, (where [ ] denotes greatest integer function and {x} denotes fractional part of x):
(A) $[1, 2)$ (B) $(1, 2]$ (C) $(1, 2)$ (D) $\left(\frac{1}{2}, 1\right)$

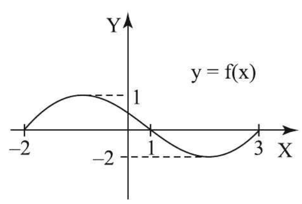
43. The number of solutions of the equation sgn $(\{x\}) = |x - 1|$ is (where {x} denotes fraction part of x)
(A) 1 (B) 2 (C) 3 (D) infintite

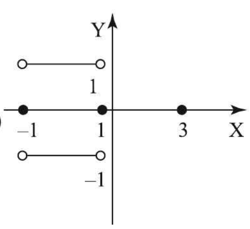
44. The number of solutions of $x^2 - 2[x] + 3 = 0$ is (where, [ ] denotes the greatest integer function).
(A) 1 (B) 2 (C) 3 (D) None

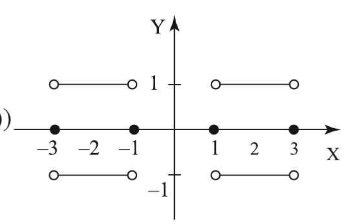
45. The number of solutions of the equation $x^2 - 2 - 2[x] = 0$ ([.] denotes greatest integer function) is
(A) one (B) two (C) zero (D) infinity

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

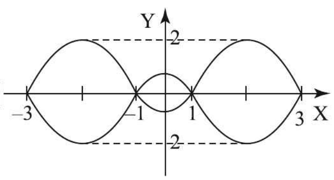
46. The graph of the function $y = f(x)$ is as shown in the figure. The which of the following graphs are correct ?



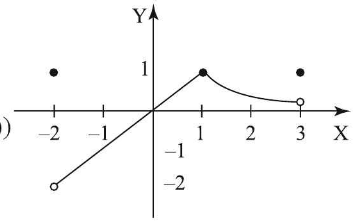
(A) $|y| = \text{sgn}(f(x))$


(B) $|y| = \text{sgn}(-f(|x|))$


(C) $|y| = |f(|x|)|$



(D) $y = x\,\mathrm{sgn}(f(x))$



47. The function $f(x) = \begin{cases} x+2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 1 \\ (x-2)^2 & \text{if } x \ge 1 \end{cases}$

(A) is continuous for all $x \in R$
(B) is continuous but not differentiable $\forall\, x \in R$
(C) is such that $f'(x)$ changes its sign exactly twice
(D) has two local maxima and two local minima.

48. Which of the following statements concerning the function

$f(x) = \begin{cases} \dfrac{5x^2 - 10x - 15}{x^2 - x - 6}, & \text{if } x \ne 3 \\ 4, & \text{if } x = 3 \end{cases}$ are correct

(A) $f(x)$ is continuous for all $x \ne -2$
(B) $f(x)$ is always decreasing
(C) $f(x)$ has no vertical asymptotes
(D) $f(x)$ has one horizontal asymptote.

49. Let $f(x) = \dfrac{x-1}{x^2}$ then which of the following is correct.

(A) $f(x)$ has minima but no maxima.
(B) $f(x)$ increases in the interval $(0, 2)$ and decreases in the interval $(-\infty, 0) \cup (2, \infty)$.
(C) $f(x)$ is concave down in $(-\infty, 0) \cup (0, 3)$.
(D) $x = 3$ is the point of inflection.

50. Which of the following statement(s) is/are true for the function $f(x) = (x-1)^2(x-2) + 1$ defined on $[0, 2]$?

(A) Range of f is $\left[\dfrac{23}{27}, 1\right]$.
(B) The extrema in the graph of $y = f(x)$ occur at $(1, 1)$ and $\left(\dfrac{5}{3}, \dfrac{23}{27}\right)$.
(C) The value of p for which the equation $f(x) = p$ has 3 distinct solutions lies in interval $\left(\dfrac{23}{27}, 1\right)$.
(D) none of these

51. Suppose f is defined from $R \to [-1, 1]$ as

$f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ where R is the set of real number. Then the statement which does not hold is

(A) f is many-one and onto
(B) f increases for $x > 0$ and decrease for $x < 0$
(C) minimum value is attained
(D) maximum value is not attained even though f is bounded.

Comprehension - 1

Let $x^+ = \begin{cases} x & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$ and

let $x^- = \begin{cases} -x & \text{if} \quad x \leq 0 \\ 0 & \text{if} \quad x > 0 \end{cases}$

52. The incorrect statement is

(A) $x = x^+ - x^-$ (B) $x^+ = \dfrac{|x| + x}{2}$ (C) $|x| = x^+ + x^-$ (D) None of these

53. The number of solutions of the equation

$(x - 1)^+ + (x - 1)^- = 2x^+ - 2x^-$ is

(A) zero (B) one (C) two (D) infinite

54. The complete solution set of the inequality $(x + 1)^+ - 2\,(x - 2)^- \geq 0$ is

(A) $[1, \infty)$ (B) $[2, \infty)$ (C) $(-\infty, -1]$ (D) None of these

Comprehension - 2

Consider the quadratic function $q : R \to R$ with $q(x) = x^2 - 6x + 5$

55. How many of the following assertions is (are) true ?

(a) q is concave up

(b) q is invertible over R

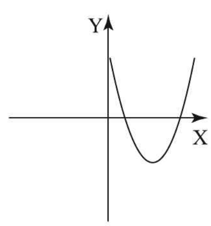
(c) the graph q has vertex $(-3, -4)$

(d) the graph of q has y-intercept $(0, 5)$ and x-intercepts $(-1, 0)$ and $(5, 0)$

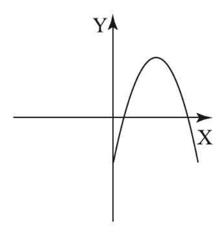
(A) none (B) exactly one (C) exactly two (D) exactly three

56. Which one most resembles the graph of $y = -q(x + 3)$?

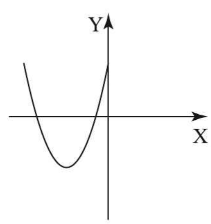
(a)



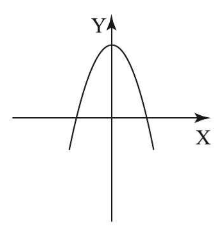
(b)



(c)

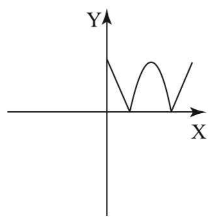


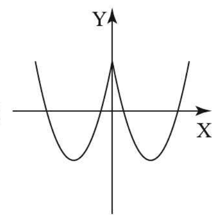
(d)

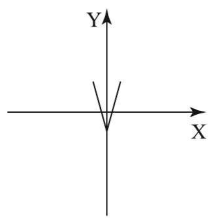


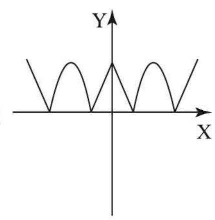
(A) a (B) b (C) c (D) d

57. Which one most resembles the graph of $y = q(|x|)$?

(a)


(b)


(c)


(d)


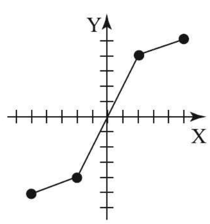
(A) a (B) b (C) c (D) d

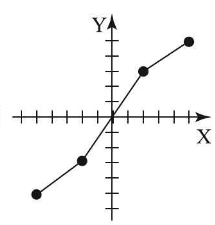
Comprehension - 3

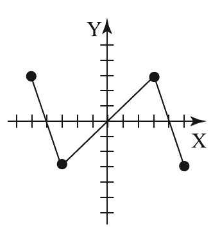
Consider the function f :

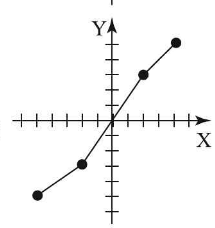
$$y = f(x) = \begin{cases} \dfrac{x}{3} - \dfrac{10}{3} \text{ if } x \in [-5, -2) \\ 2x \text{ if } x \in [-2, 2] \\ \dfrac{x}{3} + \dfrac{10}{3} \text{ if } x \in (2, 5] \end{cases}$$

58. Which one most resembles the graph of f ?

(a)


(b)


(c)


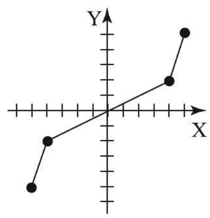
(d)


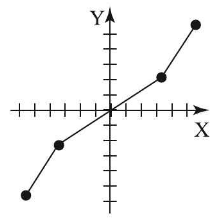
(A) a (B) b (C) c (D) d

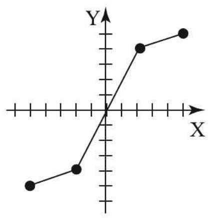
59. Which one could not possibly be a possible value for $\underbrace{(f \text{ o} \ldots \text{o} f)}_{\text{n compositions}}(a)$, where n is a positive integer and $a \in [-5, 5]$?

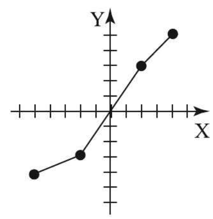
(A) 0 (B) –5 (C) 5 (D) 6

60. Which one most resembles the graph of f–1 ?

(A) 

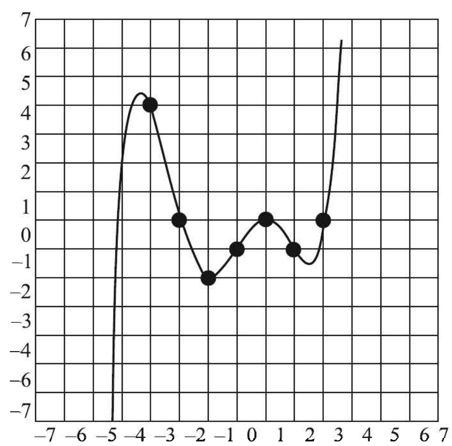
(B) 

(C) 

(D) 

Comprehension - 4

Consider to the polynomial p in figure. The polynomial has degree 5. You may assume that the points marked with dots have integer coordinates.



61. Determine the value of p(–3) p(0)

(A) 0 (B) 4 (C) –4 (D) None of these

62. Determine p(x)

(A) $\dfrac{(x-3)(x+2)(x+4)(x-1)^2}{24}$ (B) $(x-3)(x+2)(x+4)(x-1)^2$

(C) $\dfrac{(x-3)(x+2)(x+4)(x-1)}{24}$ (D) $(x-3)(x+2)(x+4)(x-1)$

63. Determine the value of (p o p) (–3)

(A) 4 (B) 18 (C) 20 (D) 24

Comprehension - 5

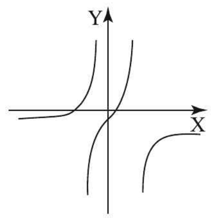
Consider rational function f, with

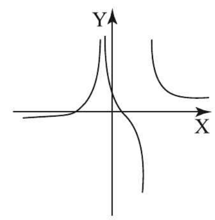
$$f(x) = \frac{(x-1)(x+2)}{(x+1)(x-2)}.$$

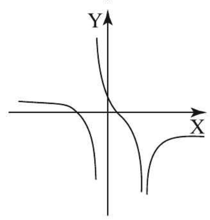
64. Which of the following is a horizontal asymptote for?
(A) $y = -1$ (B) $y = 1$ (C) $y = 0$ (D) $y = 2$

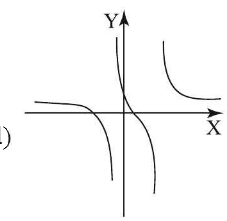
65. Where are the poles of f ?
(A) $x = 1$ and $x = -2$ (B) $x = -1$ and $x = -2$ (C) $x = -1$ and $x = 2$ (D) $x = 1$ and $x = 2$

66. The graph of $y = f(x)$ most resembles

(a)


(b)


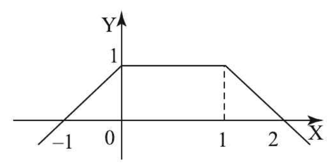
(c)


(d)


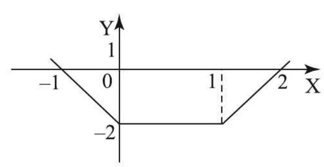
(A) a (B) b (C) c (D) d

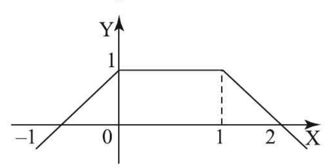
MATCH THE COLUMNS FOR JEE ADVANCED

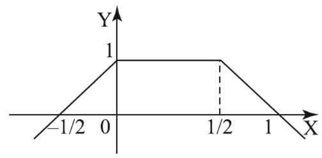
67. Given the graph of $y = f(x)$, match the following graph.

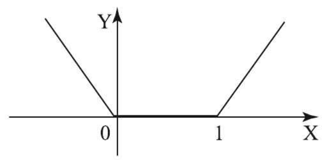


Column - I **Column - II**

(A) $y = f(1-x)$ (P)


(B) $y = f(2x)$ (Q)


(C) $y = -2\,f(x)$ (R) 

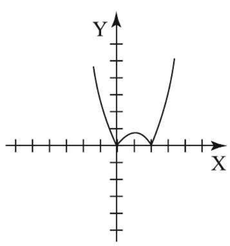
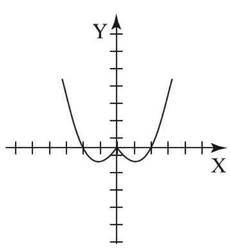
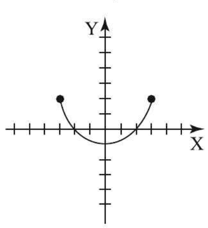
(D) $y = 1 - f(x)$ (S) 

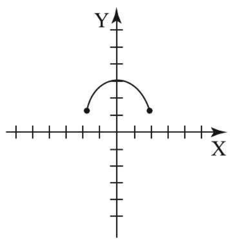
68. Consider the equation $(x - 1)2 = |x - k|$.

Match the value of parameter k in the column - I with the number of solutions in Column II.

Column - I	**Column - II**
(A) $k = \frac{3}{4}$	(P) exactly one solution
(B) $k = 1$	(Q) exactly two solutions
(C) $k = \frac{5}{4}t$	(R) exactly three solutions
(D) $k = 0$	(S) exactly four solutions

69. Match the letter of figures with the equations

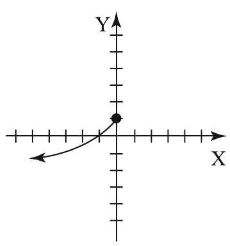
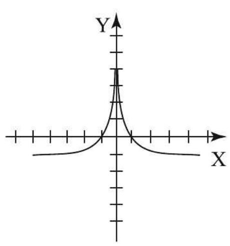
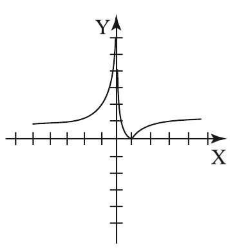
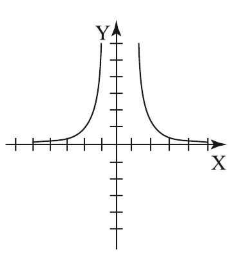
Column - I	**Column - II**
(A) 	(P) $y = 1 + \sqrt{4 - x^2}$
(B) 	(Q) $y = 2 - \sqrt{9 - x^2}$
(C) 	(R) $y = x^2 - 2\|x\|$

(D) 

(S) $y = 2 + \sqrt{9 - x^2}$

(T) $y = |x^2 - 2x|$

70. Match the letter of figures with the equations

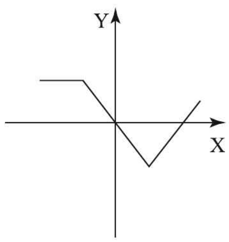
Column - I	**Column - II**
(A) 	(P) $y = \dfrac{1}{\lvert x\rvert - 1}$
(B) 	(Q) $y = 1\sqrt{-x}$
(C) 	(R) $y = \dfrac{1}{\sqrt{x^2 - 1}}$
(D) 	(S) $y = \dfrac{1}{\lvert x\rvert} - 1$
	(T) $y = \left\lvert \dfrac{1}{x} - 1 \right\rvert$

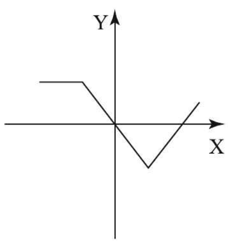
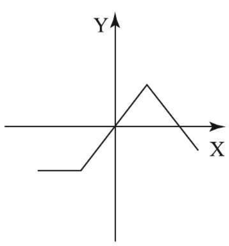
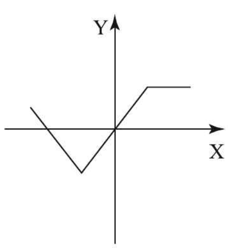
71.

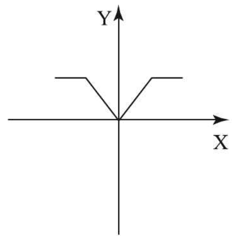
Column - I	Column - II
[.] and { . } represent the greatest integer and fractional part functions respectively.	
(A) Number of solutions of [x] = cos–1 x	(P) 3
(B) Number of solutions of sin–1 x = sgn(x)	(Q) 2
(C) Number of solution of $\{x\} = e^{x^2}$	(R) 1
(D) Number of solutions of $\dfrac{\sin^{-1} x + \cos^{-1} x}{2} = (x)$	(S) 0

72. There are five graphs shown below. The first graph is that of the original curve y = f(x), and the other four are various transformations of the original graph. You are to match each graph letter below with the appropriate equation

y = f(x)



Column - I	Column - II
(A) 	(P) y = f(–x)
(B) 	(Q) y = –f(x)
(C) 	(R) y = f(|x|)
	(S) y = |f(x)|

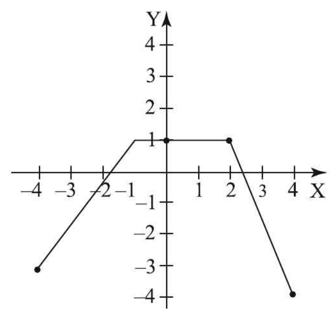
(D) 

(T) $y = f(-|x|)$

REVIEW EXERCISES for JEE ADVANCED

1. Consider the function $f : R - \{-1, 1\} \to R$ where
$$f(x) = \begin{cases} -x & \text{if } x < -1 \\ x^2 & \text{if } -1 < x < 1 \\ x & \text{if } x > 1 \end{cases}$$. Draw its graph.

2. Consider the function $f : [-4, 4] \to [-5, 1]$ whose graph is made of straight lines, as shown in the figure. Find a piecewise formula for f.



3. Sketch the graph of the functions:
(i) $y = 1 + \ln(x - 2)$ (ii) $y = 3 + e^{x-2}$ (iii) $y = 1 - e^{-x+1}$ (iv) $y = 3 \ln \sqrt[3]{|1-x|}$

4. Construct the graphs of the following functions :
(i) $y = -\sin \dfrac{x}{3}$ (ii) $y = \sin\left(x - \dfrac{\pi}{3}\right)$ (iii) $y = \cos\left(x + \dfrac{\pi}{4}\right)$ (iv) $y = \tan 3x$

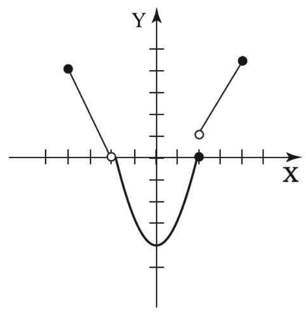
5. Construct the graphs of the following functions:
(i) $y = -\tan \dfrac{x}{2}$ (ii) $y = \tan\left(\dfrac{\pi}{6} - x\right)$

6. Construct the graphs of the following functions:
(i) $y = \sin^{-1}(x - 2)$ (ii) $y = \cos^{-1}\left(\dfrac{x}{2}\right)$ (iii) $y = \tan^{-1}(x + 1)$ (iv) $y = \cot^{-1}(3 - x)$

7. Sketch the graph of the following :
(i) $y = \left(\dfrac{1}{2}\right)^{x-1} - 1$ (ii) $y = \ln|x|$ (iii) $y = \sin|x|$ (iv) $y = \tan|x|$

8. Show that if the graph of a curve has x-axis symmetry and y-axis symmetry then it must also have symmetry about the origin.

9. Consider the graph of the curve $y = f(x)$. It may be assumed that the domain of f can be written in the form $[a, b) \cup (b, c]$, where a, b, c are integers, and that its range can be written in the form $[u, v]$, with u and v integers. Find a, b, c, u and v.



10. What is the equation of the resulting curve after $y = x^2 - x$ has been, successively, translated one unit up and reflected about the y-axis ?

11. Suppose $\phi : R \to R$ is defined to be $\phi(t) = |t|$ for $-1 \le t \le 1$ and $\phi(t + 2) = \phi(t)$ for $t \in R$. Draw its graph.

12. Let f(x) = distance of x from the nearest integer. Draw the graph of f.

13. Identify the curves given by the following parametric equaion and write equations for the curves in terms of x and y.
   (i) $x = 3t + 5, y = 4t - 1$ (ii) $x = t + 2, y = t2$
   (iii) $x = t - 2, y = \dfrac{t}{t-2}$ (iv) $x = 5 \cot t, y = 5 \sin t$

14. Construct the graph of the following functions:
   (i) $y = \dfrac{x+1}{x-1}$ (ii) $y = y = \dfrac{2x-2}{x+3}$

15. Draw the graph of the following functions:
   (i) $y = (x + 3)/(x - 1)$ (ii) $y = (2\,|\,x\,| - 1)/(x - 3)$

16. Construct the graph of the following functions:
   (i) $y = |\,|x - 1| - 2\,|$ (ii) $y = x + |\,x - 1\,| + \dfrac{|\,x-2\,|}{x-2}$
   (iii) $y = \sqrt{(x^2+1)^2 - 4x^2}$ (iv) $y = |x|^2 - 2|x| - 3$.

17. Let $f(x) = [x]$ and let $g(x) = [2x]$ for all real x. In each case, draw the graph of the function h defined over the interval $[-1, 2]$ by the formula given.
   (i) $h(x) = f(x) + g(x)$ (ii) $h(x) = f(x) + g(x/2)$ (iii) $h(x) = f(x)\,g(x)$.

18. Draw the graphs of
   (i) $f(x) = x + [x]$ (ii) $g(x) = [-x]$
   (iii) $f(x) = 2[x]$ (iv) $f(x) = [x] + \left[x + \dfrac{1}{2}\right]$

19. In each case, sketch the graph of the function f defined by the formula given.
   (i) $f(x) = [\sqrt{x}]$ for $0 \le x \le 10$.
   (ii) $f(x) = \sqrt{[x]}$ for $0 \le x \le 4$.
   (iii) $f(x) = [x]^2$ for $0 \le x < 3$.
20. Determine any intercepts with the axes and any symmetries of the curve $y^2 = |x^3 + 1|$.
21. Sketch the graph of the curve $y = \sqrt{\dfrac{1-x}{x+1}}$ and label the axis intercepts and asymptotes.
22. Draw the graph of the curves
   (i) $(x + y)^2 = 1$ (ii) $x^2 - y^2 = 4$ (iii) $\sqrt{y^2} = x^2 - 3x + 2$.
23. Determine the number of roots of
   $\sin x = x$, $\sin x = 1/3\ x$, $\sin x = 1/8\ x$, $\sin x = 1/120\ x$.
24. Draw the graph of the following functions:
   (i) $y = 2^{(|x|+x)/x}$ (ii) $y = 2^{|\log_2 x|}$ (iii) $y = \log_x \sqrt{x}$ (iv) $y = |\log_2 x| / \log_2 x$
25. Sketch the region $\{(x, y) : x^2 \le y \le |x|\}$.
26. Draw the graph of the function
   $$y = \begin{cases} 1 - \sin x, & (2n+1)\pi < x < 2n\pi \\ \sin^2 x, & 2n\pi \le x \le (2n+1)\pi \end{cases}, n \in I,$$
   in the interval $[0, 3\pi]$.
27. Give piecewise definition of
   $g(x) = \max\left\{\dfrac{1}{x^2}, \dfrac{1}{x^3}, \dfrac{1}{2}\right\}$, in the interval [1/2, 2] and draw its graph.
28. Given in figures (i) and (ii) are the graphs of two curves, $y = f(x)$ and $y = f(ax)$ for some real constant a. Determine the value of C.

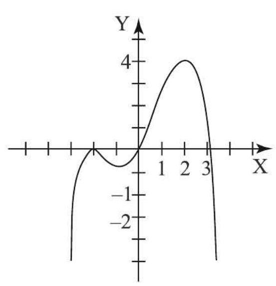


Fig (i)

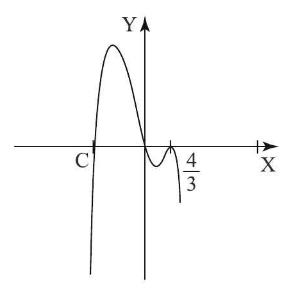
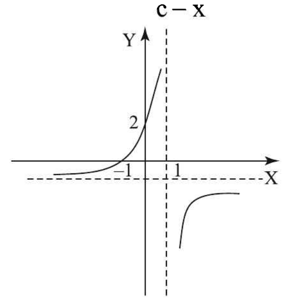


Fig (ii)

29. The figure below shows a curve of the form $y = \dfrac{ax + b}{c - x}$. Find (a, b, c).



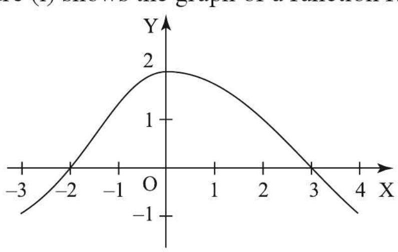
30. If the graph of $f(x) = \dfrac{2x^2+6x}{x^2+3x-4}$ is symmetric about the line x = a, find a.
31. Draw the graph of f(x) = max {sinx, cosx} $\forall$ x $\in$ R. Find the least value of f(x).
32. Draw the polynomial $P(x) = \dfrac{1}{2}x^4 - 2x^2$.
33. Find the least and the greatest value of the function $f(x) = \dfrac{1}{5}x^3 + \dfrac{9}{20}x^2 - \dfrac{1}{2}x$ on the interval [– 2, 1] and construct its graph on that interval.
34. Investigate the behaviour of the fuction $y = (x^4 - 2x^2)/4$ with the aid of the derivative and construct its graph.
35. Find the number of solutions of the following equations:
    (i) $2\cos 6x - \cos 2x + 1 = 0, x \in [0, \pi]$
    (ii) $2^{\cos x} = \sin x, x \in [-\pi, \pi]$
    (iii) $\sin\left(\dfrac{\pi x}{2}\right) = \dfrac{99x}{500}, x \in R$
    (iv) $\dfrac{3}{4\pi}x^2 - \cos \pi x = 0, x \in R$

TARGET EXERCISES for JEE ADVANCED

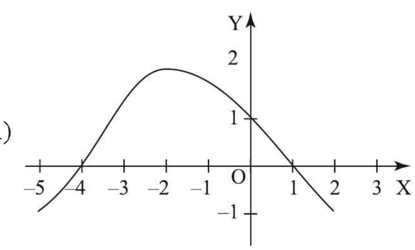
1. The curve $y = \dfrac{x}{x^2-x-1}$ undergoes the following successive transformations :
    (i) a translation one unit right,
    (ii) a reflection about the x-axis,
    (iii) a translation two units up,
    (iv) a reflection about the y-axis.

    Find the equation of the resulting curve.
2. Figure (i) shows the graph of a function f(x) with domain [–3, 4] and range [–1, 2].

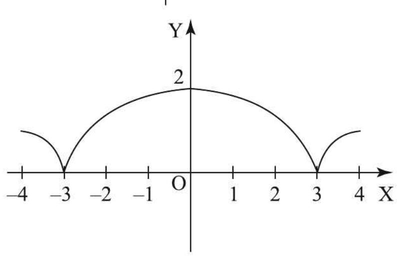
(i)



(ii)



(iii)



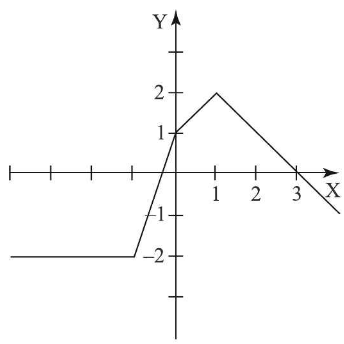
    (i) Find the function represented by Figure (ii).
    (ii) Find the function represented by Figure (ii).
    (iii) Find the domain and range of y = 1 – f(x + 1)
    (iv) Find the number of points of intersection of the curve in figure (iii) and $(2x-6)^2 + 4y^2 = 49$.

3. Draw the graph of the following functions:
   (i) $y = |x^2 - 4x + 3| + 2x$ (ii) $y = x^2 - 2|x + 1| - 1$ (iii) $y = (|1 - x| + 2)(x + 1)$

4. Draw the graph of the following functions:
   (i) $y = 2 - \dfrac{4}{|x-1|}$ (ii) $|y| = 2 - \dfrac{4}{|x-1|}$ (iii) $|y| = \left|2 - \dfrac{4}{|x-1|}\right|$

5. Draw the graph of the following curves:
   (i) $y = \log_2 |x - 1| - 1$ (ii) $|y| = |e^{|x|} - 2|$
   (iii) $|y| = |x - 1| + |x + 1|$ (iv) $y = \left|\dfrac{1}{|x|-2}\right|$

6. Given below is the graph of a curve $y = f(x)$. Find the graphs of
   (i) $y = -f(x)$ (ii) $y = |f(x)|$ (iii) $y = |f(x)| - 1$
   (iv) $y = |f(x - 1)|$ (v) $y = f(|x|)$.



$y = f(x)$

7. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} 1 & \text{for } 2n < x \le 2n+1, (n \in I) \\ 2 & \text{for } 2n+1 < x \le 2n+2 \end{cases}$$

   Draw its graph.

8. Draw the graphs of the following functions :
   (i) $f(x) = \text{sgn}(1-|x|)$ (ii) $f(x) = \text{sgn}([x^2 - x])$
   (iii) $f(x) = \min. \{|x|, x^2, 2\}$, $[-4, 4]$ (iv) $f(x) = \max(|\tan x|, \cos|x|)$, $x \in [-\pi, \pi]$.

9. Let $f(x) = x^2 - 4|x|$ and

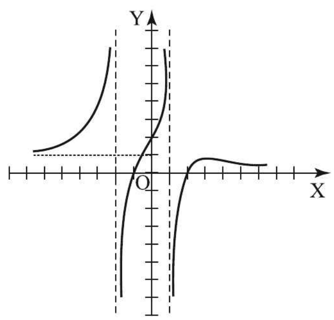
$$g(x) = \begin{cases} \min\{f(t) : -6 \le t \le x\}, & x \in [-6, 0) \\ \max\{f(t) : 0 \le t \le x\}, & x \in [0, 6] \end{cases}$$

   then find the value of $g(-1) + g(1) + g(5)$.

10. Draw the graph of the following functions :
   (i) $y = \dfrac{e^x + e^{-x}}{2}$ (ii) $y = \dfrac{e^x - e^{-x}}{2}$ (iii) $y = [x] + \sqrt{\{x\}}$

11. Draw the graphs of the function
   (i) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (ii) $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

12. Draw the graph of the following functions :
 (i) $y = \max \{x + |x|, x - [x]\}$ (ii) $y = \sin (\pi \{ |x| + [x]\})$
 (iii) $y = e^x \sin x$
13. The polynomial $p(x) = x^4 - 4x^3 + 4x^2 - 1$ has a local maximum at (1, 0) and local minima at (0, –1) and (2, –1).
 (a) Factor the polynomial completely and sketch its graph.
 (b) Determine how many real zeros the polynomial $q(x) = p(x) + c$ has for each constant c.
14. The rational function q given in the figure has only two simple singularities and satisfies $q(x) \to 1$ as $x \to \pm \infty$. You may assume that the singularities and zeroes of q are located at integer points.



 (a) Find q(0). (b) Find q(x) for arbitrary x.
 (c) Find q(–3). (d) Find $\lim\limits_{x \to -2^+} q(x)$.
15. Draw the graph of the following functions:
 (i) $y = \dfrac{2\tan(x/2)}{1 - \tan^2(x/2)}$ (ii) $y = \ln \tan x + \ln \cot x$
16. Draw the graph of the following functions:
 (i) $y = \sin^2 x$ (ii) $y = \sqrt{\sin x}$
17. Draw the graph of the following functions:
 (i) $y = |\sin x| + |\cos x|$ (ii) $y = 2x - 1 + \dfrac{1}{(x+1)}$
18. Draw the graph of the following functions:
 (i) $y = \log_2(4x - x^2)$ (ii) $y = \log_{1/2}\left|\dfrac{2x-1}{x+1}\right|$ (iii) $y = \log_{1/2} \sin x$
19. Plot the region given by the following:
 (i) $y = \{(x, y) \mid (x - |x|)^2 + (y - |y|)^2 \le 4 \}$ (ii) $y = \{(x, y) \mid |x + y| + |y - x| \le 4 \}$
 (iii) $y = \{(x, y) \mid \log_{(|x| - 0.5)}(x^2 + y^2) \le \log_{(|x| - 0.5)} 4\}$
20. Sketch the region in the plane defined by each of the following equations.
 (i) $[x]^2 + [y]^2 = 1$ (ii) $[x]^2 - [y]^2 = 3$ (iii) $[x + y]^2 = 1$
21. Construct the graphs of the following :
 $\sin y = \sin x$.

22. Draw the graph of the following functions:

 (i) $y = \dfrac{1+x^2}{2+x^2}$ (ii) $y = \dfrac{x-1}{x^2}$ (iii) $y = \dfrac{x+1}{x^2+1}$

23. Draw the graph of the following functions:

 (i) $y = \sin^2 x - 2\sin x$ (ii) $y = \dfrac{\cos x}{\cos 2x}$ (iii) $y = \tan^{-1}\left(\dfrac{1}{x}\right)$

24. Draw the graph of the following curves:

 (i) $y = \dfrac{2x}{1+x^2}$ (ii) $y = \dfrac{(1+x)^4}{(1-x)^4}$ (iii) $y = \dfrac{x}{(1-x^2)^2}$ (iv) $y^2 = x^2\left(\dfrac{a+x}{b-x}\right)$

25. Show that the equaton $2x = (2n + 1)\,\pi(1 - \cos x)$, where n is a positive integer, has $2n + 3$ roots.

26. Show that the equation $(2/3)\, x \sin x = 1$ has four roots between $-\pi$ and π.

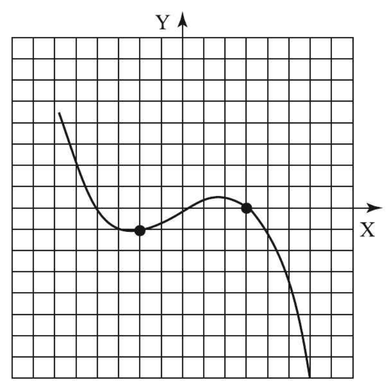
27. The curve $y = |x| + x$ undergoes the following successive transformations : (i) a translation 1 unit down, (ii) a reflection about the y-axis, (iii) a translation 2 units right. Find the equation of the resulting curve.

28. How many solutions does the system of equations

$$\begin{cases} |x| + |y| = 1 \\ x^2 + y^2 = a^2 \end{cases}, \text{ possess ?}$$

29. Find the graphical solution of the inequality $\sqrt{x-y} \ge \sqrt{x+y}$

30. The polynomial p shown in the figure has degree 3. You may assume that all its roots are integers.



 (a) Find p(–2), assuming it is an integer. (b) Find a formula for p(x).

31. Solve graphically

 (i) $\dfrac{x}{2} > 1 + \dfrac{4}{x}$ (ii) $\dfrac{3}{x-1} < \dfrac{2}{x+1}$

32. Solve for x, y : $|y| = 1 - x^2$ and $|x^2 - 2x| + y = 1$.

33. Draw the curves $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$ and find the number of integral coordinates lying within the region enclosed between the curves.

34. Discuss the number of the roots of the equations

 (i) $\cot x + x - 3/2\,\pi = 0$, (ii) $x^2 + \sin 2x = 1$,

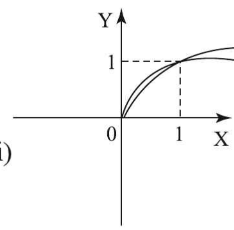
 (iii) $(1 + x^2)\tan x = 2x$, (iv) $\sin x - x + 1/6\, x^3 = 0$,

35. Given a function $y = (-x^2 + 3x - 1)/x$. Investigate its behaviour and make a rough drawing of the graph.
36. Investigate the behaviour of the function
    $y = (x + 4)/(x + 1)^3$ and construct its graph. How many solutions does the equation
    $(x^3 + 4)/(x + 1)^3 = c$ possess?

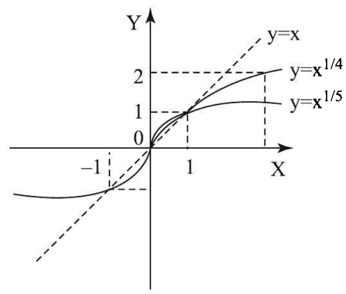
ANSWERS

PRACTICE PROBLEMS [A]

1. (i)



(ii)



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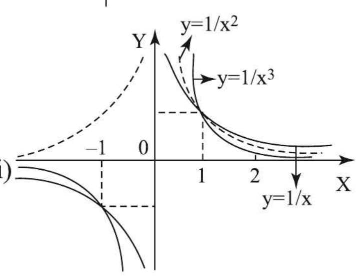
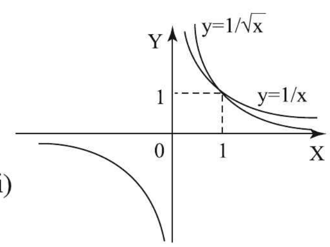
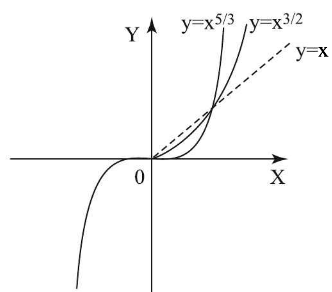


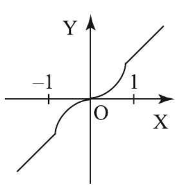
2. (i)



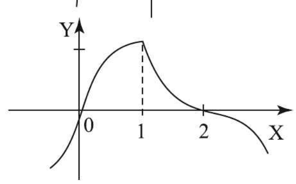
(ii)



3. (i)

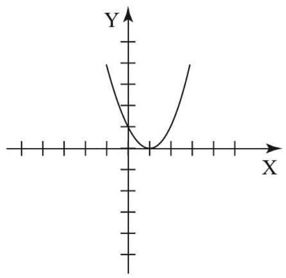


(ii)

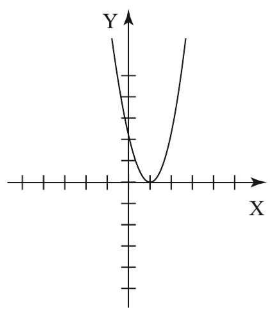


4. (i) 3 (ii) $[-5, -1) \cup (2, 5]$ (iii) $[-5, -3) \cup (2, 5]$

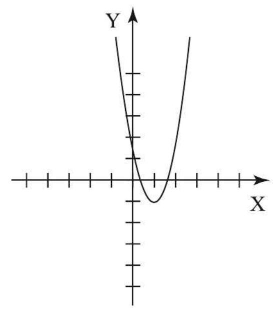
PRACTICE PROBLEMS [B]

1. 

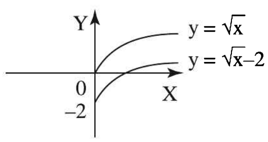
$y = (x-1)^2$

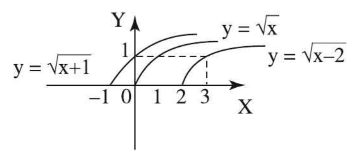


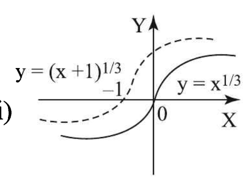
$y = 2(x-1)^2$

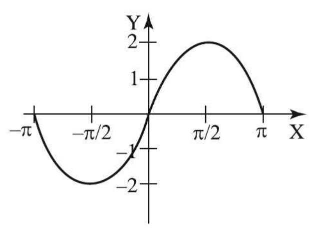


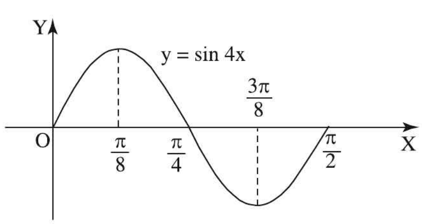
$y = 2(x-1)^2 - 1$

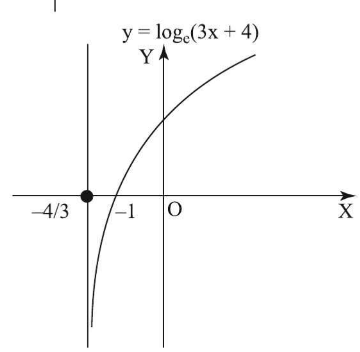
2. (i) 

(ii) 

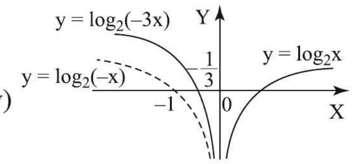
(iii) 

3. (i) 

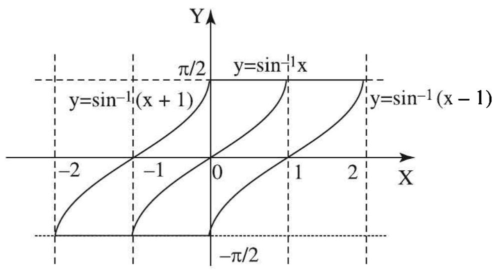
(ii) 

(iii) 

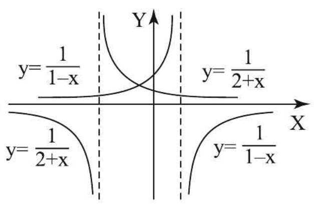
(iv)



4.



5. (i), (ii)



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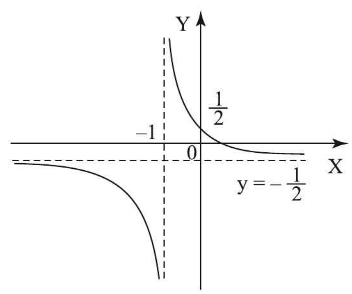
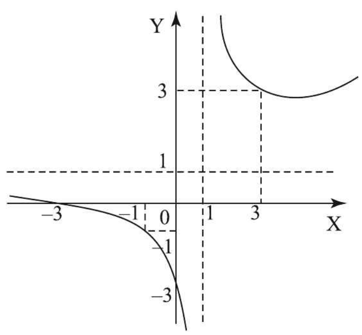


6. (i)



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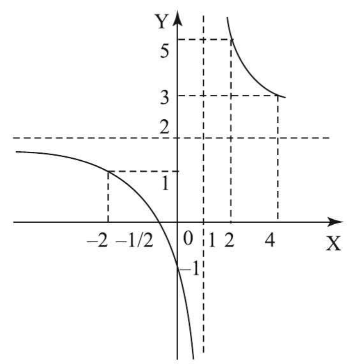
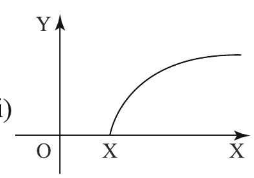


7. (i)



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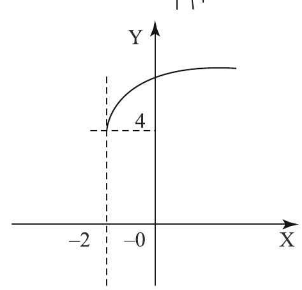

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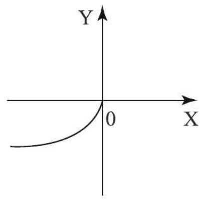

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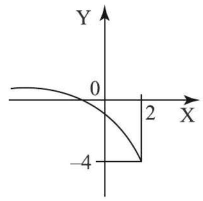
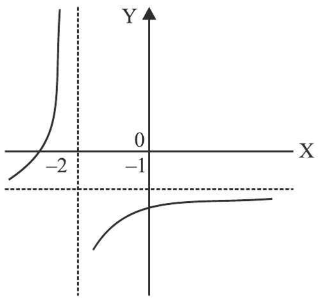


8. (i)



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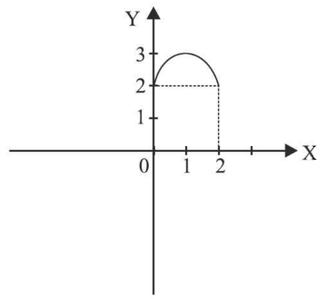

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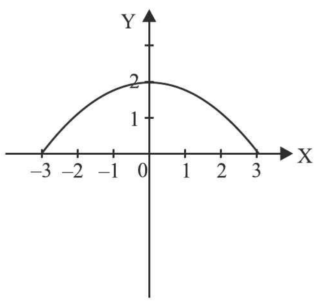

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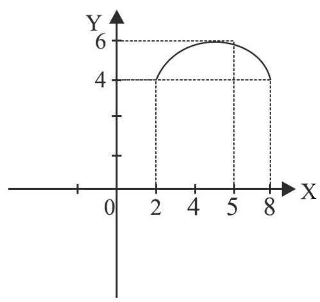
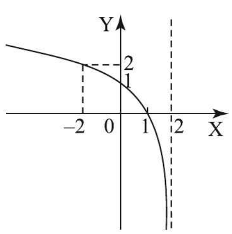


9. (i)



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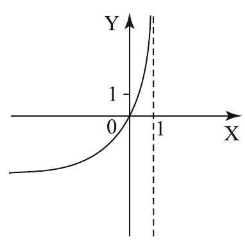
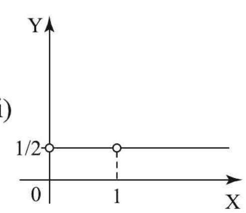


(iii)



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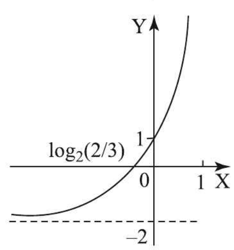
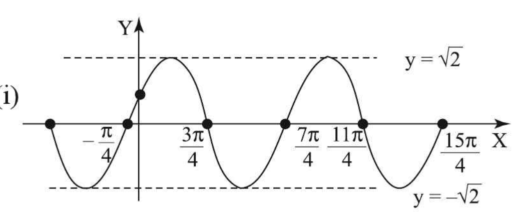
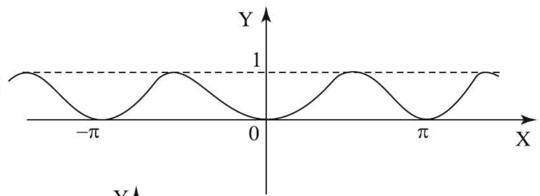


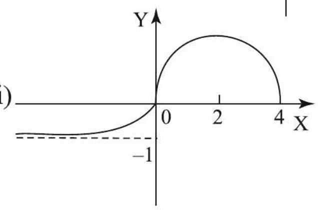
10. (i)



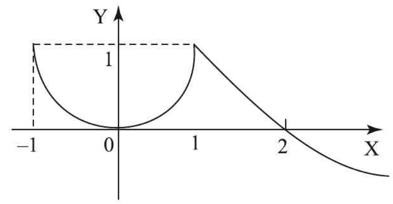
(ii)



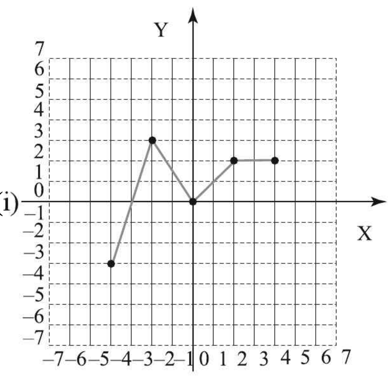
11. (i)



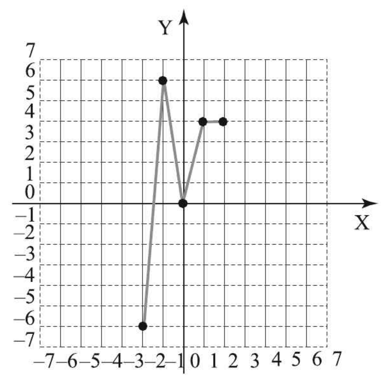
(ii)



12. (i)



(ii)

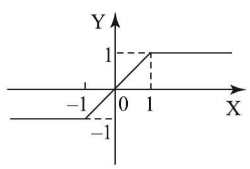


13. (a) $x^2 + 3xy + 2y^2 = 1$; (b) same as (a); (c) C itself.

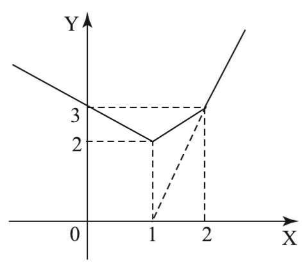
14. (a) $y = -(x + 1)(x + 2) - 2$ (b) $y = -2x - 7$ (c) $y = |1 - x| - 1$

PRACTICE PROBLEMS [C]

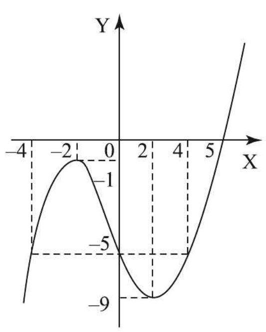
1. (i)



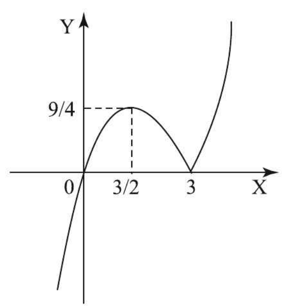
(ii)



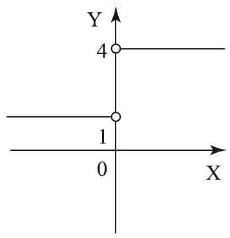
(iii)



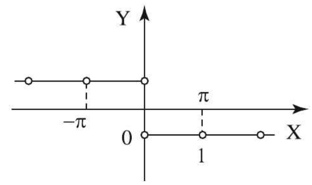
(iv)



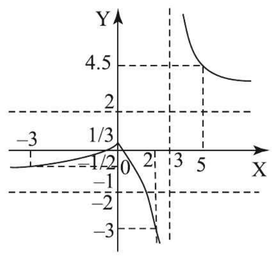
2. (i)



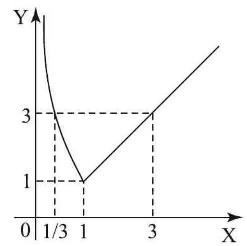
(ii)



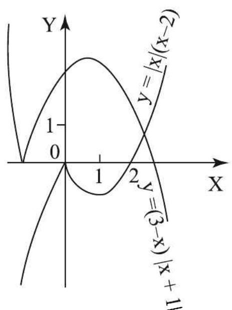
(iii)



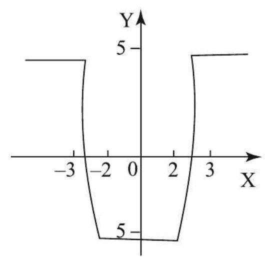
(iv)



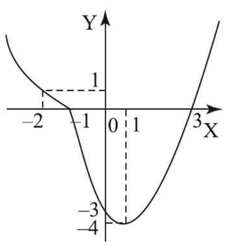
3.



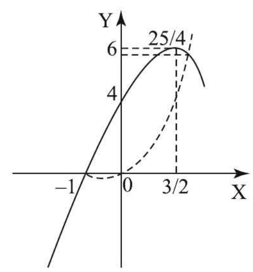
4.



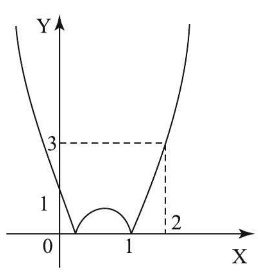
5. (i)



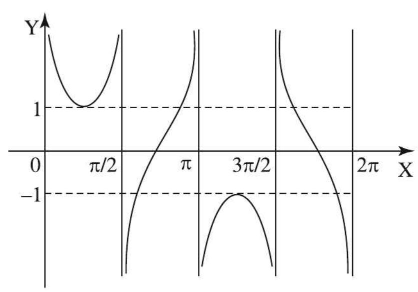
(ii)



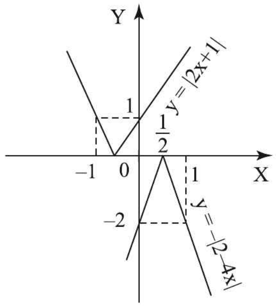
(iii)

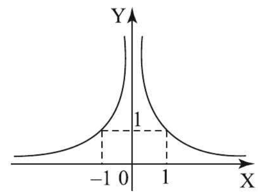


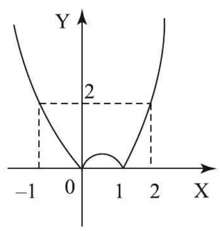
6.

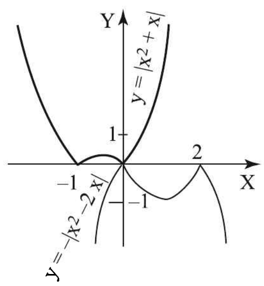


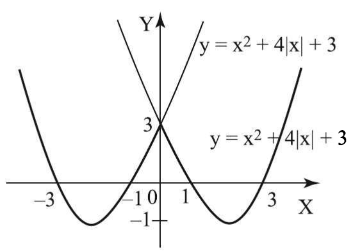
PRACTICE PROBLEMS [D]

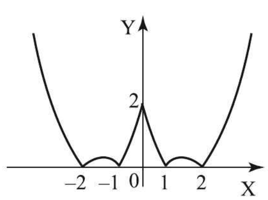
1. 

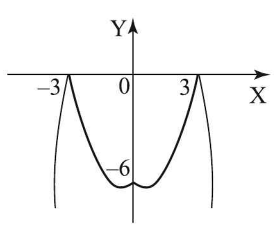
2. (i) 

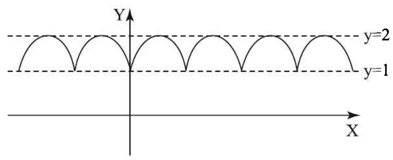
(ii) 

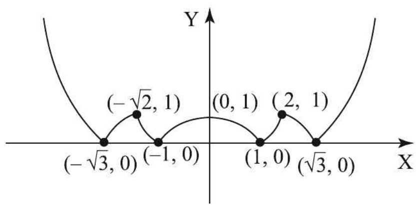
(iii), (iv) 

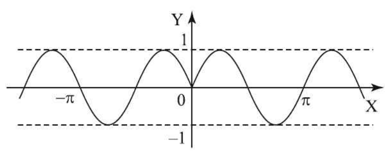
3. 

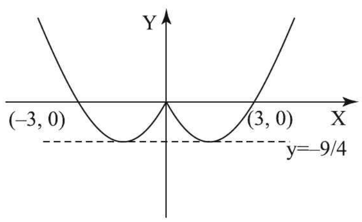
4. (i) 

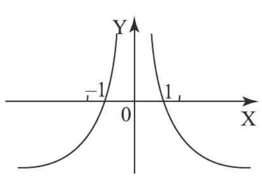
(ii) 

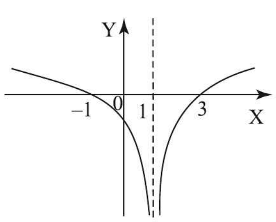
5. (i) 

(ii) 

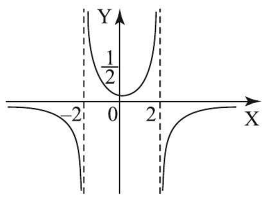
6. (i) 

(ii) 

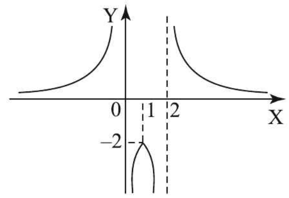
(iii) 

(iv) 

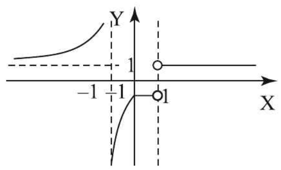
7. (i)



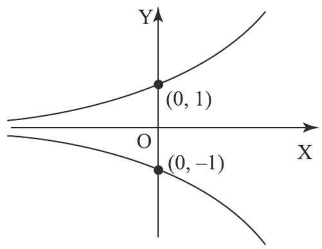
(ii)



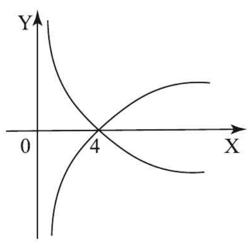
(iii)



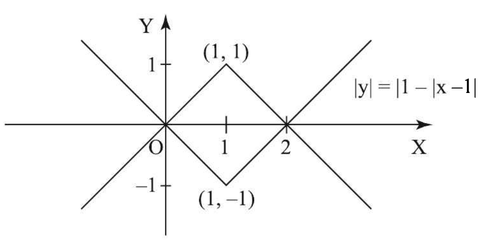
8. (i)



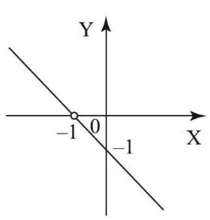
(ii)



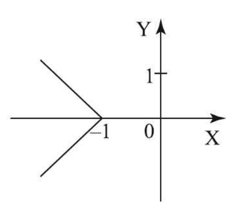
(iii)

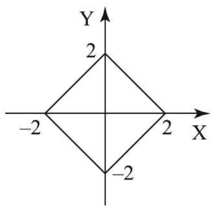


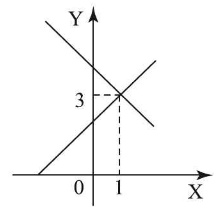
9. (i)

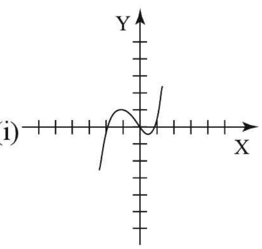


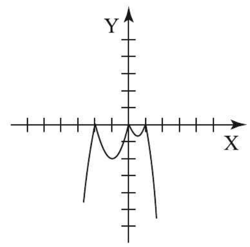
(ii)

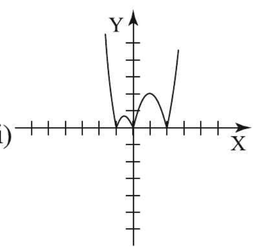


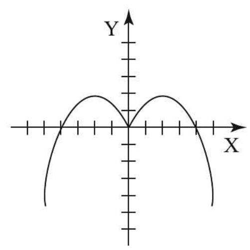
(iii) 

(iv) 

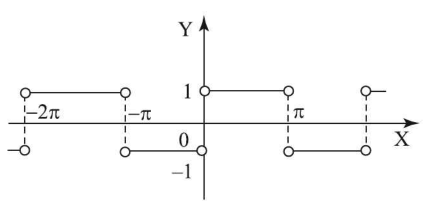
10. (i) 

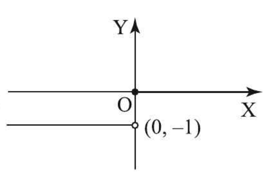
(ii) 

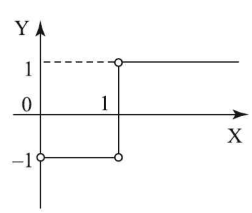
(iii) 

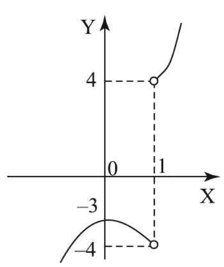
(iv) 

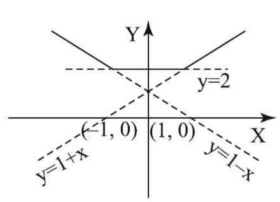
Practice Problems [E]

1. (i) 

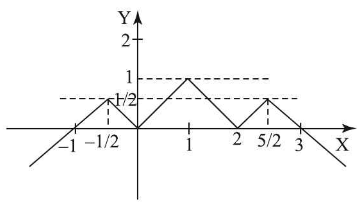
(ii) 

(iii) 

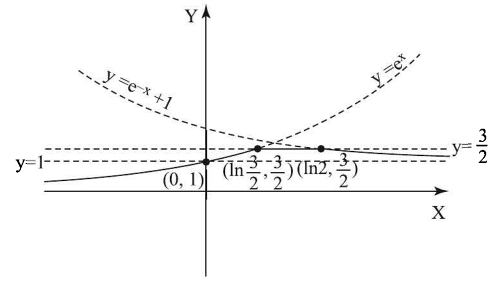
(iv) 

2. (i) 

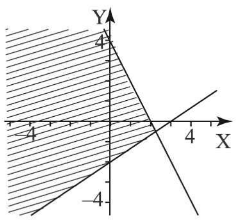
(ii)



(iii)

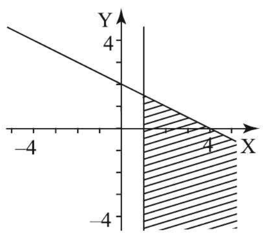


3. (i)



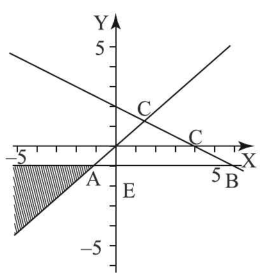
Solution of $\begin{cases} 2x - 3y \le 6 \\ 2x + y \le 4 \end{cases}$

(ii)

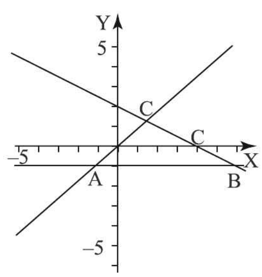


Solution of $\begin{cases} 2x + 5y \le 10 \\ x \ge 1 \end{cases}$

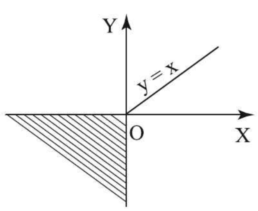
4. (i)



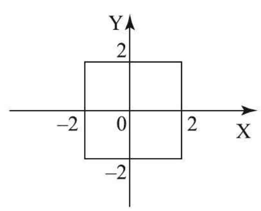
(ii)



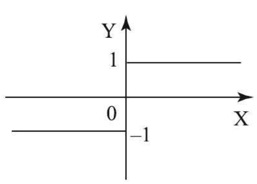
5. (i)



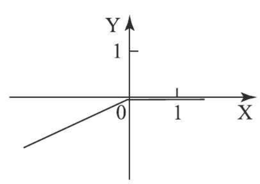
(ii)



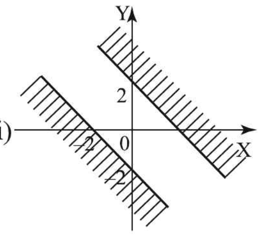
(iii)



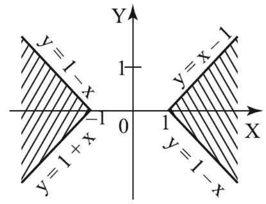
(iv)



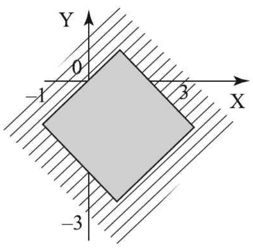
6. (i)



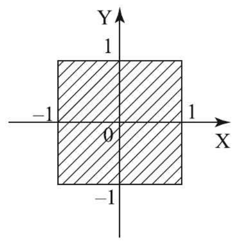
(ii)



(iii)

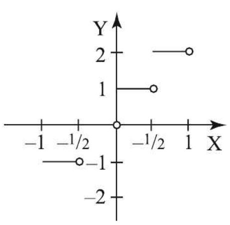


(iv)

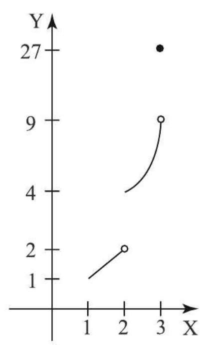


Practice Problems [F]

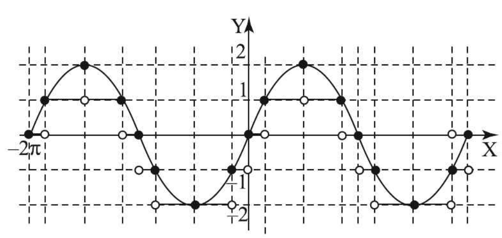
1. (i)



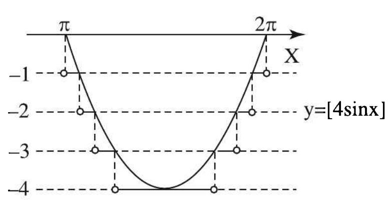
(ii)



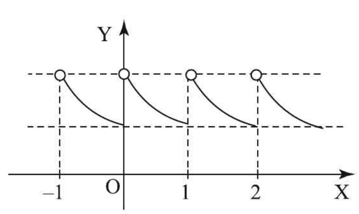
2.

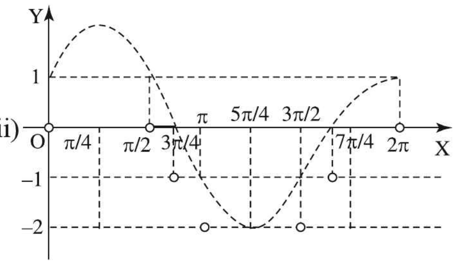


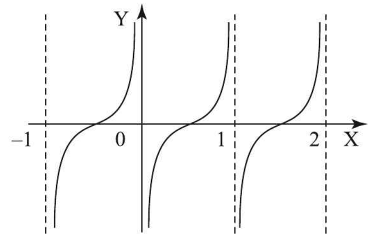
3.

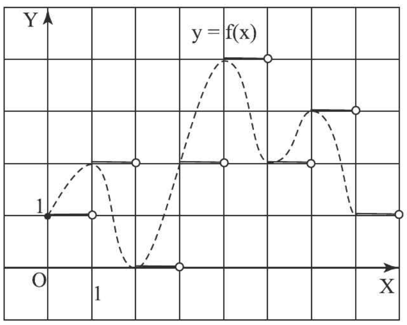


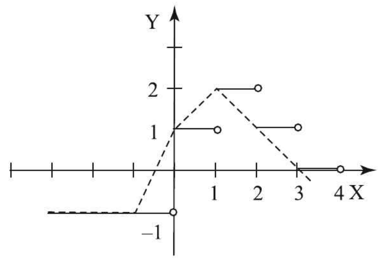
4. (i)

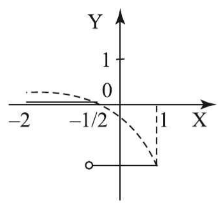


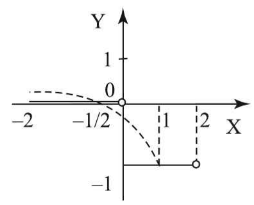
(ii) 

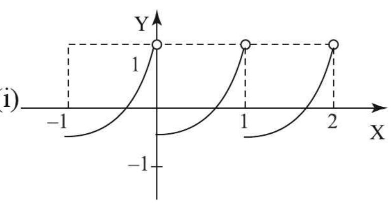
5. 

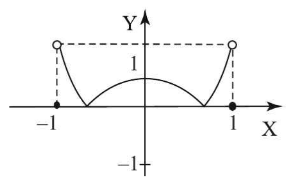
6. 

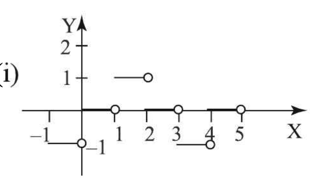
7. 

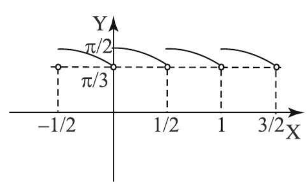
8. (i) 

(ii) 

9. (i) 

(ii) 

10. (i) 

(ii) 

11. 2

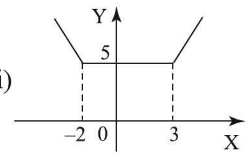
12. $(-\infty, -2] \cup \left\{0, \dfrac{\pi}{2}\right\}$

13. 6

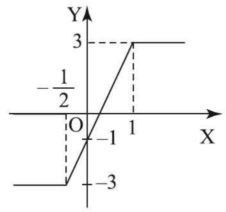
14. $\dfrac{3}{2}$

PRACTICE PROBLEMS [G]

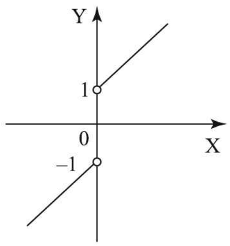
1. (i)



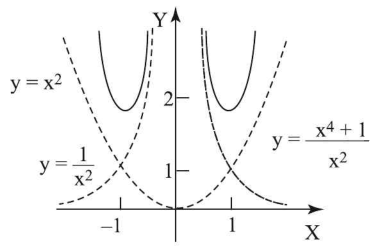
(ii)



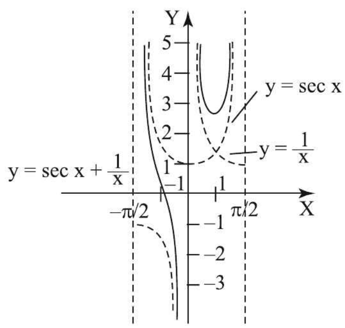
(iii)



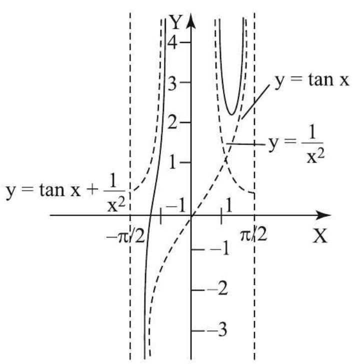
2. (i)



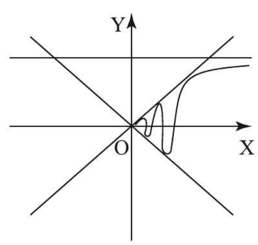
(ii)



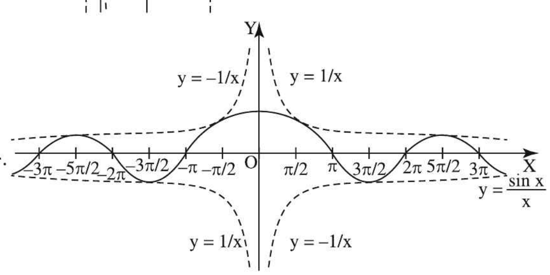
(iii) $y = \tan x + \frac{1}{x^2}, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$



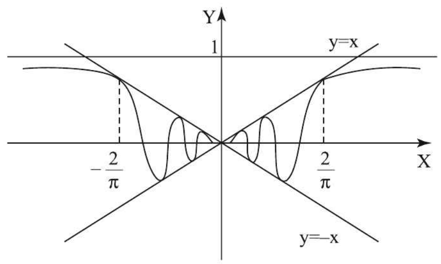
3.



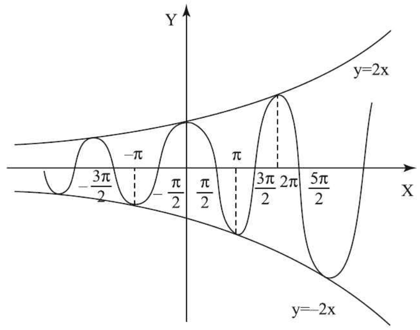
4.



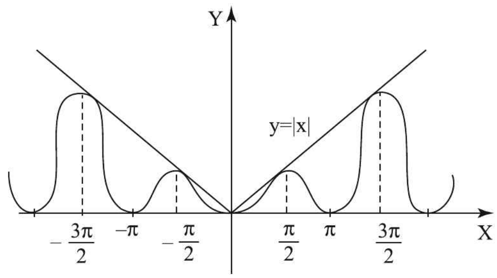
5. (i)



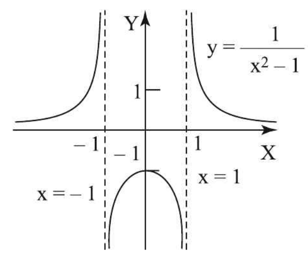
(ii)



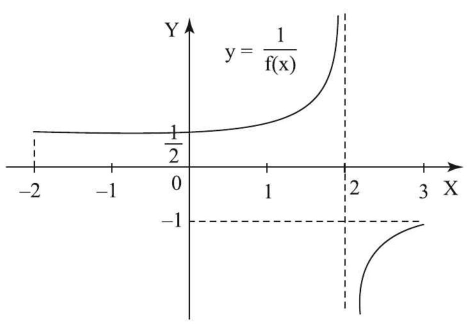
(iii)



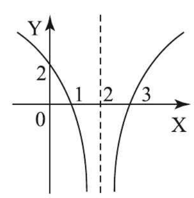
6.



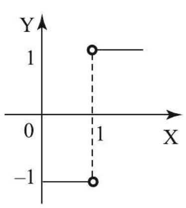
7.



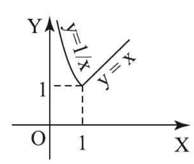
8. (i)



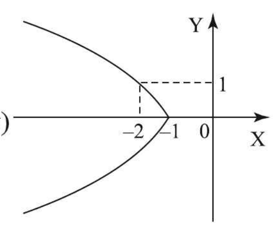
(ii)



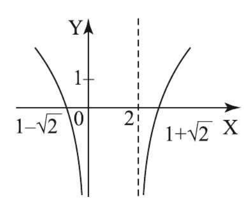
(iii)

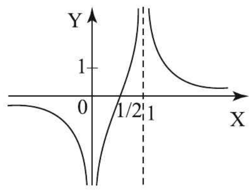


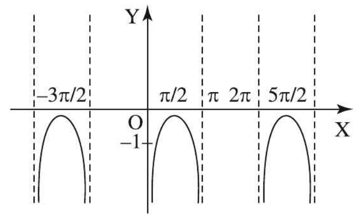
(iv)

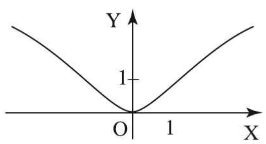


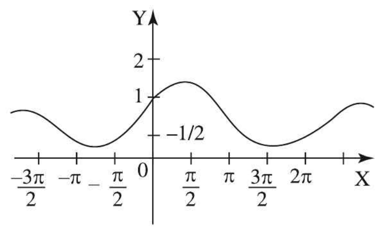
9. (i)

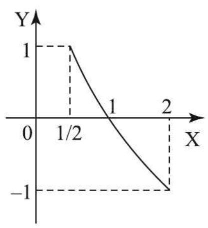


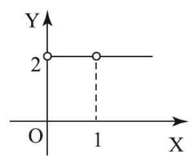
(ii) 

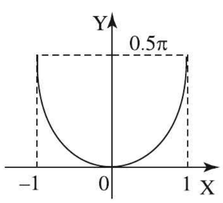
(iii) 

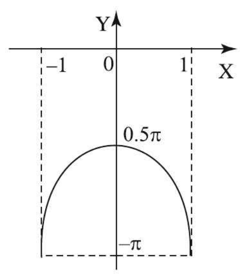
10. (i) 

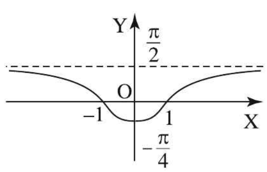
(ii) 

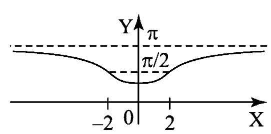
(iii) 

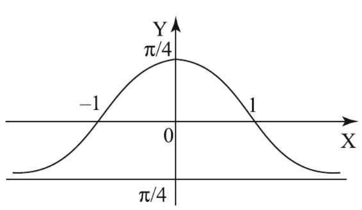
(iv) 

11. (i) 

(ii) 

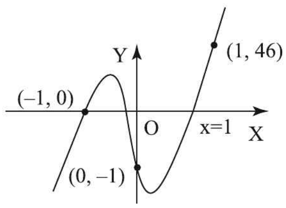
(iii) 

(iv) 

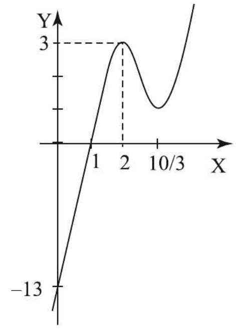
12. 

Practice Problems [H]

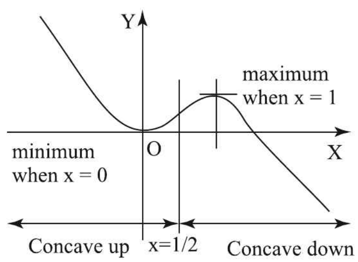
1. (i)



(ii)

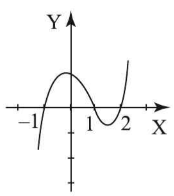


(iii)

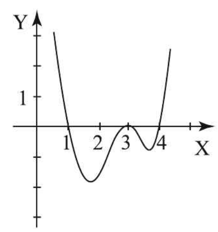


2. (i) –1, 0, 1 (ii) (–1, 0), (0, 0), (1, 0) (iii) (0, –1)

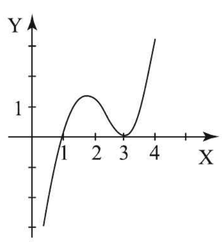
3. (i)



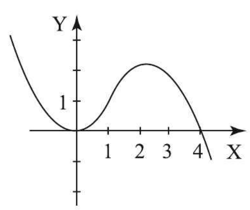
(ii)



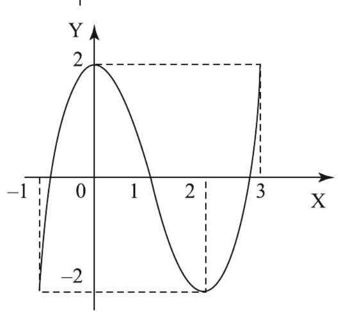
(iii)



(iv)

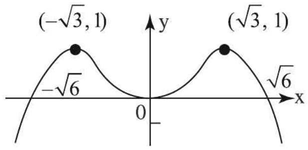


4.

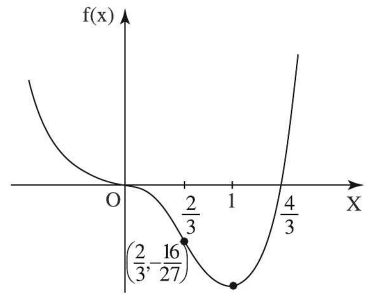


$f(x)_{min} = -2$; $f(x)_{max} = 2$

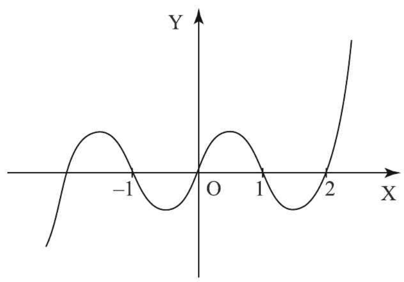
5.



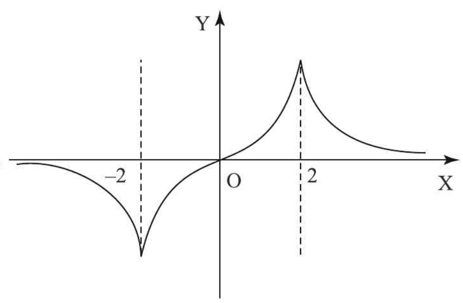
6. (i)



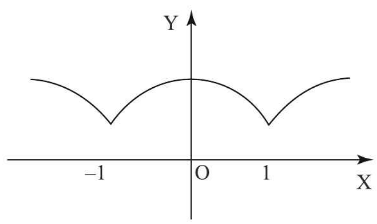
(ii)



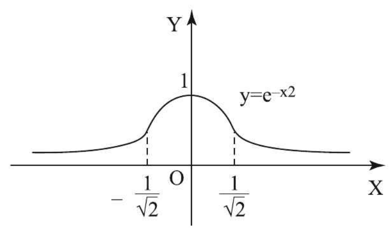
(iii)



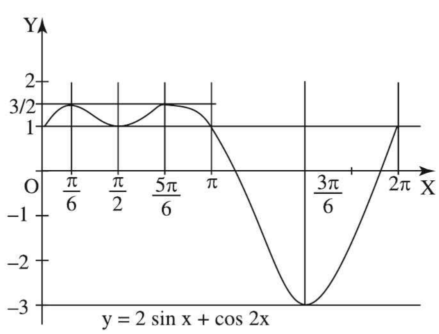
(iv)



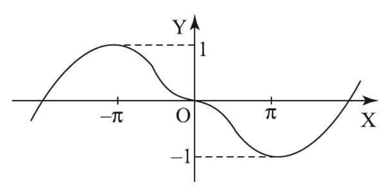
7.



8.

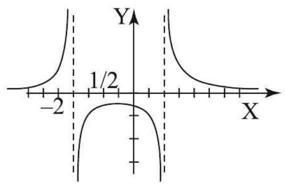


9.

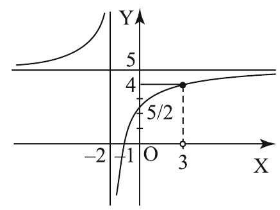


PRACTICE PROBLEMS [I]

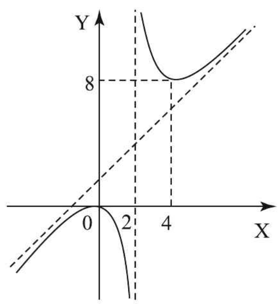
1. (i)



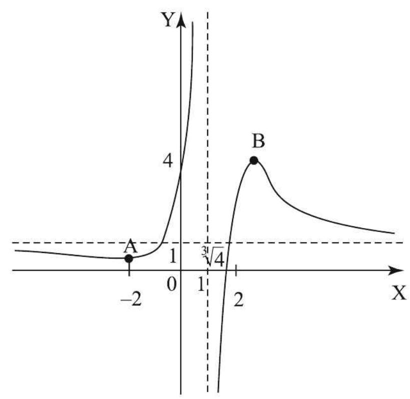
(ii)



2.

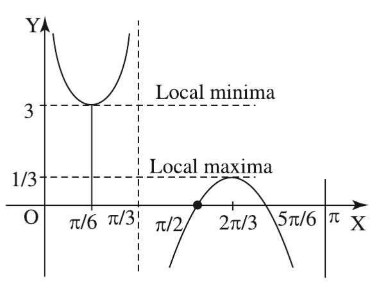


3.

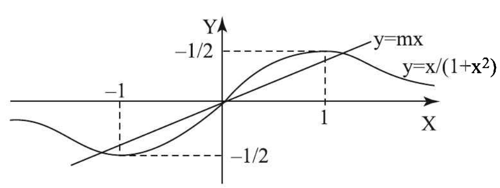


One root for c ∈ (–∞, 4/9) ∪ (4, ∞); two roots for c ∈ {4/9, 1, 4}; three roots for c ∈ (4/9, 1) ∪ (1, 4).

4.

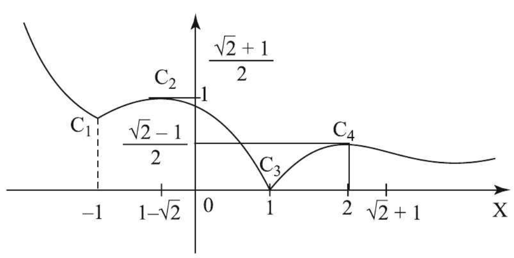


5.



m ∈ (0, 1)

6.

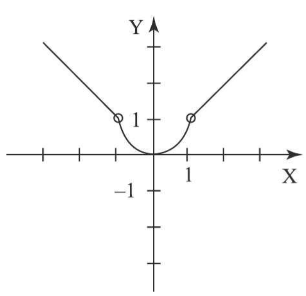


Objective Exercises

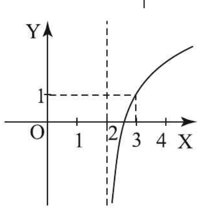
1. B	2. B	3. C	4. C
5. B	6. C	7. C	8. B
9. C	10. D	11. C	12. C
13. B	14. A	15. A	16. B
17. D	18. B	19. A	20. C
21. A	22. B	23. C	24. B
25. D	26. C	27. B	28. C
29. B	30. B	31. B	32. AC
33. A	34. B	35. D	36. D
37. C	38. AC	39. A	40. A
41. C	42. C	43. A	44. D
45. A	46. ABC	47. ABD	48. AD
49. BCD	50. BC	51. ACD	52. D
53. B	54. A	55. B	56. D
57. B	58. A	59. D	60. A
61. C	62. A	63. B	64. B
65. C	66. D	67. (A)–(Q) ; (B)–(R) ; (C)–(P) ; (D)–(S)	

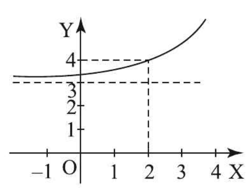
68. (A)–(R) ; (B)–(R) ; (C)–(R) ; (D)–(Q)
69. (A)–(T) ; (B)–(R) ; (C)–(Q) ; (D)–(P)
70. (A)–(Q) ; (B)–(S) ; (C)–(T) ; (D)–(R)
71. (A)–(S) ; (B)–(P) ; (C)–(S) ; (D)–(Q)
72. (A)–(R) ; (B)–(Q) ; (C)–(P) ; (D)–(T), (E)–(T)

Review Exercises for JEE Advanced

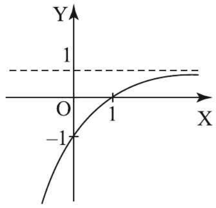
1. 

2. $$f(x)=\begin{cases}(4x+13)/3, & -4\le x<-1\\ 1, & -1\le x<2\\ -5x/2\ +6, & 2\le x\le 4\end{cases}$$

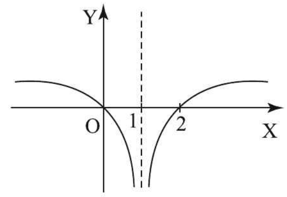
3. (i) 

(ii) 

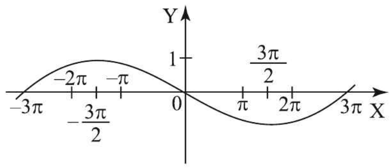
(iii)



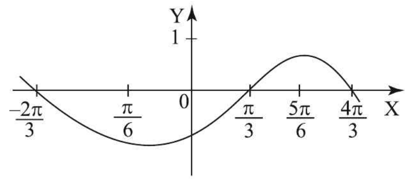
(iv)



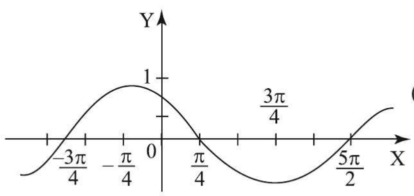
4. (i)



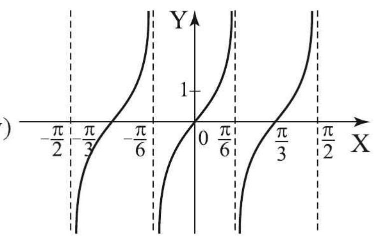
(ii)



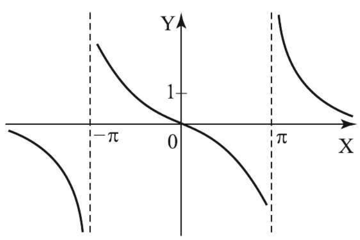
(iii)



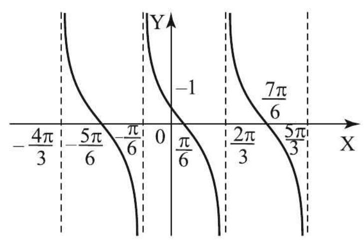
(iv)



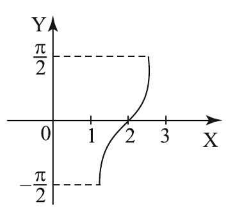
5. (i)



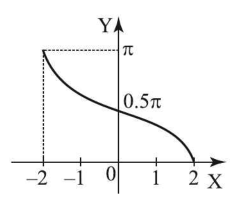
(ii)



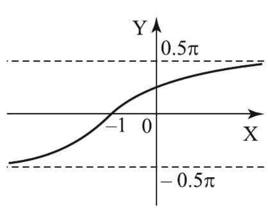
6. (i)



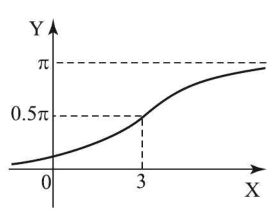
(ii)



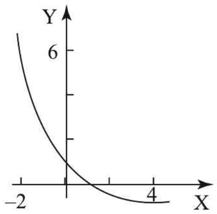
(iii)



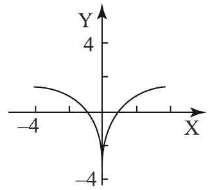
(iv)



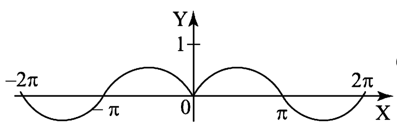
7. (i)



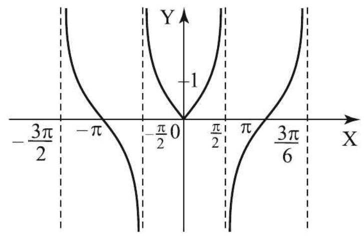
(ii)



(iii)



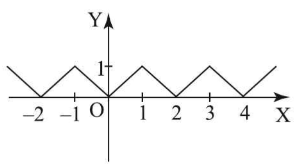
(iv)



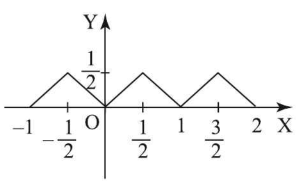
9. $[-4, -2) \cup (-2, 4]; [-4, 4]$

10. $y = x^2 + x + 1$

11.



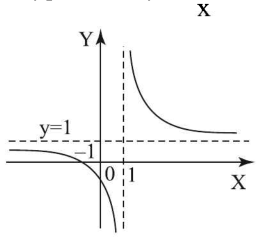
12.



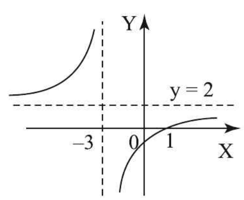
13. (i) Straight line : $4x - 3y = 23$ (ii) Parabola : $y = (x - 2)^2$

(iii) Hyperbola : $y = \dfrac{2}{x} + 1$ (iv) Circle : $x^2 + y^2 = 25$

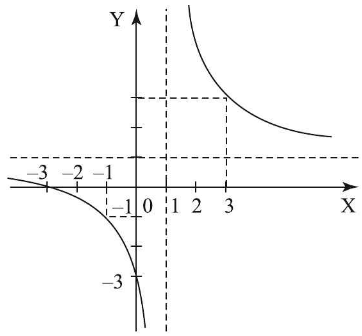
14. (i)



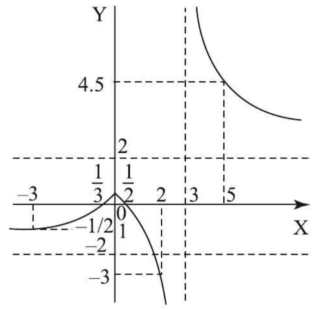
(ii)



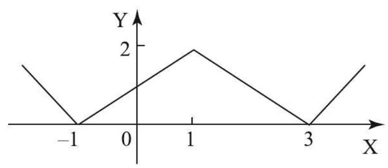
15. (i)



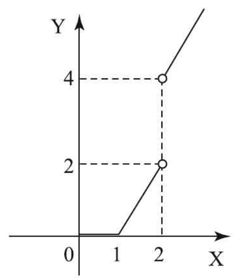
(ii)



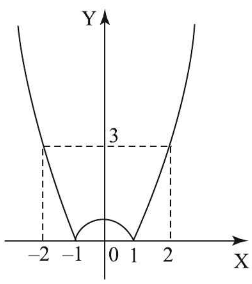
16. (i)



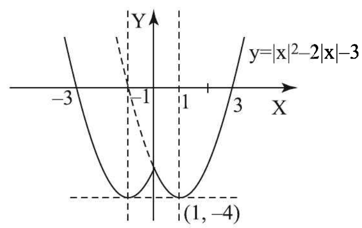
(ii)



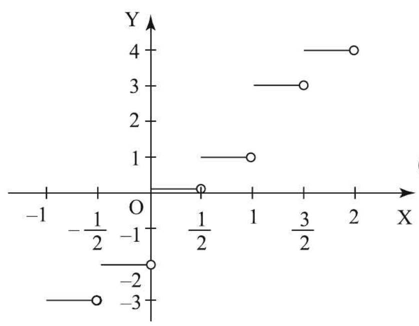
(iii)



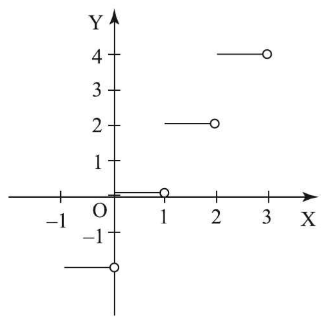
(iv)



17. (i)



(ii)



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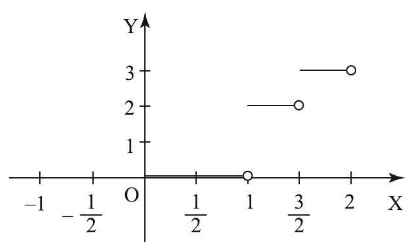
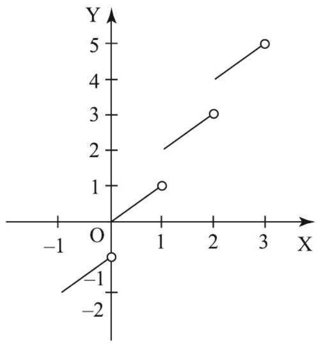
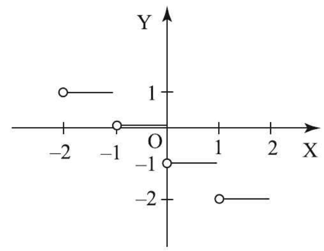


18. (i)



(ii)



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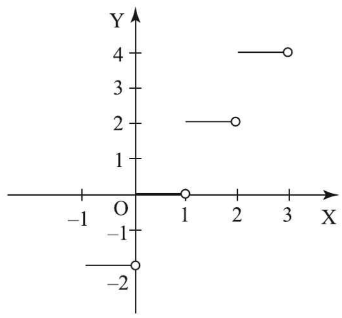
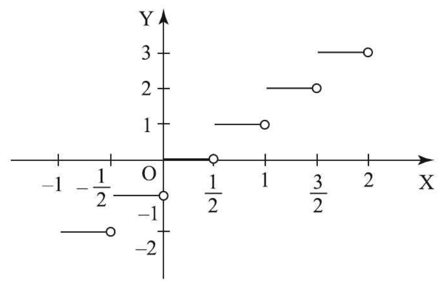
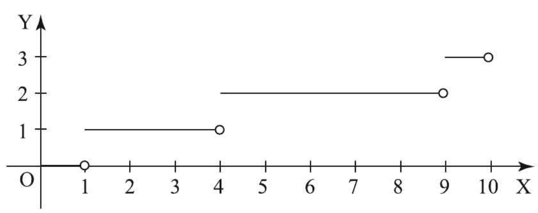


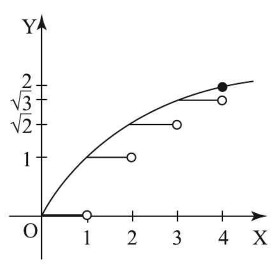
(iv)



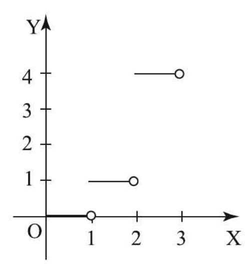
19. (i)



(ii)

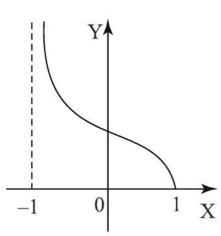


(iii)

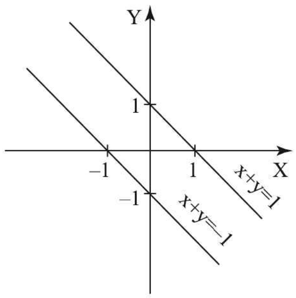


20. (–1,0); (0, 1) and (0, –1); symmetric about x axis.

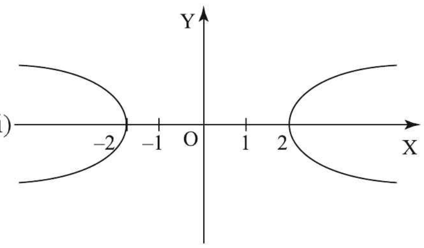
21.



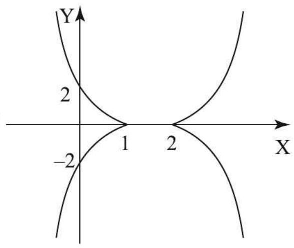
22. (i)



(ii)

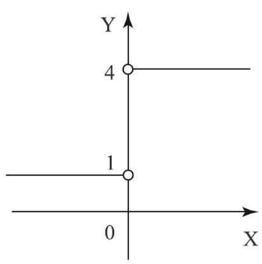


(iii)

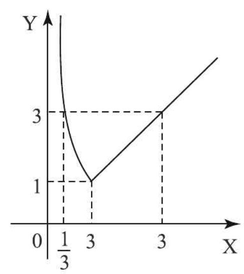


23. 1, 3, 7, 11.

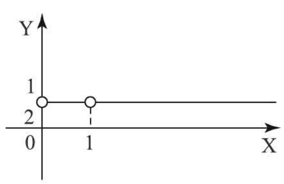
24. (i)



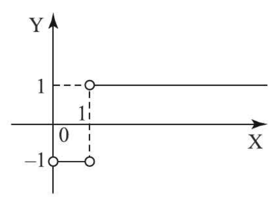
(ii)



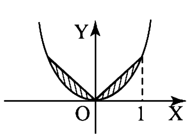
(iii)



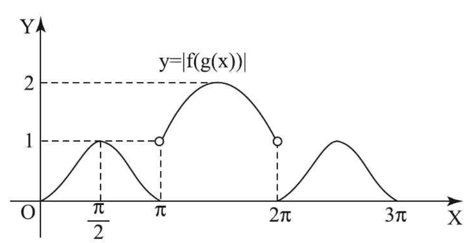
(iv)



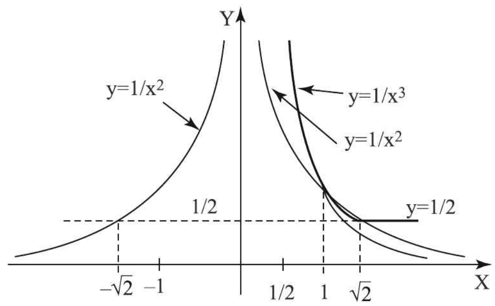
25.



26.



27. $g(x) = \begin{cases} \dfrac{1}{x^3} & \dfrac{1}{2} \le x \le 1 \\ \dfrac{1}{x^2} & 1 < x \le \sqrt{2} \\ \dfrac{1}{2} & \sqrt{2} < x \le 2 \end{cases}$

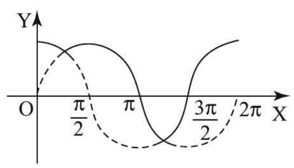


28. $a = -\dfrac{3}{2}, C = -2$

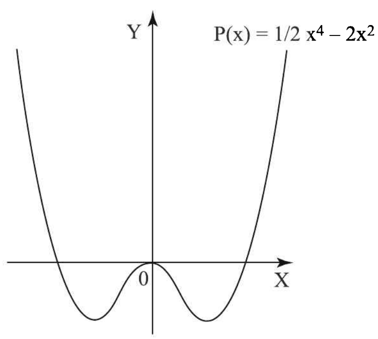
29. (2, 2, 1)

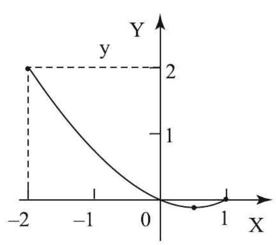
30. $a = -\dfrac{3}{2}$

31. The bold curve represents the graph of f(x) in $[0, 2\pi]$

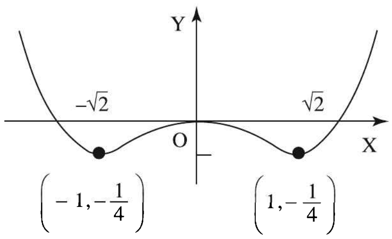


Clearly, the least value of f(x) is $-\dfrac{1}{\sqrt{2}}$, which is obtained at $\dfrac{5\pi}{4}$.

32. 

33. 

$f(x)_{min} = -31/240$; $f(x)_{max} = 34/15$

34. 

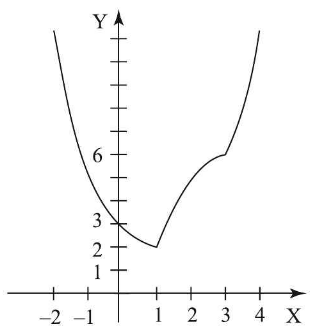
35. (i) 5 (ii) 2 (iii) 7 (iv) 6

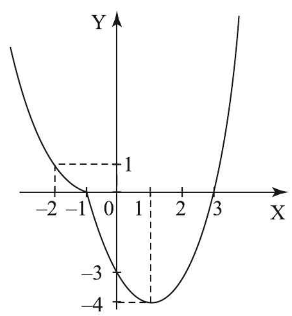
Target Exercises for JEE Advanced

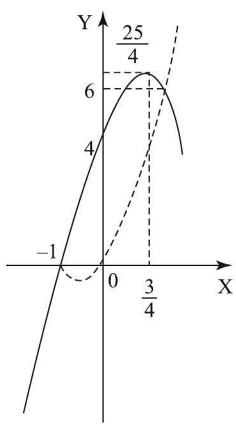
1. $y = 2 + \dfrac{x+1}{x^2+3x+1}$

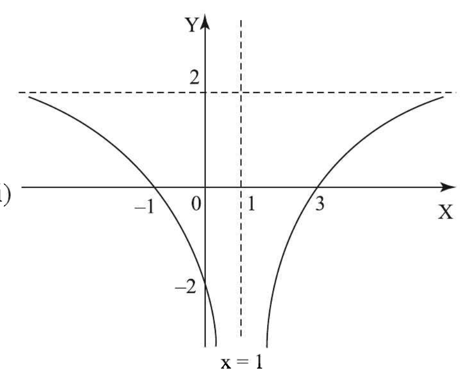
2. (i) $y = f(x+2)$ (ii) $y = |f(|x|)|$

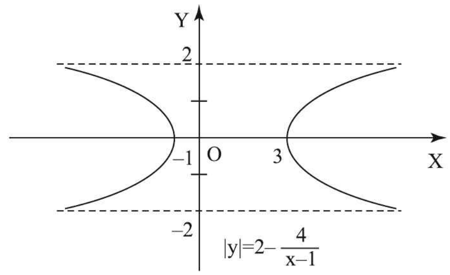
   (iii) $[-4, 3]$ and $[-1, 2]$ (iv) 1

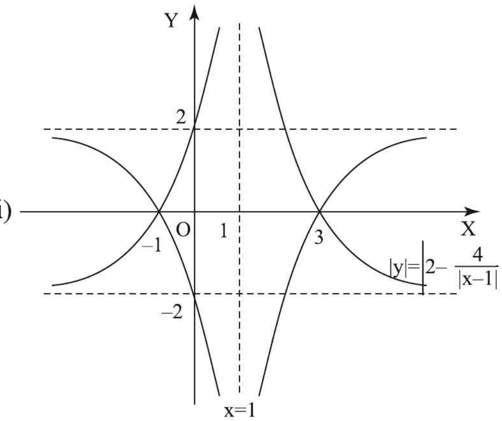
3. (i) 

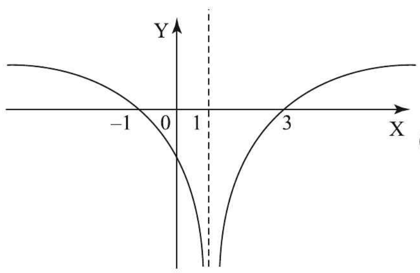
(ii) 

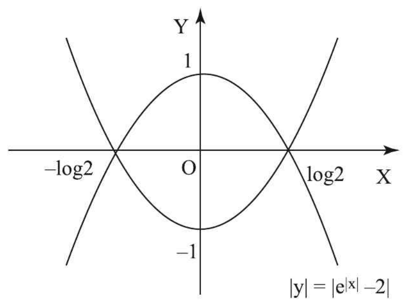
(iii) 

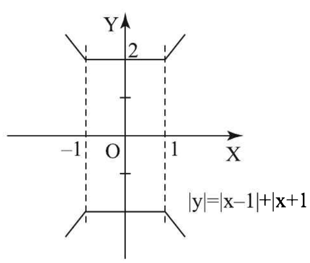
4. (i) 

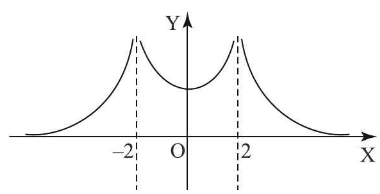
(ii) 

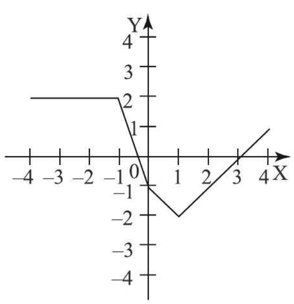
(iii) 

5. (i) 

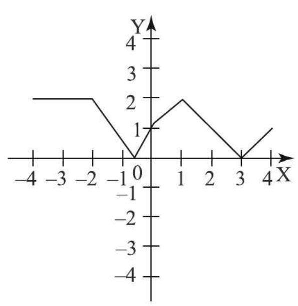
(ii) 

(iii) 

(iv) 

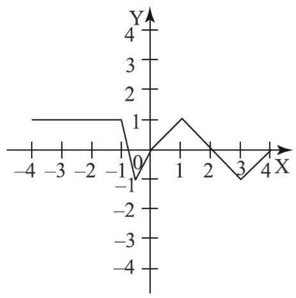
6. (i) 

y = –f(x)

(ii) 

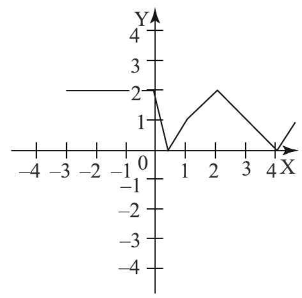
y = |f(x–1)|

(iii)



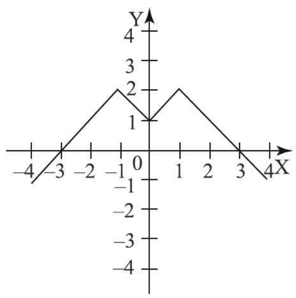
y = |f(x)|–1

(iv)



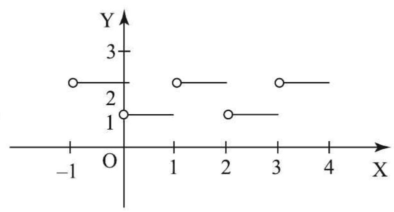
y = |f(x–1)|

(v)

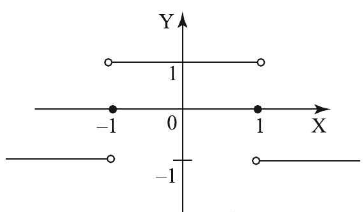


y = f(|x|)

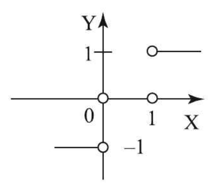
7.



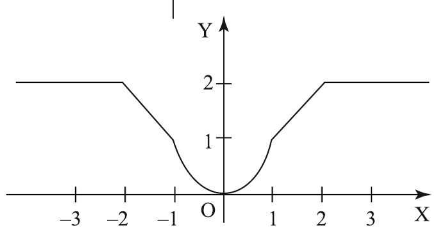
8. (i)



(ii)

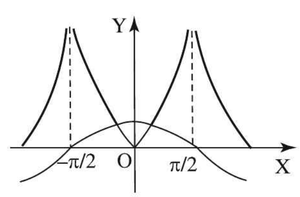


(iii)



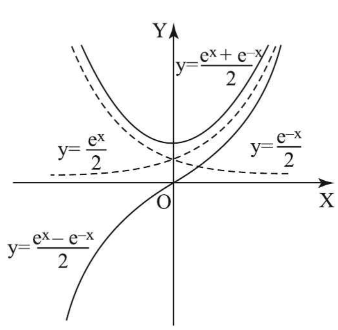
y=f(x)

(iv)

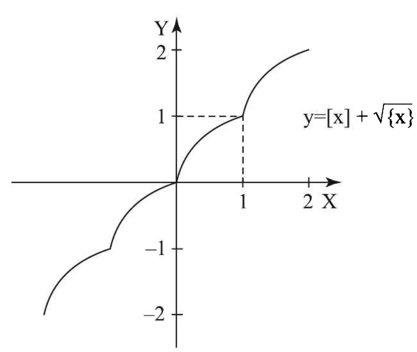


9. 1

10. (i) and (ii)

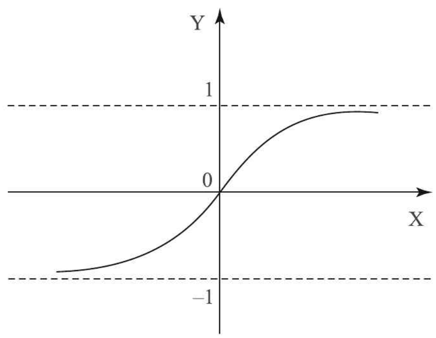


(iii)

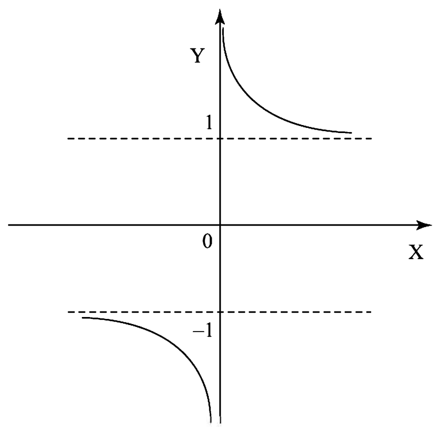


11.

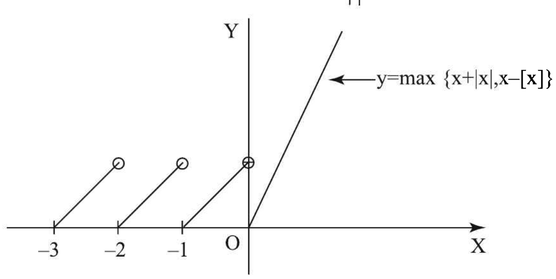
(i) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$



(ii) $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$



12. (i)



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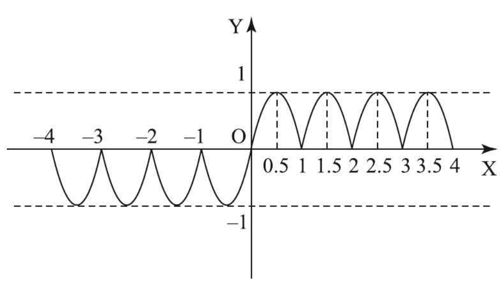

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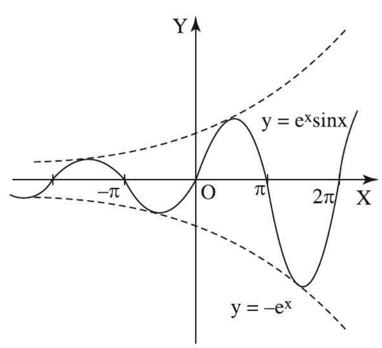
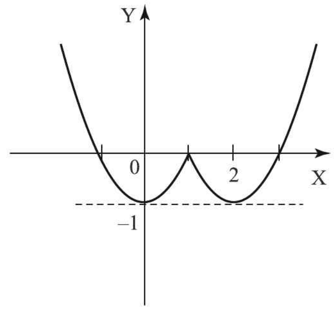


13. (a) $p(x) = (x-1)^2\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$

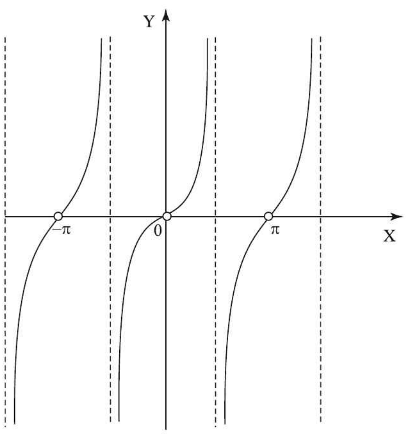


(b) ϕ if $c > 1$; 2 if $c < 0$; 4 if $0 \le c \le 1$.

14. (a) 1 (b) $q(x) = \frac{(x-2)(x+1)}{(x-1)(x+2)}$

(c) $\frac{5}{2}$ (d) $-\infty$

15. (i)



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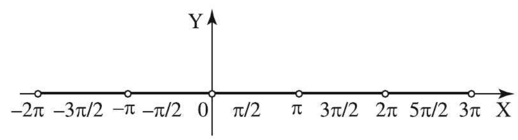
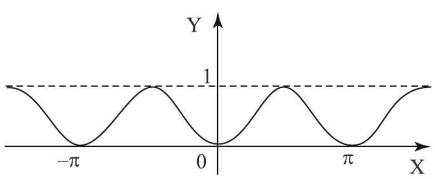


16. (i)



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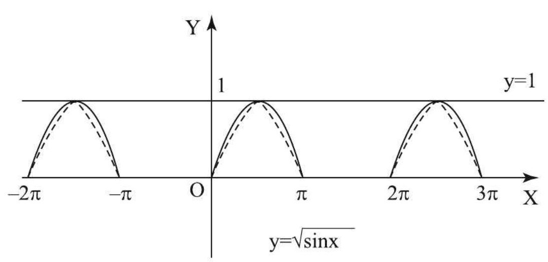
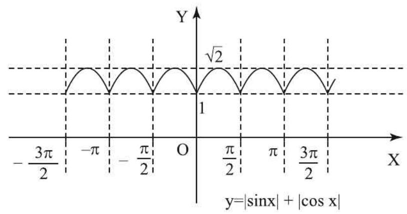


17. (i)



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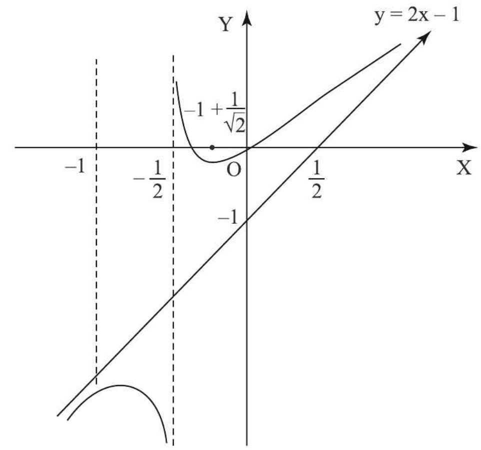
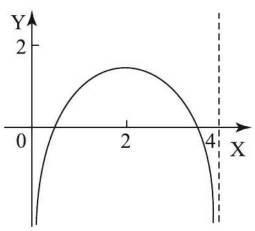


18. (i)



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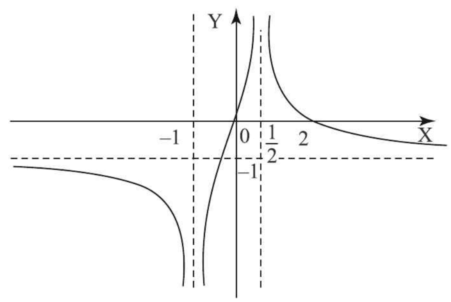

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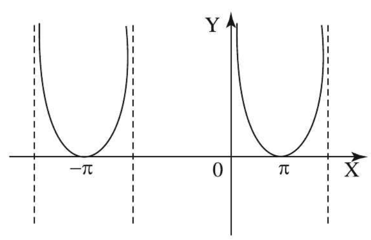
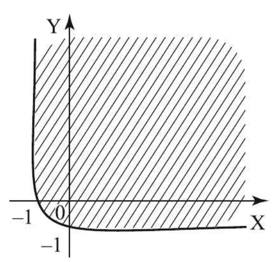


19. (i)



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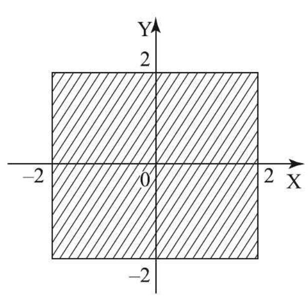

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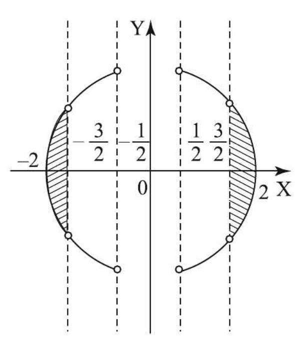
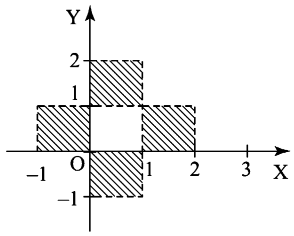


20. (i)



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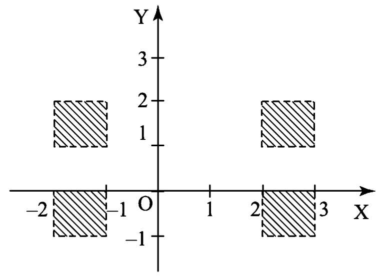

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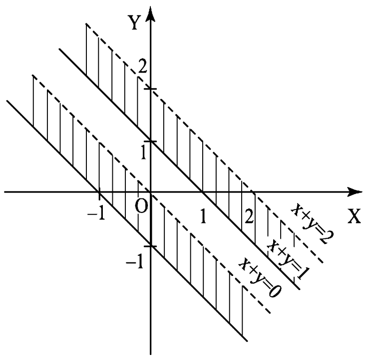

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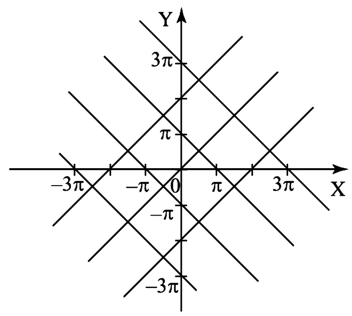
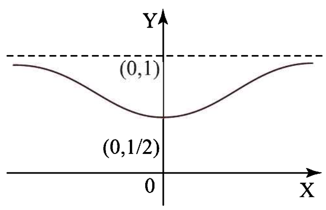
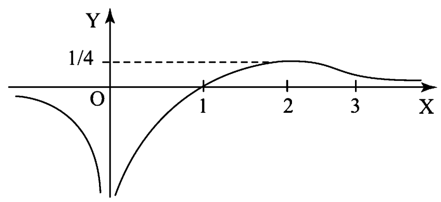


22. (i)



(ii)



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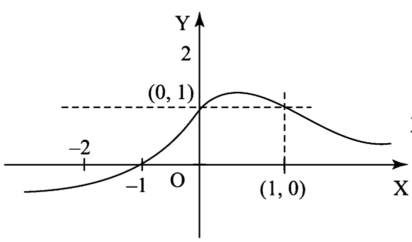
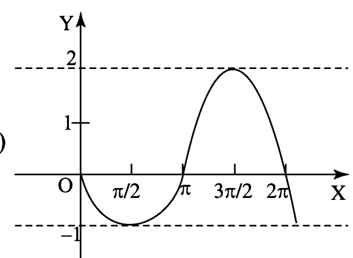
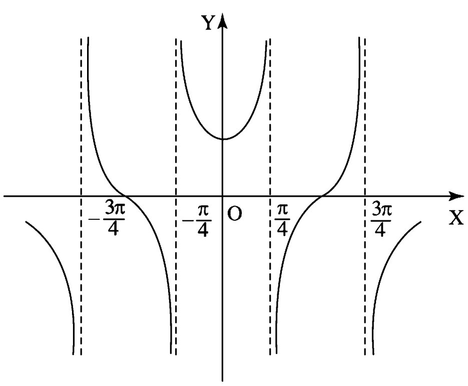


23. (i)



(ii)



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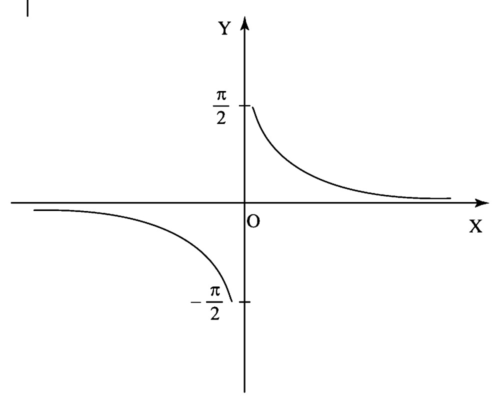
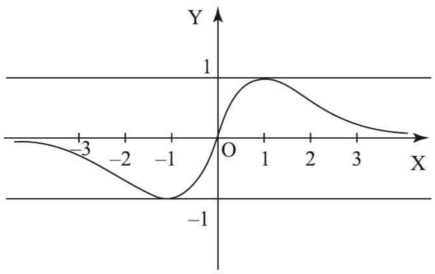


24. (i)



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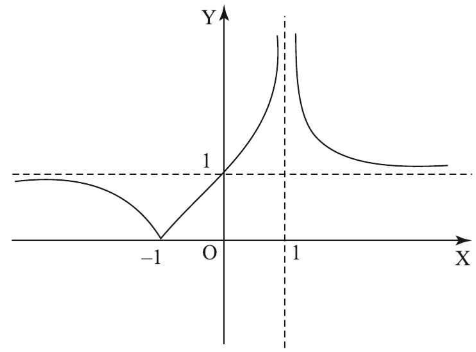

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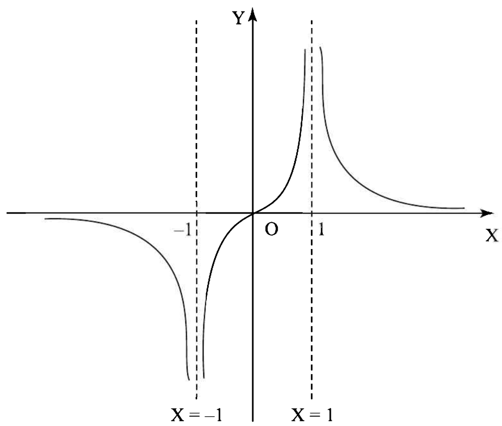

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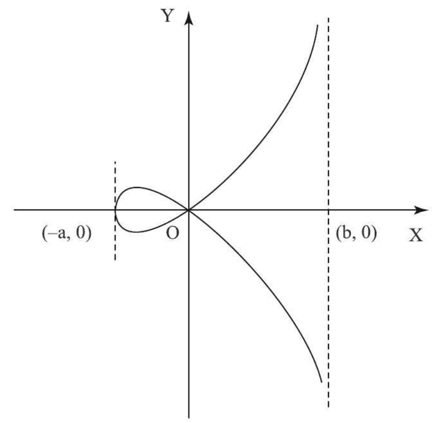
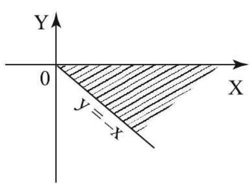


27. $y = |x-2| - x + 1$

28. ϕ if $|a| < \sqrt{2}/2$, or $|a| > 1$; four if $|a| = \sqrt{2}/2$, or $|a| = 1$; eight if $\sqrt{2}/2 < |a| < 1$.

29.

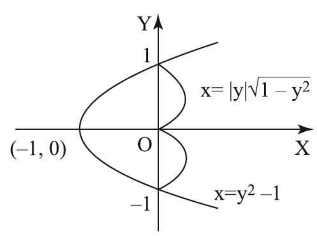


30. (a) –1 (b) $p(x) = \dfrac{-x(x+4)(x-3)}{20}$.

31. (i) $(-2, 0) \cup (4, \infty)$ (ii) $(-\infty, -5) \cup (-1, 1)$

32. (0, 1) (1, 0), $\left(\dfrac{1-\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right)$

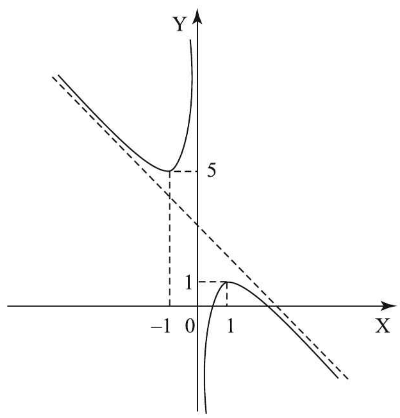
33.



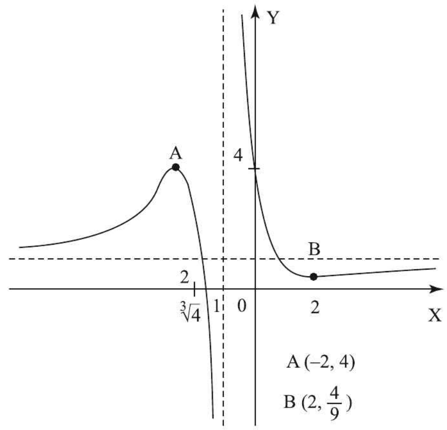
; no integral coordinates.

34. (i) infinite (ii) 2 (iii) infinite (iv) 1

35.



36.



One root for c ∈ $(-\infty, 4/9) \cup (4, \infty)$; two roots for c ∈ $\{4/9, 1, 4\}$; two roots for c ∈ $(4/9, 1) \cup (1, 4)$.

CHAPTER 3

SETS AND RELATIONS

3.1 | SETS

Sets are used to study the concepts of relations and functions. The study of geometry, sequences, probability, etc. also requires the knowledge of sets. The theory of sets was developed by German mathematician Cantor.

In everyday life, we often speak of collections of objects of a particular kind, such as, a pack of cards, a crowd of people, a cricket team, etc. In mathematics also, we deal with objects of different kinds; like numbers, points, lines, planes, triangles, circles, angles, equations, functions and many more. Often, objects of a similar nature or with a common property are collected into sets. The objects which are collected in a set are called the elements of that set. For example, the set of natural numbers, the set of prime numbers, the set of points, etc.

Definition A set is a collection of well defined objects. In other words, the member of set is clearly identifiable.

The following points may be noted :

(i) Objects, elements and members of a set are synonymous terms.

(ii) Sets are usually denoted by capital letters A, B, C, X, Y, Z, etc.

(iii) The elements of a set are represented by small letters a, b, c, x, y, z, etc.

If ‘a’ is an element of a set A, we say that "a belongs to A". The Greek symbol $\in$ (episilon) is used to denote the phrase 'belongs to'. Thus, we write $a \in A$.

If 'b' is not an element of a set A, we write $b \notin A$ and read "b does not belong to A".

Consider the following examples :

(i) If Q is the set of all quadrilaterals, and A is a parallelogram, then $A \in Q$. If C is a circle, then $C \notin Q$.

(ii) If G is the set of all even numbers, then $16 \in G$, and $3 \notin G$.

(iii) If L is the set of all solutions of the equation $x^2 = 1$, then 1 is an element of L, while 2 is not.

Representation of sets

The set is represented in two ways :

(i) Roaster form

(ii) Set builder form

Roaster form

All elements of the set are listed with a comma (",") in between and the listing itself is enclosed within brackets "{" and "}".

For example, {0}, {2, 6, 7, 9}, {x, y, z}.

The set of numbers, which divide 12, is written as :

A = {1, 2, 3, 4, 6, 12}

Note that this is convenient (or indeed possible) only for sets with relatively few elements. If there are more elements and one wants to list the elements “explicitly” sometimes periods are used; for example,

{0, 1, 2, 3,.....,}, {2, 4, 6,...., 20}.

The meaning should be clear from the context.

In this descriptive or explicit method, an element may be listed more than once, and the order in which the elements appear is irrelevant though desirable. Thus, the following all describe the same set :

{1, 2, 3}, {2, 3, 1}, {1, 1, 3, 2, 3}.

The roaster form is limited in certain circumstance. For example, we cannot represent set of real numbers in roaster form. Real numbers is an infinite set, but the elements of this set do not follow a pattern or have a particular sequence. As such, we can not define same with the help of ellipsis.

Every member of the set is unique and distinct. However, we encounter situations in which collection can have repeated elements.

Set builder form

In set builder form, we give a rule which determines if a given object is in the set or not; this is also called implicit description; for example,

{x : x is a natural number and x > 3}

Collections are often characterised by a common property. We can, therefore, define members of a set in terms of the common property. However, we need to ensure that objects outside the collection do not have the same common property.

The construction of qualification for the common property is quite flexible. Only thing is that we need to be explicit in what we mean. Generally, we denote the member by a symbol like "x" and then define the membership. Consider the examples :

A = {x: x is a vowel in English alphabet}

B = {x: x is an integer and 0 < x < 10}

The roaster equivalents of two sets are :

A = {a, e, i, o, u} B = {1, 2, 3, 4, 5, 6, 7, 8, 9}

Can we write set "B" as the one which comprises single digit natural number? Yes. Thus, we can see that there are indeed different ways to define and identify members and hence the flexibility in defining collection. We should be careful in using words like "and" and "or" in writing qualification for the set. Consider the example here :

$C = \{x : x \in Z \text{ and } 2 < x < 4\}$

Both conditional qualifications are used to determine the collection. The elements of the set as defined above are integers. Thus, the only member of the set is "3".

Now, let us consider an example, which involves "or" in the qualification,

$C = \{x: x \in A \text{ or } x \in B\}$

The member of this set can be elements belonging to either of two sets "A" and "B". The set consists of elements (i) belonging exclusively to set "A", (ii) elements belonging exclusively to set "B", and (iii) elements common to sets "A" and "B".

EXAMPLE 3.1 A set in roaster form is given as:

$$A = \left\{\frac{5^2}{6}, \frac{6^2}{7}, \frac{7^2}{8}\right\}$$

Write the set in set builder form.

SOLUTION We see here that we are dealing with natural numbers. The numerators are square

of natural numbers in sequence. The number in denominator is one more than numerator for each member. We can denote natural number by "n". Clearly, if numerator is "n^2", then denominator is "n+1". Therefore, the expression that represents a member of the set is :

$$x = \frac{n^2}{n+1}$$

However, this set is not an infinite set. It has exactly three members. Therefore, we need to specify "n" so that only members of the set are exclusively denoted by the above expression. We see here that "n" is greater than 4, but "n" is less than 8. For representing three elements of the set, we have

$$5 \le n \le 7$$

We can write the set, now in the builder form as :

$A = \{x : x = \dfrac{n^2}{n+1}$, where n is a natural number and $5 \le n \le 7\}$

In set builder form, the sequence within the range is implied. It means that we start with the first valid natural number and proceed sequentially till the last valid natural number.

Some important sets representing numbers

Few key number sets are used regularly in mathematical context. As we use these sets often, it is convenient to have predefined symbols :

- P (prime numbers)
- N (natural numbers)
- I or Z (integers)
- Q (rational numbers)
- R (real numbers)

We put a superscript "+", if we want to specify membership of only positive numbers, where appropriate. " I^+ ", for example, means set of positive integers.

Empty set

Consider a definition : "the set of integers between 1 and 2". There is no integer within this range. Hence, the set corresponding to this definition is an empty set.

An empty set has no member or object. It is denoted by symbol "ϕ" and is represented by a pair of brackets without any member or object.

$$\phi = \{ \}$$

The empty set is also called "null" or "void" set.

It can also be described by a property, e.g.

$$\phi = \{x : x \ne x\}.$$

Consider some examples :

(i) $A = \{x$: x is an even prime number greater than $2\}$. Then A is an empty set, because 2 is the only even prime number.

(ii) $B = \{x : x^2 = 4$ and x is odd$\}$

An odd integer squared cannot be even. Hence, set "B" also does not have any element in it.

There is a bit of paradox here. If the definition does not yield an element, then the set is not well defined. We may be tempted to say that empty set is not a set in the first place. However, there is a practical reason to have an empty set. It enables mathematical operations. We shall find many examples as we study operations on sets.

Finite and infinite sets

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, e, g\}$

We observe that A contains 5 elements and B contains 6 elements.

Consider the set of natural numbers. We see that the number of elements of this set is not finite since there are infinite number of natural numbers. We say that the set of natural numbers is an infinite set. The sets A and B given above are finite sets and $n(A) = 5$, $n(B) = 6$.

A set which is empty or consists of a definite number of elements is called **finite**, otherwise, the set is called **infinite**.

Definition A set A is called finite, if A = f, or if there is natural number n such that the elements of A can be numbered 1, . . . , n in such a way that every element of A appears exactly once in the list. Otherwise, A is called infinite.

Consider the following examples :

(i) The set {a, b, c, d} is finite. A possible numbering might look like this: Assign 1 to a, assign 2 to b, assign 3 to c, assign 4 to d. Ofcourse, there are other possibilities of listing the above set.

(ii) The set of all solutions of the equation $x^2 + 23x - 17 = 0$ is finite, since the number of solutions of a polynomial is atmost equal to the degree.

(iii) The sets N, Z, Q, R are infinite.

(iv) The set of all multiples of 5 is infinite.

When we represent a set in the roaster form, we write all the elements of the set within brackets { }. It is not possible to write all the elements of an infinite set within brackets { } because the numbers of elements of such a set is not finite. So, we represent some infinite set in the roaster form by writing a few elements which clearly indicate the structure of the set followed (or preceded) by three dots.

For example, (1, 2, 3, ...) is the set of natural numbers, {1, 3, 5, 7, ...) is the set of odd natural numbers, {..., –3, –2, –1, 0, 1, 2, 3, ...} is the set of integers. All these sets are infinite.

Note All infinite sets cannot be described in the roaster form. For example, the set of real numbers cannot be described in this form. Because the elements of this set do not follow any particular pattern.

Cardinality

The cardinality of a set "A" is equal to number of elements in the set. It is denoted by n(A).

The cardinality of an empty set is zero. The cardinality of a finite set is some positive integers. The cardinality of a number system like integers is infinity. Curiously, the cardinality of some infinite set can be compared. For example, the cardinality of natural numbers is less than that of integers. However, we cannot make such deduction in most cases of infinite sets.

If a set A has only one element, we call it a **singleton** set. Thus, {a} is a singleton set.

EXAMPLE 3.2 Consider the set

$A = \{2, 9, 16, ..., 716\}$,

where the elements are in arithmetic progression. How many elements does it have ? Is $401 \in A$?

SOLUTION Observe that the elements have the form $2 = 2 + 7 . 0$, $9 = 2 + 7.1$, $16 = 2 + 7 . 2$, ...

thus the general element term has the form $2 + 7n$. Now,

$$2 + 7n = 716 \Rightarrow n = 102.$$

This means that there are 103 elements, since we started with n = 0.

If $2 + 7k = 401$, then $k = 57$, so $401 \in A$. On the other hand , $2 + 7a = 514 \Rightarrow a = \frac{512}{7}$ which is not integer, and hence $514 \notin A$.

PRACTICE PROBLEMS [A]

1. Write the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$ in the set-builder form.

2. Write the following sets in roster form :
   (i) A = {x : x is a two-digit natural number such that sum of its digits is 8}
   (ii) B = {x : x is a prime number which is divisor of 60}
   (iii) C = The set of all letters in the word TRIGONOMETRY.
3. Write the following sets in the set-builder form :
   (i) {2, 4, 8, 16, 32}
   (ii) {1,4,9,...,100}
4. List all the elements of the set
   $\{x \in I : 1 \le x^2 \le 100$, x is divisible by 3$\}$.
5. Determine all the fractions lying strictly between 2 and 3 that have denominator 6, that is , determine the set $\{x \in N : 2 < \frac{x}{6} < 3\}$ explicitly.
6. Describe the following sets explicitly:
   $A = \{x \in I : x \le 5\}$
   $B = \{x \in N : x \text{ divides } 24\}$
7. Find a property P and a set M such that you can write the following sets in the form $\{x \in M : P(x)\}$:
   A = {1, 2, 4, 8, 16}
   B = {6, 4, 8, 2, 0}
   C = {1, 3, 5, 7, ...}
   D = {3, 5, 11, 2, 7, 13}
8. Which of the following are examples of the null set?
   (i) Set of even prime numbers
   (ii) {y : y is a point common to any two parallel lines}
9. State whether each of the following set is finite or infinite :
   (i) The set of lines which are parallel to the x–axis.
   (ii) The set of letters in the English alphabet.
   (iii) The set of numbers which are multiple of 5.
10. Are the following sets finite or infinite ?
    (i) Let W be the set of the days of the week.
    (ii) Let S be the set of solutions of the equation $x^2 - 16 = 0$.
    (iii) Let G be the set of points on a line.
11. Are the following sets finite or infinite ?
    (i) $\{x \in R : x^2 + 2x - 1 = 0\}$
    (ii) $\{x \in N : x \le 0\}$
    (iii) $\{x \in Q : 0 \le x \le 1\}$

3.2 | SUBSETS

The collections are generally linked in a given context. If we think of ourselves, then we belong to a certain society, which in turn belongs to a province, which in turn belongs to a country and so on. In the context of a school, all students of a school belong to school. Some of them belong to a certain class. If there are sections within a class, then some of these belong to a section.

We need to have a mathematical relationship between different collections of similar types. In set theory, we denote this relationship with the concept of "subset".

A set, "A" is a subset of set "B", if each member of set "A" is also a member of set "B".

If a set A is a subset of a set B, it is written as $A \subseteq B$. The relation $\subseteq$ is called the inclusion relation.

So, $A \subseteq B$ whenever $x \in A$ implies $x \in B$.

Consider some examples :

(i) If $A = \{2, 4, 8\}$ and B is the set of natural numbers less than 10, then A is a subset of B and we write $A \subseteq B$.

(ii) If A is the set of all divisor of 56 and B the set of all prime divisors of 56, then $B \subseteq A$.

If set "A" is not a subset of "B", then we write this symbolically as : $A \not\subseteq B$

Note

1. Every set is subset of itself. $A \subseteq A$.
   This is so because every element is present in itself.
2. Empty set ϕ is a subset of every set. This deduction is direct consequence of the fact that empty set has no element. As such, this set is subset of all sets.
   Proof: Let us assume that $\phi \not\subseteq A$. Then, by our definition of a subset $\exists$ atleast one element x such that $x \in \phi$ but $x \notin A$. But $\forall$ x, $x \notin \phi$ since ϕ is the null set.
   Therefore $\exists$ no element x such that $x \in \phi$ and $x \notin A$. Thus our assumption that $\phi \not\subseteq A$ is wrong. Hence we must have $\phi \subseteq A$.
3. If A is a subset of B, and B is a subset of C, then A is a subset of C.
   Proof: We want to show that $A \subseteq C$, and we have as hypothesis that $A \subseteq B$ and $B \subseteq C$. Using our definition of $\subseteq$, we have to show that if $x \in A$, then x is an element of C. Thus, let x be an arbitrary element of A. Our hypothesis $A \subseteq B$ tells us that $x \in B$, and, since $B \subseteq C$, we also have $x \in C$. This is what we wanted to prove.
4. Observe carefully the difference between $\subseteq$ and $\in$. It should be noted that $\in$ connects an element and a set, while $\subseteq$ connects two sets.

If $B = \{1, 2, 3\}$, then 1 is an element of B, but 1 is not a subset of B. The set $A = \{1\}$, which has 1 as its only element, however, is a subset of B, since it fulfills the definition: whenever $x \in A$, then $x \in B$. Note that there is only one possibility for x, namely, $x = 1$. It is very important that you learn to distinguish between an object (which may of course be a set itself), and the set formed from objects, e.g.

- 1 is different from $\{1\}$,
- $\{1\}$ is different from $\{\{1\}\}$.

Observe that the unique element of the last set is the set $\{1\}$.

Equal sets

When shall we call two sets equal? An intuitive definition is to say that two sets are equal if they contain the same elements. We shall use our already defined notion $\subseteq$ to define equality of sets:

Definition Two sets A and B are equal, if $A \subseteq B$ and $B \subseteq A$. If A and B are equal, we write $A = B$.

Observe that this definition expresses just what we would intuitively mean by equality of sets: If every element of A is an element of B, and if every element of B is an element of A, then they must contain the same elements. To prove that two sets A and B are equal, we must show that A is a subset of B, and that B is a subset of A.

Let $A = \{x \in R : x^2 = 1\}$, and $B = \{1, -1\}$; we want to show that $A = B$.

"$A \subseteq B$" : Let $x \in A$; then, by the definition of A, x solves the equation $x^2 = 1$, hence, $x = 1$ or $x = -1$. In either case, $x \in B$.

"$B \subseteq A$": Let $x \in B$; then, by definition of B, $x = 1$ or $x = -1$. In either case, x is a real number and solves the equation $x^2 = 1$, hence, it fulfils the defining properties of A. This implies that $x \in A$.

Let A = {1, 3, 5} and B = {x : x is an odd natural number less than 6}. The $A \subseteq B$ and $B \subseteq A$ and hence $A = B$.

The sets $A = \{x \in I : x^2 \le 9\}$, $B = \{x \in I : |x| \le 3\}$,

$C = \{-3, -2, -1, 0, 1, 2, 3\}$ are equal. The first set is the set of all integers whose square lies between 1 and 9 inclusive, which is precisely the second set, which again is the third set.

We need to know two special aspects of this equality. We mentioned about repetition of elements in a set. We observed that repetition of elements does not change the set. Consider an example here : A = {1, 5, 5, 8, 7} = {1, 5, 8, 7}

Another point is that sequence does not change the set. Therefore,

A = {1, 5, 8, 7} = {5, 7, 8, 1}

In the nutshell, when we have to compare two sets we look for distinct elements only. If they are same, then two sets in question are equal.

EXAMPLE 3.1 Which of the following pairs of sets are equal? Justify your answer.

(i) X, the set of letters in "ALLOY" and B, the set of letters in "LOYAL".

(ii) $A = \{n : n \in I \text{ and } n^2 \le 4\}$ and $B = \{x : x \in R \text{ and } x^2 - 3x + 2 = 0.\}$

SOLUTION

(i) We have, X = {A, L, L, O, Y}, B = {L, O, Y, A, L}. Then X and B are equal sets as repetition of elements in a set do not change a set. Thus, X = {A, L, O, Y} = B

(ii) $A = \{-2, -1, 0, 1, 2\}$, $B = \{1, 2\}$. Since $0 \in A$ and $0 \notin B$, A and B are not equal sets.

Proper subset

We can see from the deductions above that special circumstance of "equality" can blur the distinction between "set" and "subset". In order to emphasize, mother-child relation between sets, we coin the term "proper subset". If every element of set "B" is not present in set "A", then "A" is a "proper" subset of set "B"; otherwise not. This means that set "B" is a larger set, which, besides other elements, also includes all elements of set "A".

Set of vowels in English alphabet, "V", is a "proper" subset of set of English alphabet, "E". All elements of "V" are present in "E", but not all elements of "E" are present in "V".

We write a "proper" subset relation using symbol "$\subset$" and write symbol "$\subseteq$" to mean possibility of equality as well.

Definition Let A and B be two sets. If $A \subset B$ and $A \ne B$, then A is called a **proper subset** of B and B is called superset of A. For example, A = {1, 2, 3} is a proper subset of B {1, 2, 3, 4}.

EXAMPLE 3.2 Let V = {d}, W = {c, d}, X = {a, b, c}, Y = {a, b} and Z = {a, b, d}. Determine whether each statement is true or false :

(i) $Y \subset X$ (ii) $W \ne Z$ (iii) $Z \supset V$
(iv) $V \subset X$ (v) $X = W$ (vi) $W \subset Y$

SOLUTION

(i) Since each element in Y is a member of X, $Y \subset X$ is true.

(ii) Now $a \in Z$ but $a \notin W$; hence $W \ne Z$ is true.

(iii) The only element in V is d and it also belongs to Z; Z has other elements, hence $Z \supset V$ is true.

(iv) V is not a subset of X since $d \in V$ but $d \notin X$; hence $V \subset X$ is false.

(v) Now $a \in X$ but $a \notin W$; hence $X = W$ is false.

(vi) W is not a subset of Y since $c \in W$ but $c \notin W$; hence $W \subset Y$ is false.

EXAMPLE 3.3 If A = {x : x = $4^n - 3n - 1$ and $n \in N$} and B = (y : y = 9(n – 1) and $n \in N$} then prove that $A \subset B$.

SOLUTION We have $x = 4^n - 3n - 1 = (1 + 3)^n - 3n - 1$

$= 1 + {}^nC_1(3) + {}^nC_2(3)^2 + {}^nC_3(3)^3 + \ldots + {}^nC_n(3)^n - 3n - 1$

$= {}^nC_2(3)^2 + {}^nC_3(3)^3 + \ldots + {}^nC_n(3)^n$

$= 9\,\{{}^nC_2 + {}^nC_3\,(3) + \ldots + 3^{n-2}\}$

$\therefore$ A = $\{9\,({}^nC_2 + {}^nC_3 \cdot (3) + \ldots + 3^{n-2}, n \in N)\}$.

For n = 1, $x = 4^1 - 3.1 - 1 = 0$.

Thus A contains 0 and some positive integral multiples of 9.

But B = {9 (n – 1), $n \in N$}.

B contains 0 and all positive integral multiples of 9.

It is clear that B contains A.

Hence $A \subset B$.

Alternative:

Since $x = 4^n - 3n - 1$ the values of x for $n \in N$ are 0, 9, 54, 243, ...

$\therefore$ A = {0, 9, 54, 243, ...} ...(1)

and y = 9(n –1)

the values of y for $n \in N$ are

0, 9, 18, 27, 36, ...

$\therefore$ B = {0, 9, 18, 27, 36, ...} ...(2)

It is clear from (1) and (2) that all elements of A are in B, and $A \neq B$.

Hence $A \subset B$.

EXAMPLE 3.4 The set S and set E are defined as:

S = {(x, y) : |x – 3| < 1 and |y – 3| < 1};

E = {(x, y) : $4x^2 + 9y^2 - 32x - 54y + 109 \le 0$}.

Show that $S \subset E$.

SOLUTION Graph of S :

$|x - 3| < 1 \Rightarrow -1 < (x - 3) < 1 \Rightarrow 2 < x < 4$

similarly $|y - 3| < 1 \Rightarrow 2 < y < 4$

$\therefore$ S consists of all points inside the square bounded by the lines x = 2, y = 2, x = 4 and y = 4.

Graph of E :

4

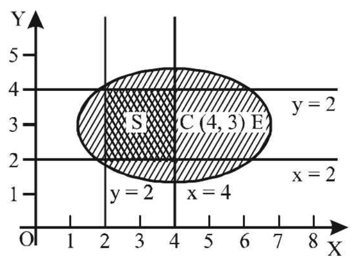
$x^2 + 9y^2 - 32x - 54y + 109 \le 0$

$\Rightarrow 4\,(x^2 - 8x) + 9\,(y^2 - 6y) + 109 \le 0$

$\Rightarrow 4\,(x - 4)^2 + 9\,(y - 3)^2 \le 36$

$$\Rightarrow \frac{(x-4)^2}{3^2} + \frac{(y-3)^2}{2^2} \le 1.$$

$\therefore$ E consists of all points inside and on the ellipse with centre (4, 3) and semi major and semi minor axes as 3 and 2 respectively.



From the above diagram it is evident that the double-hatched region (S) is within the region represented by E i.e., $S \subset E$.

Power set

The collection of all subsets of a set "A" is called power set. It is denoted as P(A).

For example, consider a set given by :

A = {1, 3, 4}

What are the possible subsets? There are three subsets consisting of individual elements: {1}, {3} and {4}. Then, elements taken two at a time form the following subsets : {1,3}, {1,4} and {3,4}. Since order or sequence does not matter in set representation, there are only three subsets of two elements taken together. Now, the elements taken three at a time form only one subset : {1,3,4}. Remember, a set is a subset of itself. Further, empty set (ϕ) is subset of any set. Hence, ϕ is also a subset of the given set "A".

The set comprising of all possible subsets of given set "A" is :

P (A) = {ϕ, {1},{3},{4},{1, 3},{1, 4},{3, 4},{1, 3, 4}}

We note two important points from this representation of power set :

1. The elements of a power set are themselves sets. In other words, every element of a power set is a set.
2. If the numbers of elements (cardinality) in a set is "n", then numbers of elements in power set is equal to $1 + {}^nC_1 + {}^nC_2 + ... + {}^nC_r + ... + 1 = (1 + 1)^n = 2^n$.

For a set having three elements, the total numbers of elements in the power set is :

$\Rightarrow \quad m = 2^n = 2^3 = 8$

We can see that this result is consistent with the illustration given above. We should, here, emphasize to avoid confusion that counting of elements of a set (cardinality) excludes empty set. It is, however, counted as members of power set.

Let us look at some power sets:

(i) A = ϕ: Since ϕ is a subset of every set, we must have $\phi \subseteq \phi$, i.e. $\phi \in P(\phi)$. But ϕ has no elements, and therefore no other subsets; hence, we have $P(\phi) = \{\phi\}$. Observe that ϕ is different from $\{\phi\}$: while ϕ has no elements, $\{\phi\}$ has exactly one element, namely the empty set ϕ.

(ii) A = {x}: A has two subsets, namely the empty set ϕ, and A itself. Since A has only one element, there can be no other subsets, thus, P(A) = {ϕ, {x}}.

(iii) A = {x, y}: using the previous lines of reasoning we obtain that

P(A) = {ϕ, {x}, {y}, {x, y}}.

So, we see that P(A) has four elements.

(iv) A = {x, y, z}: Except for the subsets ϕ and A, we can pick one or two elements of A at a time to form a subset. Hence,

P(A) = {ϕ, {x}, {y}, {z}, {x, y}, {x, z}, {y, z},
{x, y, z}}.

Observe that P(A) has eight elements.

EXAMPLE 3.5 List all the subsets of the set {–1, 0, 1}.

SOLUTION Let A = {–1, 0, 1}. The subset of A having no element is the empty set ϕ. The subsets of A having one element are {–1}, {0}, {1}. The subsets of A having two elements are {–1, 0}, {–1, 1}. The subset of A having three elements of A is A itself. So, all the subsets of A are ϕ, {–1}, {0}, {1}, {–1, 0}, {–1, 1}, {0, 1} and {–1, 0, 1}.

EXAMPLE 3.6 The finite sets "A" and "B" have "m" and "n" numbers of elements respectively. The total numbers of subsets of "A" is 56 more than the total numbers of subsets of "B". Find "m" and "n".

SOLUTION According to relation obtained for power set, the total numbers of subsets of "A" and "B" are :

$k_A = 2^m$

$k_B = 2^n$

According to question,

$k_A - k_B = 56$

$\Rightarrow \quad 2^m - 2^n = 56$

We need to find two equations to find "m" and "n". For this we seek expansion of "56" in terms of powers of "2".

$56 = 8.7 = 8\ (8 - 1) = 2^3\ (2^3 - 1)$

In order to get this form, we rearrange the expression on the LHS of the earlier equation as :

$\Rightarrow \quad 2^n\ (2^{m-n} - 1) = 2^3\ (2^3 - 1)$

Equating powers of similar base,

n = 3 and m = 6

Class and Family

Frequently, the members of a set are sets themselves.

If the elements of a set are sets themselves, then such a set is said to be a 'class of sets'.

For example, each line in a set of lines is a set of points. To help clarify these situations, words such as "class" and "family" are used. Usually we use class for a set of sets, and family for a set of classes. The words subclass and subfamily have meanings analogous to subset.

The set A = {ϕ, {a}, {b}, {a, b}} is a class of sets. Its members are ϕ, {a}, {b}, {a, b} which are sets themselves.

PRACTICE PROBLEMS [B]

1. Let A = {1, 2, ...8, 9}, B = {2, 4, 6, 8}, C = {1, 3, 5, 7, 9}, D = {3, 4, 5} and E = {3, 5}. Which sets can equal X if we are given the following information ?
   (i) X and B are disjoint.
   (ii) $X \subset D$ but $X \not\subset B$.
   (iii) $X \subset A$ but $X \not\subset C$
   (iv) $X \subset C$ but $X \not\subset A$.

2. Make correct statements by filling in the symbols $\subset$ or $\not\subset$ in the blank spaces:
   (i) (x : x is a circle in the plane} ..... (x : x is a circle in the same plane with radius 1 unit}
   (ii) {x : x is a triangle in a plane} .... {x : x is a rectangle in the plane}.

3. Are the following pair of sets equal ? Give reasons.
   (i) A = {2, 3}, B = {x : x is solution of $x^2 + 5x + 6 = 0$}
   (ii) A = {x : x is a letters in the word FOLLOW}
   B = {y : y is a letter in the word WOLF}

4. Let E = {x : $x^2 - 3x + 2 = 0$}, F = {2, 1} and G = {1, 2, 2, 1, 6/3}. Then show that E = F = G.

5. Find the power set of {a, b, c, d}; how many elements does it have ?

6. Write three members of the class {{2, 3}, {2}, {5, 6}}.

3.3 | SET OPERATIONS

We are familiar with basic algebraic operations. These basic mathematical operations, however, are not valid in all contexts. For example, algebraic operation such as addition has different details, when operated on vectors. Clearly, we expect that these operations will also be not same in the case of sets which are collections and not individual elements.

Nevertheless, set operations bear resemblance to algebraic operation. For example, when we combine (not add) two sets, then the operation involved is called "union". We can see that there is resemblance of the intent of addition, subtraction etc. in the case of sets also.

Venn diagrams

Venn diagrams are pictorial representation of sets/subsets and relationship that the sets/ subsets have among them. It helps us to analyze relationship and carry out valid set operations in a relatively easier manner vis -a- vis symbolic representation.

Universal set

Usually, in a particular context, we have to deal with the elements and subsets of a basic set which is relevant to that particular context. Universal set is the largest set among collection of sets. Importantly, it is not the collection of everything

as might be conjectured by the nomenclature. For example, "R", is universal set comprising of all real numbers. The rational numbers, integers and natural numbers are its subsets. In other consideration, we can call integers as universal set. In that case, sets such as {1,2,3}, prime numbers, even numbers, odd numbers are subset of the universal set of integers.

The universal set is pictorially represented by a region enclosed within a rectangle on Venn diagram. For illustration, consider the universal set of first 10 natural numbers as shown in figure 3.1.

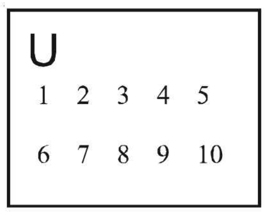


Figure 3.1

Many times, however, we may not be required to list elements of a universal set. In such case, we represent the universal set simply by a rectangle and the symbol for universal set, "U", in the corner. This is particularly helpful, where number of elements in universal set are very large.

The subsets of the universal set are represented by closed curves usually circles. The subset of vowels (V) is shown here within the circle with the listing of elements. Note that we have not listed all the alphabets for universal set and used the symbol "U" in the corner only.

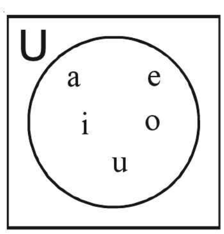


Figure 3.2

The subset of the universal set is represented by a closed curve – usually a circle.

Union of Sets

Union works on two operands, each of which is a set. The operation is denoted by symbol " ∪". Now, the question is : what do we expect when two sets are combined? Clearly, we need to enlist all the elements of two sets in the resulting set.

Definition The union of sets "A" and "B" is a third set, which consists all the elements of two sets. In symbol,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

The word "or" in the set builder form defining union is important. It means that the element "x" belongs to either "A" or "B'. The element may belong to both sets (common to two sets), but not necessarily. We can, therefore, infer that union set consists of :

1. elements exclusive to A
2. elements exclusive to B
3. elements common to A and B

As a set includes only distinct elements, the common elements are represented only once in the union set. Thus, union set consists of elements of both sets without repeating an element. Now, the set is represented on Venn diagram as shown in figure 3.3.

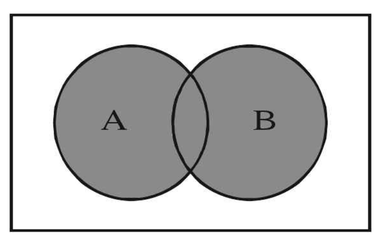


Figure 3.3

$A \cup B$ is the set of all elements in A or B, or in both A and B.

For illustration of working with union, let us consider two sets of positive integers as given here,

$A = \{1, 2, 3, 4, 5, 6\}$
$B = \{4, 5, 6, 7, 8\}$

The union of two sets is :

$\Rightarrow \quad A \cup B = \{1, 2, 3, 4, 5, 6, 4, 5, 6, 7, 8\}$

But repetition of elements in a set does not change it. Hence, we need not repeat elements in the resulting union.

$\Rightarrow \quad A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Here, universal set is natural numbers. The representation of union of joint sets is shown in the figure 3.4. We can observe that every construction of union on Venn diagram ensures that elements are not repeated.

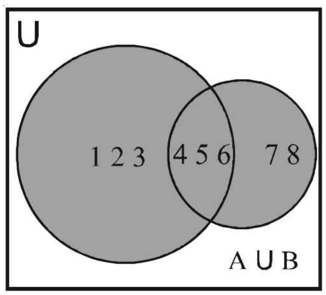


Figure 3.4

EXAMPLE 3.1 Let $A = \{2, 4, 6, 8\}$ and $B = \{6, 8, 10, 12)$. Find $A \cup B$.

SOLUTION We have $A \cup B = \{2, 4, 6, 8, 10, 12\}$

Note that the common elements 6 and 8 have been taken only once while writing $A \cup B$.

EXAMPLE 3.2 Let $A = \{a, e, i, o, u\}$ and $B = \{a, i, u\}$. Show that $A \cup B = A$.

SOLUTION We have, $A \cup B = \{a, e, i, o, u\} = A$.

This example illustrates that union of set A and its subset B is the set A itself, i.e., if $B \subseteq A$, then $A \cup B = A$.

Now we shall discuss some of the important characteristics / deductions for the union operation.

Idempotent law

The literal meaning of the word "idempotent" is "unchanged when multiplied by itself". Following the clue, the union of a set with itself is the set itself. This is an equivalent statement conveying the meaning of "idempotent" in the context of union. Symbolically,

$$A \cup A = A$$

The union set consists of distinct elements and common elements taken once. Between two sets here, all elements are common. The union set consists of all elements of either set.

Identity law

The algebraic operators like addition and multiplication have defined identities, which does not change the other operand of the operator. For example, if we add "0" to a number, it remains same. Hence, "0" is additive identity. Similarly, "1" is multiplicative identity.

In the case of union, we find that union of a set with empty set does not change the set. Hence, empty set is union identity.

$$A \cup \phi = A$$

As there is no element in empty set, union has same elements as that in "A".

Law of U

All sets are subsets of universal set for a given context. We have seen that union with subset results in the set itself. Clearly, union of universal set with its subset will result in the universal set itself.

$$U \cup A = U$$

Commutative law

In order to assess whether commutative property holds or not, we consider the example, used earlier. Let the sets be >

$A = \{1, 2, 3, 4, 5, 6\}$
$B = \{4, 5, 6, 7, 8\}$

Then,

$A \cup B = \{1,2,3,4,5,6,4,5,6,7,8\}$
$= \{1,2,3,4,5,6,7,8\}$
$B \cup A = \{4,5,6,7,8,1,2,3,4,5,6\}$
$= \{1,2,3,4,5,6,7,8\}$

Thus, we see that order of operands with respect to the union operator is not differentiating. We can also appreciate this law on Venn diagram, which does not change by changing positions of sets across union operator.

Union of three sets

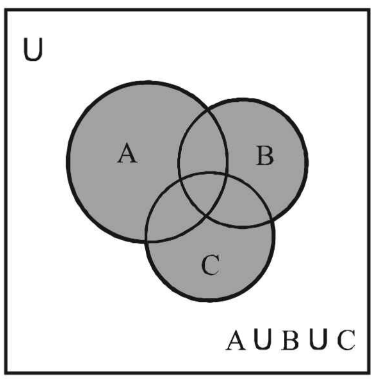


Figure 3.5 *Multiple unions*

If A_1, A_2, A_3........., A_n is a finite family of sets, then their unions, one after another, is denoted as :

$$A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots \cup A_n$$

Associative law

The associative property also holds with respect to union operator. We know that associative property is about changing the place of parentheses as here :

$(A \cup B) \cup C = A \cup (B \cup C)$

The parentheses simply change the precedence of operation. On Venn diagram, union involving three sets appears same, irrespective of whether we apply union operation in a particular sequence.

Intersection of sets

We understand that a set representing a real situation is not an isolated collection. Sets, in general, overlaps with each other. It is primarily because a set is defined on few characteristics, whereas elements generally can possess many characteristics. Unlike union, which includes all elements from two sets, the intersection between two sets includes only common elements.

Definition The intersection of sets "A" and "B" is the set of all elements common to both "A" and "B".

The use of word "and" between two sets in defining an intersection is quite significant. Compare it with the definition of union. We used the word "or" between two sets. Pondering on these two words, while deciding membership of union or intersection, is helpful in application situation.

The intersection operation is denoted by the symbol, " $\cap$ ". We can write intersection in set builder form as :

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

Again note use of the word "and" in set builder qualification. We can read this as "x" is an element, which belongs to set "A" and set "B". Hence, it means that "x" belongs to both "A" and "B".

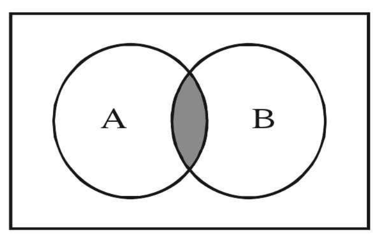


Figure 3.6

$A \cap B$ is the set of elements common in both A and B.

In order to understand the operation, let us consider the earlier example again,

$A = \{1, 2, 3, 4, 5, 6\}$
$B = \{4, 5, 6, 7, 8\}$

Then,

$A \cap B = \{4, 5, 6\}$

On Venn diagram, an intersection is the region intersected by circles, which represent the two sets.

EXAMPLE 3.3 Consider the sets of arithmetic progressions

$A = \{3, 9, 15, ... 681\}$, $B = \{9, 14, 19,...., 564\}$.

How many elements do they share, that is, how many elements does $A \cap B$ have ?

SOLUTION The members of A have common difference 6 and the members of B have common difference 5. Since the least common multiple of 6 and 5 is 30, and 9 is the smallest element that A and B have in common, every element in $A \cap B$ has the form $9 + 30k$. We then need the largest $k \in N$ satisfying the inequality

$9 + 30k \le 564 \quad \Rightarrow \quad k \le 18.5,$

and since k is integral, the largest value it can achieve is 18. Thus $A \cap B$ has $18 + 1 = 19$ elements, where we have added 1 because we started with $k = 0$.

In fact, $A \cap B = \{9, 39, 69,....., 549\}$.

Intersection with a subset

Since all elements of a subset is present in the set, it emerges that intersection with subset is the subset itself. Hence, if "A" is subset of set "B", then :

$B \cap A = A.$

EXAMPLE 3.4 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{2, 3, 5, 7\}$. Find $A \cap B$ and hence show that $A \cap B = B$.

SOLUTION We have $A \cap B = \{2, 3, 5, 7\} = B$. We note that $B \subseteq A$ and that $A \cap B = B$.

Disjoint sets

If two sets A and B have no elements in common, i.e., if no element of A is in B and no element of B is in A, then we say that A and B are disjoint, or mutually exclusive.

$A \cap B = \phi$

The sets $A = \{1, 3, 7, 8\}$ and $B = \{2, 4, 7, 11\}$ are not disjoint, since the element 7 is common to both A and B. On the other hand, the sets $C = \{a, b, c\}$ and $D = \{x, y, z\}$ are disjoint, since they have no elements in common.

Multiple intersection

If $A_1, A_2, A_3, \ldots, A_n$ is a finite family of sets, then their intersections one after another is denoted as :

$A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n$

Here are some of the important characteristics/ deductions for the intersection operation.

Idempotent law

The intersection of a set with itself is the set itself.

$A \cap A = A$

This is because intersection is a set of common elements. Here, all elements of a set is common with itself. The resulting intersection, therefore, is the set itself.

Identity law

The intersection with universal set yields the set itself. Hence, universal set functions as the identity of the intersection operator.

$A \cap U = A$

It is easy to interpret this law. Only the elements in "A" are common to universal set. Hence, intersection, being the set of common elements, is set "A".

Commutative law

The order of sets around the intersection operator does not change the intersection.

Hence, commutative property holds in the case of intersection operation.

$$A \cap B = B \cap A$$

Associative law

The associative property holds with respect to intersection operator.

$$(A \cap B) \cap C = A \cap (B \cap C)$$

The intersection of sets "A" and "B" on Venn diagram is shown in figure 3.7.

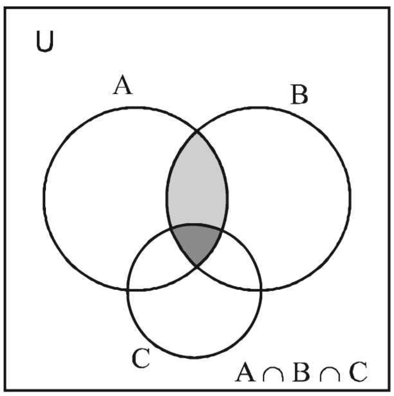


Figure 3.7

$A \cap B$ is the set of common elements as shown as coloured region.

In turn, the intersection of set "$A \cap B$" and set "C" is the small region in the centre.

It is easy to visualise that the ultimate intersection is independent of the sequence of operation.

Distributive law

The intersection operator ($\cap$) is distributive over union operator ($\cup$) :

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

We can check out this relation with the help of Venn diagram. For convenience, we have not shown the universal set in figure 3.8. In the first diagram on the left, the coloured region shows the union of sets "B" and "C" ie. $B \cup C$. The coloured region in the second diagram on the right shows the intersection of set "A" with the union obtained in the first diagram i.e. $B \cup C$.

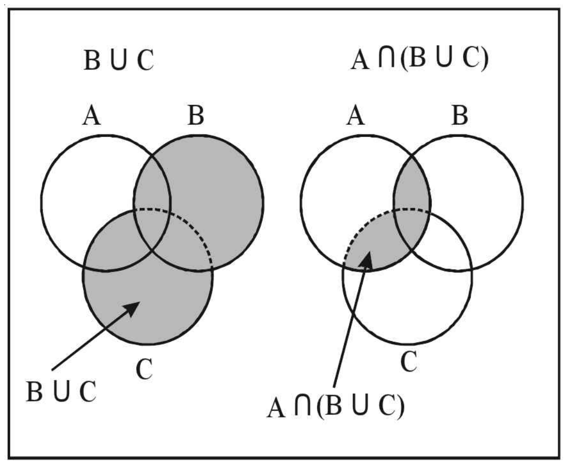


Figure 3.8

We can now interpret the coloured region in the second diagram from the point of view of expression on the right hand side of the equation:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

The coloured region is indeed the union of two intersections : "$A \cap B$" and "$A \cap C$" . Thus, we conclude that distributive property holds for "intersection operator over union operator".

In the same manner, we can prove distribution of "union operator over intersection operator" :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Analytical proof

Distributive properties are important and used in problems. Now we shall prove the same in analytical manner. For this, let us consider an arbitrary element "x", which belongs to set " $A \cap (B \cup C)$ " :

$$x \in A \cap (B \cup C)$$

Then, by definition of intersection :

$\Rightarrow \quad x \in A$ and $x \in (B \cup C)$

$\Rightarrow \quad x \in A$ and ($x \in B$ or $x \in C$)

$\Rightarrow$ $(x \in A \text{ and } x \in B)$ or $(x \in A \text{ and } x \in C)$

$\Rightarrow$ $(x \in A \cap B)$ or $(x \in A \cap C)$

$\Rightarrow$ $x \in (A \cap B)$ or $(A \cap C)$

$\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$

But, we had started with "$A\cap (B \cup C)$ " and used its definition to show that "x" belongs to another set. It means that the other set consists of the elements of the first set at the least. Thus,

$\Rightarrow$ $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

Similarly, we can start with "$(A \cap B) \cup (A \cap C)$" and reach the conclusion that :

$$(A \cap B) \cup (A \cap C) \subset A \cap (B \cap C)$$

If sets are subsets of each other, then they are equal. Hence,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proceeding in the same manner, we can also prove other distributive property of "union operator over intersection operator" :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Difference of sets

The difference of sets "A – B" is the set of all elements of "A", which do not belong to "B".

In the set builder form, the difference set is :

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

and

$$B - A = \{x : x \in B \text{ and } x \notin A\}$$

On Venn diagram, the difference "A – B" is the region of "A", which excludes the common region with set "B".

EXAMPLE 3.5 Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9\}$. Then

SOLUTION $A\cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}$, $A\cap B = \{1, 3, 5\}$, $A - B = \{2, 4, 6\}$, $B - A = \{7, 9\}$

From Venn diagram, we observe that if we derive union of $(A \cap B)$ to either of the difference sets, then we get the complete individual set.

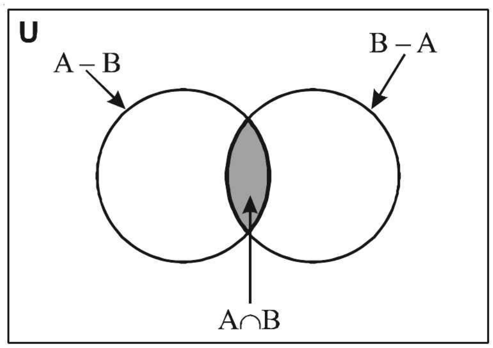


Figure 3.9

$A = (A - B) \cup (A \cap B)$ and

$B = (B - A) \cup (A \cap B)$

Difference of sets is not commutative

The positions of sets about minus operator affect the result. It is clear from the figure 3.9, where "A – B" and "B – A" represent different regions on Venn diagram. As such, the difference of set is not commutative. Let us consider the example used earlier, where :

$A = \{1, 2, 3, 4, 5, 6\}$

$A = \{4, 5, 6, 7, 8\}$

Then,

$A - B = \{1, 2, 3\}$

and

$B - A = \{7, 8\}$

Clearly,

$A - B \neq B - A$

EXAMPLE 3.6 Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$.

Find $A - B$ and $B - A$.

SOLUTION We have, $A - B = \{1, 3, 5\}$, since the elements 1, 3, 5 belong to A but not to B and $B - A = \{8\}$, since the element 8 belongs to B and not to A. We note that $A - B \neq B - A$.

Symmetric Difference of two sets

Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ or

$(A \cup B) - (A \cap B)$ and is denoted by $A \Delta B$ or $A \oplus B$ (A direct sum B).

$$A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Note A D B = B D A {commutative}
For example, let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7\}$

then $A - B = \{2, 4\}$, $B - A = \{7\}$

$\therefore \quad A \Delta B = (A - B) \cup (B - A) = \{2, 4, 7\}$

Note The sets A – B, A Ç B and B – A are mutually disjoint sets, i.e., the intersection of any of these two sets is a null set.

Complement of a set

The complement of a set A consists of elements, which are elements of U, but not the elements of A.

We write the complement set in terms of set builder form as :

A' or $\overline{A} = \{x : x \in U \text{ and } x \notin A\}$

Note that elements of A' does not belong to set A. On Venn diagram, the complement of A is the remaining region of the universal set.

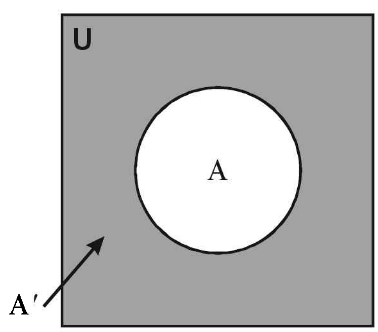


Figure 3.10

Consider some examples :

Let $U = N$, and $A = \{x \in N : x \text{ is prime}\}$.

Then $\overline{A}$ is the set of all those natural numbers which are less than 2 or have atleast three divisors.

If B be the set of all even natural numbers.

Then $\overline{B}$ is the set of all odd natural numbers.

Interpretation of complement

We can read the conditional statement for the complement with the help of two ways arrow as :

$x \in A' \Leftrightarrow x \in U$ and $x \notin A$

In terms of minus or difference operation,

$A' = U - A$

It is clear from the representation on Venn diagram that the universal set comprises of two distinct sets – set A and complement set A'.

$\Rightarrow \quad U = A \cap A'$

Complement of universal set

The complement of universal set is empty set. It is so because difference of union set with itself is the empty set.

$U' = \{x : x \in U \text{ and } x \notin U\} = \phi$

Complement of complement set is set itself

The complement of complement set is set itself. The complement set is defined as :

$A' = U - A$

Now, complement of complement set is :

$\Rightarrow \quad (A')' = (U - A)'$

Let us consider the example, where :

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1, 2, 3, 4, 5, 6\}$

Then

$A' = \{1,2,3,4,5,6,7,8\} - \{1,2,3,4,5,6\}$

$= \{7,8\}$

Again taking complement, we have :

$(A')' = \{1,2,3,4,5,6,7,8\} - \{7,8\}$

$= \{1,2,3,4,5,6\} = A$

Union with complement set

The union of set with its complement is universal set :

$A \cup A' = \{x : x \in U \text{ and } x \in A\}$

$\cup \{x : x \in U \text{ and } x \in A\} = U$

From Venn diagram also, we see that universal set consists of set A and component A′.

$U = A \cup A'$

The two sets on the right side of the equation are disjoint sets. Hence,

$A \cup A' = U$

Intersection with complement set

There is nothing common between set A and its complement A′. Thus, intersection of a set with its complement yields the empty set,

$A \cap A' = \phi$

EXAMPLE 3.7 Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 3, 5, 7, 9}. Find A'.

SOLUTION We note that 2, 4, 6, 8, 10 are the only elements of U which do not belong to A.

Hence A' = {2, 4, 6, 8, 10}.

De-Morgan's laws

In the real world situation, we want to negate a condition of incidence. For example, consider a class in the school. Some students play either basketball or football or both, but there are students, who play neither basketball nor football. We have to identify later category of students as a set.

Let the set of students playing basketball be "B" and that playing football be "F". Then, students who do not play basketball is complement set B′ and students who do not play football is complement set F′. These complement sets are drawn to the same universal set, "U".

The two complement sets are overlapping sets. There are students in the set B' who play football and there are students in the set F', who play basketball. In order to remove those students playing other game, we intersect two complements. The members of the intersection of two complements, therefore, represent students who play neither basketball nor football.

Looking at the intersection of two complement sets, however, we observe that this is equal to the complement of union "B ∪ F". This conclusion can be derived from basic interpretation as well. We know that union "B ∪ F" represents students, who play either or both games. The complement of the union, therefore, represents, who neither play basketball nor football.

This fact, as a matter of fact, is the first De-Morgan's law. Symbolically,

$B' \cap F' = (B \cup F)'$

The second De-Morgan's law is :

$B' \cup F' = (B \cap F)'$

In the parlance of illustration given earlier, let us interpret right hand side of the second De-Morgan's law. The intersection "B ∩ F" represents students playing both games. Its complement, therefore, represents students who do not play both games, but may play one of them.

Thus, $A' \cap B' = (A \cup B)'$

$A' \cup B' = (A \cap B)'$

Analytical proof

Here, we shall prove the first De-Morgan's law. The second law can be proved in similar fashion. Let us consider an arbitrary element "x" belonging to set (A ∪ B)′.

$x \in (A \cup B)'$

$\Rightarrow \quad x \notin (A \cup B)$

Then, by definition of union,

$\Rightarrow \quad x \notin \{x : x \in A \text{ or } x \in B\}$

Here, "not or" is interpreted same as "and",

$\Rightarrow \quad x \notin A \text{ and } x \notin B$

$\Rightarrow \quad x \in A' \text{ and } x \in B'$

$\Rightarrow \quad x \in A' \cap B'$

But, we had started with (A ∪ B)′ and used its definition to show that "x" belongs to another set. It means that the other set consists of the elements of the first set at the least. Thus,

$(A \cup B)' \subseteq A' \cap B'$

Similarly, we can start with $A' \cap B'$ and reach the conclusion that :

$A' \cap B' \subseteq (A \cup B)'$

If sets are subsets of each other, then they are equal. Hence,

$A' \cap B' = (A \cup B)'$

EXAMPLE 3.8 Let U = {1, 2, 3, 4, 5, 6}, A = {2, 3} and B = {3, 4, 5}.Find A', B', $A' \cap B'$, $A \cup B$ and hence show that $(A \cup B) = A' \cap B'$.

SOLUTION Clearly A' = {1, 4, 5, 6}, B' = {1, 2, 6}. Hence $A' \cap B'$ = {1, 6}

Also $A \cup B$ = {2, 3, 4, 5}, so that $(A \cup B)'$ = {1, 6}

$(A \cup B)' = \{1, 6\} = A' \cap B'$

Number of elements in the union of two sets

We want to know the numbers of elements in the union of two sets in terms of the numbers of elements in individual sets.

The sum of the numbers in the individual sets is generally greater than the number in the union. The reason is that union includes common elements only once. On the other hand, sum of the numbers of individual sets counts common elements once with each set, in total two times. Clearly, it is required that we deduct the numbers of elements, which are common to each set, from the sum of numbers of elements in individual sets. Hence,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Here, $n(A \cap B)$ represents the numbers of elements common to two sets.

Alternatively, we can approach this formula in yet another way. See the representation of intersection of two sets. The union of two sets can be considered to comprise of three distinct regions which are three "disjoint" sets. Clearly,

$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

However, we observe that if we add $n(A \cap B)$ to either of the two difference sets, then we get the complete individual set.

$n(A) = n(A - B) + n(A \cap B)$

$\Rightarrow$ $n(A - B) = n(A) - n(A \cap B)$

and

$n(B) = n(B - A) + n(A \cap B)$

$\Rightarrow$ $n(B - A) = n(B) - n(A \cap B)$

Substituting for the numbers of the difference set in the equation for the numbers in the union set, we have :

$\Rightarrow$ $n(A \cup B) = n(A) - n(A \cap B) + n(A \cap B) + n(B) - n(A \cap B)$

$\Rightarrow$ $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Now let us find the numbers of elements in the union of "disjoint" sets. Since, there are no common elements between two disjoint sets, the intersection between disjoint sets is an empty set. Hence,

$n(A \cup B) = n(A) + n(B)$

Suppose, we wish to find the number of elements in the union of three sets. Proceeding as in the case of union of two sets, we deduct the intersections between each pair of sets as :

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

We can achieve this result analytically as well. Here, we consider "A" as one set and "$(B \cup C)$ " as other set. Then, we apply the relation, which has been obtained for the numbers in the union of two sets as :

$n(A \cup B \cup C) = n(A) + n(B \cup C) - n[A \cap (B \cup C)]$

Applying result for the union of two sets for $n(B \cup C)$, we have :

$n(B \cup C) = n(B) + n(C) - (B \cap C)$

Putting in the expression for $n(A \cup B \cup C)$,

$\Rightarrow$ $n(A \cup B \cup C) = n(A) + n(B) + n(C) - (B \cap C) - n[A \cap (B \cup C)]$

At this stage, our task is to evaluate $n[A \cap (B \cup C)]$, Following the distributive property,

$n[(A \cup (B \cup C)] = n[(A \cap B) \cup (A \cap C)]$

We can treat each of the terms in the small bracket on the right hand side of the above equation as a set. Applying relation obtained for the numbers in the union of two sets again, we have:

$n[(A \cap (B \cup C)] = n(A \cup B) + n(A \cup C) - n[(A \cap B) \cap (A \cap C)]$

The last term in above equation is :

$[(A \cap B) \cap (A \cap C)] = (A \cap B \cap C)$

Hence, $n[(A \cap (B \cup C)]$

$= n(A \cup B) + n(A \cup C) - n[(A \cap B \cap C)]$

Now, putting this expression in the expression of the numbers in the union involving three sets and rearranging terms, we have :

$\Rightarrow \quad n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

In a nutshell, we find that number of elements in the union, here, is equal to the sum of numbers in the individual sets, minus elements common to two sets taken at a time, plus elements common to all three sets.

EXAMPLE 3.9 In a group of students, 40 students study either English or Mathematics. Of these 25 students study Mathematics, 10 students study both Mathematics and English. How many students study English?

SOLUTION The word "or" in the first sentence indicates that union of students studying Mathematics (M) or English (E) or both is 40. Using formula, we have :

$$n(M \cup E) = n(M) + n(E) - n(M \cap E)$$

$$\Rightarrow \quad n(E) = n(M \cup E) - n(M) + n(M \cap E)$$

Putting values,

$$n(E) = 40 - 25 + 10 = 25$$

EXAMPLE 3.10 In the house of total 200 students, 140 students play basketball and 80 students play football. Each student of the house plays atleast one of these two games. How many students play both basketball and football?

SOLUTION The individual sets here are students playing basket ball (B) and football (F). Hence,

$$n(B) = 140$$

$$n(F) = 80$$

Clearly, there is no bar that a students playing basketball cannot play football. This is also evident from the sum of the numbers in each set. The sum is 140 + 80 = 220, whereas total numbers of students in the house is 200 only. Thus, there are students who play both games. We can interpret the total number as the union of two individual sets. Hence, applying expansion for the numbers of a union :

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

The students who play both games constitute the intersection of two individual sets. Putting values,

$$\Rightarrow \quad n(B \cup F) = 140 + 80 - 200 = 20$$

EXAMPLE 3.11 In a house of total 200 students, 100 students play basketball, 60 students play football and 20 play both games. How many students play neither basketball nor football?

SOLUTION We have already discussed that "neither nor" condition is same as that of De-Morgan's law :

$$n(B' \cap F') = n(B \cup F)'$$

Now expanding the right hand term, we have:

$$n(B' \cap F') = n(B \cap F)' = U - n(B \cup F)$$

Further using formula for the numbers in a union,

$$n(B' \cap F') = n(U) - n(B) - n(F) + n(B \cup F)$$

Putting values,

$$\Rightarrow \quad n(B' \cap F') = 200 - 100 - 60 + 20 = 60$$

EXAMPLE 3.12 In a house of 200 students, 120 students study Mathematics, 60 students study English and 40 students study both Mathematics and English. Find : (i) students who study Mathematics but not English (ii) students

who study English, but not Mathematics (iii) students who study either Mathematics or English and (iv) students who neither study Mathematics nor English.

SOLUTION Let us first characterise collections as given in the question. Two sets are given one for those who study Mathematics (M) and other for those who study English (E). The addition of numbers of individual sets is 120 + 60 = 180, which is less than total numbers of students. Hence, total numbers of 200 corresponds to universal set. Here,

$n(U) = 200$; $n(M) = 120$; $n(E) = 60$ and $n(M \cup E) = 40$

(i) Students studying Mathematics, but not English means that we need to find the numbers in the difference of set, i.e M – E.

$$n(M - E) = n(M) - n(M \cap E)$$

$$\Rightarrow \quad n(M - E) = 120 - 40 = 80$$

(ii) Students studying Mathematics, but not English means that we need to find the numbers in the difference of set, i.e E – M.

$$n(E - M) = n(E) - n(M \cap E)$$

$$\Rightarrow \quad n(E - M) = 60 - 40 = 20$$

(iii) Students who study either Mathematics or English is equal to the numbers in the union of two sets.

$$n(M \cup E) = n(M) + n(E) - n(M \cap E)$$

Putting values,

$$n(M \cup E) = 120 + 60 - 40 = 140$$

(iv) Students who study neither Mathematics nor English is equal to the numbers in the compliment of the union of two sets.

$$n(M \cup E)' = n(U) - n(M \cup E) = 200 - 140 = 60$$

EXAMPLE 3.13 A market research group conducted a survey of 1000 consumers and reported that 720 consumers like product A and 450 consumers like product B, what is the least number that must have liked both products ?

SOLUTION Let U be the set of consumers questioned, P be the set of consumers who liked the product A and Q be the set of consumers who like the product B. Given that

$$n(U) = 1000,\ n(P) = 720,\ n(Q) = 450$$

So $\quad n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

$$= 720 + 450 - n(P \cap Q) = 1170 - n(P \cap Q)$$

Therefore, $n(P \cup Q)$ is maximum when $n(P \cap Q)$ is least. But $P \cup Q \subseteq U$ implies $n(P \cup Q) \le n(U) = 1000$. So maximum values of $n(P \cup Q)$ is 1000. Thus, the least value of $n(P \cap Q)$ is 170. Hence, the least number of consumers who liked both products is 170.

EXAMPLE 3.14 Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is this data correct?

SOLUTION Let U be the set of car owners investigated, P be the set of persons who owned car A and Q be the set of persons who owned car B.

Given that $n(U) = 500$, $n(P) = 400$, $n(Q) = 200$ and $(P \cap Q) = 50$.

Then $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

$$= 200 + 400 - 50 = 550$$

But $P \cup Q \subseteq U$ implies $n(P \cup Q) \le n(U)$.

This is a contradiction. So, the given data is incorrect.

EXAMPLE 3.15 If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in $A \cup B$? Find also, the maximum number of elements in $A \cup B$.

SOLUTION We have

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$,

$n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum respectively.

Case I : If $n(A \cap B)$ is maximum, i.e., $n(A \cap B) = 3$ such that $A = \{a,b,c,d,e,f\}$ and $B = \{d,a,c\}$ then

$n(A \cup B) = n(A) + n(B) - n(A \cup B) = 6 + 3 - 3 = 6$.

Case II : If $n(A \cap B)$ is minimum, i.e., $n(A \cap B) = 0$ such that $A = \{a,b,c,d,e,f\}$ and $\{g,h,i\}$

$n(A \cup B) = n(A) + n(B) - 6 + 3 = 9.$

EXAMPLE 3.16 Suppose $A_1, A_2, \ldots.. A_{30}$ are thirty sets each with five elements and $B_1, B_2, \ldots.$ B_n are n sets each with three elements.

$$\text{Let } \bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S.$$

Assume that each element of S belongs to exactly ten of the A_i's and exactly to nine of the B_j's. Find n.

SOLUTION Given A_i's are thirty sets with five elements each, so

$$\sum_{i=1}^{30} n(A_i) = 5 \times 30 = 150 \quad \ldots(1)$$

If there are m distinct elements in S and each element of S belongs to exactly 10 of the A_i's, we have

$$\sum_{i=1}^{30} n(A_i) = 10m \quad \ldots(2)$$

$\therefore$ From (1) and (2), we get $10m = 150$

$\Rightarrow \quad m = 15$

Similarly $\sum_{j=1}^{n} n(B_j) = 3n$ and $\sum_{j=1}^{n} n(B_j) = 9m$

$\therefore 3n = 9m$

$\Rightarrow \quad n = \dfrac{9m}{3} = 3m = 3 \times 15 = 45$

Hence, $\quad n = 45.$

EXAMPLE 3.17 A survey of 500 television watchers produced the following information : 285 watch football, 195 watch hockey, 115 watch basket ball, 45 watch football and basket ball, 70 watch football and hockey, 50 watch hockey and basket ball, 50 do not watch any of the three games. How many watch all the three games? How many watch exactly one of the three games?

SOLUTION Let F, H and B be the sets of television watchers who watch football, hockey and basket ball respectively. Then according to the problem, we have

$n(U) = 500, n(F) = 285, n(H) = 195, n(B) = 115,$

$n(F \cap B) = 45, n(F \cap H) = 70, n(H \cap B) = 50$

and $n(F' \cup H' \cup B') = 50.$

where U is the set of all the television watchers.

Since

$n(F' \cup H' \cup B') = n(U) - n(F \cup H \cup B)$

$\Rightarrow \quad 50 = 500 - n(F \cup H \cup B)$

$\Rightarrow \quad n(F \cup H \cup B) = 450.$

We have : $n(F \cup H \cup B) = n(F) + n(H) + n(B) - n(F \cap H) - n(H \cap B) - n(B \cap F) + n(F \cap H \cap B)$

$\Rightarrow \quad 450 = 285 + 195 + 115 - 70 - 50 - 45 + n(F \cap H \cap B)$

$\therefore\ n(F \cap H \cap B) = 20$, which is the number of those who watch all the three games.

Also number of persons who watch football only

$= (F \cap H' \cap B')$

$= n(F) - n(F \cap H) - n(F \cap B) + n(F \cap H \cap B)$

$= 285 - 70 - 45 + 20 = 190.$

The number of person who watch hockey only

$= (F \cap H' \cap B')$

$= n(H) - n(H \cap F) - n(H \cap B) + n(H \cap F \cap B)$

$= 195 - 70 - 50 + 20 = 95$

and the number of persons who watch basket ball only

$= (B \cap H' \cap F')$

$= n(B) - n(B \cap H) - n(B \cap F) + n(H \cap F \cap B)$

$= 115 - 50 - 45 + 20 = 40$

Hence, the number of those who watch exactly one of the three games.

$= 190 + 95 + 40 = 325.$

EXAMPLE 3.18 Prove that

$$A \cap (B - C) = (A \cap B) - (A \cap C).$$

SOLUTION We have $x \in A \cap (B - C)$

$\Rightarrow$ $x \in A$ and $x \in B - C$

$\Rightarrow$ $x \in A$ and ($x \in B$ and $x \notin C$)

$\Rightarrow$ ($x \in A$ and $x \in B$) and $x \notin C$

$\Rightarrow$ $x \in A \cap B$ and $x \notin C$

$\Rightarrow$ $x \in A \cap B$ and $x \notin A \cap C$

[$\because x \notin C \Rightarrow x \notin A \cap C$]

$\Rightarrow$ $x \in (A \cap B) - (A \cap C)$

$\therefore$ $A \cap (B - C) \subseteq (A \cap C)$ ...(1)

Again $y \in (A \cap B) - (A \cap C)$

$\Rightarrow$ $y \in A \cap B$ and $y \notin A \cap C$

$\Rightarrow$ ($y \in A$ and $y \in B$) and $y \notin A \cap C$

[$\because y \in A$ and $y \notin A \cap C \Rightarrow y \notin C$]

$\Rightarrow$ $y \in A$ and ($y \in B$ and $y \notin C$)

$\Rightarrow$ $y \in A$ and $y \in (B - C) \Rightarrow y \in A \cap (B - C)$.

$\therefore$ $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$. ...(2)

From (1) and (2), we get $(A \cap B) - (A \cap C) = A \cap (B - C)$

EXAMPLE 3.19 If $A \Delta B = (A - B) \cap (B - A)$, show that

$$A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C).$$

SOLUTION We have $A \cap (B \Delta C)$

$= A \cap [(B - C) \cup (C - B)]$

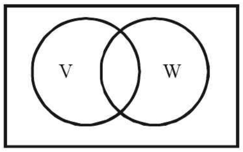
$= [A \cap (B - C)] \cup [A \cap (C - B)]$

$= [(A \cap B) - (A \cap C)] \cup [(A \cap C) - (A \cap B)]$

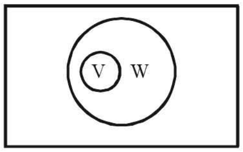
$= (A \cap B) \Delta (A \cap C)$.

PRACTICE PROBLEMS [C]

1. If U = R, the set of real numbers,
   $A = \{x : x \in R, 0 < x < 2\}$
   $B = \{x : x \in R, 1 < x \leq 3\}$,
   then find A', B', $A \cup B$, $A \cap B$, $A - B$.
2. If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{5, 6, 7, 8\}$ and $D = \{7, 8, 9, 10\}$; find
   (i) $A \cup B$ (ii) $A \cup B \cup C$
   (iii) $B \cup C \cup D$
3. If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$; find
   (i) $A \cap B$ (ii) $A \cap C \cap D$
   (iii) $A \cap (B \cup D)$ (iv) $(A \cap B) \cap (B \cup C)$
4. If $X = \{a, b, c, d\}$ and $Y = \{f, b, d, g\}$, find
   (i) $X - Y$ (ii) $Y - X$ (iii) $X \cap Y$
5. In the Venn diagrams below, shade
   (i) $W - V$ (ii) $V^c \cup W$
   (iii) $V \cap W^c$ (iv) $V^c - W^c$



(a)



(b)

6. If $U = \{a, b, c, d, e, f, g, h\}$, find the complements of the following sets :
   (i) $A = \{a, b, c\}$ (ii) $B = \{d, e, f, g\}$
7. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2\ 3, 5, 7\}$. Verify that
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

8. Let U be the set of all triangles in a plane. If A is the set of all triangles with at least one angle different from 60°, what is A' ?

9. Is it true that for any sets A and B,
$P(A) \cup P(B) = P(A \cup B)$? Justify your answer.

10. Show that for any sets A and B,
$A = (A \cap B) \cup (A - B)$ and $A \cup (B - A) = (A \cup B)$

11. If A, B and C are sets, then prove that
$(A - B) \cap (A - C) = A - (B \cup C)$. Verify the above result by Venn diagrams.

12. Let U be the set of all people and M = {Males}, S = {College students}, T = {Teenagers}, W = {people having height more than five feet}. Express each of the following in the notation of set theory.

(i) College students having heights more than five feet.
(ii) People who are not teenagers and have their heights less than five feet.
(iii) All people who are neither males nor teenagers nor college students.

13. In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. How many like tennis only and not cricket ? How many like tennis ?

14. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers. Find :

(i) The number of people who read atleast one of the newspapers.
(ii) The number of people who read exactly one newspaper.

15. At a certain conference of 100 people, there are 29 Indian women and 23 Indian men, of these Indian people 4 are doctors and 24 are either men or doctors. There are no foreign doctors. How many foreigners are attending the conference? How many women doctors are attending the conference?

16. In a group of children, 35 play football out of which 20 play football only; 22 play hockey; 25 play cricket out of which 11 play cricket only. Out of these 7 play cricket and football but not hockey, 3 play football and hockey but not cricket and 12 play football and cricket both. How many play all the three games? How many play cricket and hockey but not football, how many play hockey only? What is the total number of children in the group?

17. If A and B are subsets of a set X, then prove that
$A \subseteq B \Rightarrow X - B \subseteq X - A$.

3.4 | REAL NUMBER SYSTEM

Natural Numbers

The concept of natural numbers originated from the necessity to count objects. We can compare two natural numbers and see which of the two is the greater. All natural numbers arranged in the increasing order form a series of natural numbers, the first number being unity, the second number two, the third number three, and so on. Each natural number has a place of its own in the series. We shall denote the series of natural numbers by the letter N.

Thus, the counting numbers 1, 2, 3, 4,..... are called natural numbers.

$$N = \{1, 2, 3, 4, \ldots\ldots\}$$

Whole numbers

Let us consider the number, zero. It is designated as 0. Zero is not a natural number and is considered

to be a predecessor of all the natural numbers. The set of natural numbers including zero is called set of whole numbers. It is designated as W. Thus W = {0, 1, 2,......}.

Integers

In the set of natural numbers, for instance, we cannot subtract 5 from 3. It is therefore necessary to extend the set of natural numbers. We shall introduce new numbers into consideration, natural numbers with the minus sign, i.e. numbers of the form –m, where m is a natural number, and shall call them negative integers.

The set of numbers consisting of all natural numbers, zero, and all negative numbers, is known as the set of integers and is denoted by the letter I or Z.

The numbers..... – 3, – 2, –1, 0, 1, 2, 3...are called integers.

Thus, I (or Z) = {...–3, –2, –1, 0, 1, 2, 3...}

Note

1. Natural numbers are sometimes called positive integers and is also denoted by I^+ or Z^+.
2. The set of whole numbers is also called as the set of non-negative integers I_0^+.
3. The set of negative integers, denoted by I^- consists of {..........., –3, –2, –1}
4. The set of non-positive integers, denoted by I_0^- is {...., –3, –2, –1, 0}
5. Integers which are divisible by 2 are called **even integers**.
   i.e. 0, ± 2, ± 4,.....
6. Integers, which are not divisible by 2 are called as **odd integers**.
   i.e. ± 1, ± 3, ± 5, ± 7.......

Prime and composite numbers

The set of natural numbers consists of unity, prime, and composite numbers. A natural number which is larger than unity is a prime number if it has no divisors except for unity and itself. A natural number which is larger than unity is a composite number if it has at least one divisor different from unity and itself.

Let ‘p’ be a natural number, ‘p’ is said to be prime if it has exactly two distinct factors, namely 1 and itself.

i.e. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,..

Let ‘a’ be a natural number, ‘a’ is said to be to be composite if, it has atleast three distinct factors.

Note

1. ‘1’ is neither prime nor composite.
2. ‘2’ is the only even prime number.
3. Numbers which are not prime are composite numbers (except 1)
4. ‘4’ is the smallest composite number.

Co-prime numbers

Two natural numbers (not necessarily prime) are **co-prime**, if their G.C.D. is unity.

For example (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (5, 6), (7, 8)are co-prime numbers.

These numbers are also called **relatively prime** numbers.

Note

1. Two distinct prime numbers are always co-prime but converse need not be true.
2. Consecutive numbers are always co-prime numbers.
3. If natural numbers p_1 and p_2 are co-prime and a natural number p is divisible both by p_1 and by p_2, then p is divisible by the product p_1p_2.

Twin prime numbers

If the difference between two prime numbers is two, then the numbers are called as twin prime numbers.

For example {3, 5}, {5, 7}, {11, 13}}, {17, 19}, {29, 31} are twin prime numbers.

Note

1. An even number a can be written in the form $a = 2q$, where q is an integer
2. An odd number a can be written in the form $a = 2q + 1$, where q is an integer.
3. An integer a which is exactly divisible by three can be written in the form $a = 3q$, where q is an integer.
4. An integer a which is not exactly divisible by three can be written in one of the following forms :
   $a = 3\ell + 1$ or $a = 3n + 2$, where ℓ and n are integers.
5. An integer a which is exactly divisible by a natural number k can be written in the form $a = kq$, where q is an integer.
6. An integer a which is not exactly divisible by a natural number k can be written in the form $a = kq + r$, where r is one of the numbers 1, 2, ..., $(k - 1)$, and q is an integer.

In accordance with the divisibility of integers by a given natural number k, the set of integers can be divided into k classes.

For instance, if $k = 2$, then the set of all integers can be divided into two classes, e.g. even numbers and odd numbers.

The set of all integers can also be divided into three classes :

(a) numbers which are multiples of the number three, i.e. numbers of the form 3q, where q is an integer,
(b) numbers which when divided by three yield unity as the remainder, i.e. numbers of the form 3 + 1, where ℓ is an integer,
(c) numbers which when divided by three yield two as the remainder, i.e. numbers of the form $3n + 2$, where n is an integer.

It is now clear how we can divide the set of integers into 4 classes, 5 classes, and so on.

We find that division of integers into classes helps us to solve problems.

EXAMPLE 3.1 Every number is of one of the forms $5n$, $5n \pm 1$, $5n \pm 2$.

SOLUTION For if any number is divided by 5, the remainder is one of the numbers 0, 1, 2, $5 - 2$, $5 - 1$.

EXAMPLE 3.2 Every square number is of one the forms $5n$, $5n \pm 1$.

SOLUTION The square of every number is of one of the forms $(5m)^2$, $(5m \pm 1)^2$, $(5m \pm 2)^2$. If those are divided by 5, the remainders are 0, 1, 4; and, since $4 = 5 - 1$, the forms are $5n$, $5n + 1$, and $5n - 1$.

EXAMPLE 3.3 Consider any three consecutive natural numbers. prove that the cube of the largest cannot be the sum of the cubes of other two.

SOLUTION Let $n - 1$, n, $n + 1$ be the consecutive numbers.

If the equation $(n + 1)^3 = n^3 + (n - 1)^3$

that is,

$n^3 + 3n^2 + 3n + 1 = n^3 + n^3 - 3n^2 + 3n - 1$,

were satisfied,

then $2 = n^2 (n - 6)$

But the right side is positive only if $n > 6$, and in that case $n^2 (n - 6) > 36$ and therefore cannot be true.

EXAMPLE 3.4 Prove that the product of four consecutive integers is one less than a perfect square.

SOLUTION Let the four consecutive integers be n, $n + 1$, $n + 2$, $n + 3$.

If 1 is added to their product, we have

$$n(n + 1)(n + 2)(n + 3) + 1$$
$$= [n(n + 3)][(n + 1)(n + 2)] + 1$$
$$= (n^2 + 3n)(n^2 + 3n + 2) + 1$$

$= (n^2 + 3n)^2 + 2(n^2 + 3n) + 1 = (n^2 + 3n + 1)^2$

Therefore, the product of the four numbers is one less than the square of the integer $n^2 + 3n + 1$.

EXAMPLE 3.5 If x and y are prime numbers which satisfy $x^2 – 2y^2 = 1$, solve for x and y.

SOLUTION $x^2 – 2y^2 = 1$ gives $x^2 = 2y^2 + 1$ and hence x must be an odd number. If $x = 2n + 1$, then $x^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = 2y^2 + 1$.

Therefore $y^2 = 2n(n + 1)$. This means that y^2 is even and hence y is an even integer. Now, y is also a prime implies that y = 2. This gives x = 3. Thus the only solution is x = 3, y = 2.

EXAMPLE 3.6 Consider any three consecutive natural numbers the smallest of which is greater than 3. Then the square of the largest cannot be the sum of the squares of the other two.

SOLUTION Let n, n + 1, n + 2 be three consecutive natural numbers with n > 3.

$$(n + 2)^2 – [n^2 + (n + 1)^2]$$
$$= 3 + 2n – n^2,$$
$$= – (n + 1)\ (n – 3),$$

which cannot be zero since n > 3.

EXAMPLE 3.7 Let $n \in N$. Find all prime numbers of the form $n^3 – 8$.

SOLUTION Since $n^3 – 8 = (n – 2)(n^2 + 2n + 4)$, for it to be prime one needs either $n – 2 = 1 \Rightarrow n = 3$ or $n^2 + 2n + 4 = 1$, but this last equation does not have integral solutions. Hence $3^3 – 8 = 19$ is the only such prime.

EXAMPLE 3.8 Given that 1, 000, 002, 000, 001 has a prime factor greater than 9000, find it.

SOLUTION We have

$$1,000,002,000,001 = 10^{12} + 2.10^6 + 1$$
$$= (10^6 + 1)^2$$
$$= ((10^2)^3 + 1)^2$$
$$= (10^2 + 1)^2((10^2)^2 – 10^2 + 1)^2$$
$$= 101^2 9901^2$$

Hence the prime sought is 9901.

EXAMPLE 3.9 Find the natural number n for which the fraction $\frac{15n^2 + 8n + 6}{n}$ is a natural number.

SOLUTION We have $\frac{15n^2 \quad 8n \quad 6}{n}$

$$= 15n + 8 + \frac{6}{n}.$$

Since 15n + 8 is a natural number, and $\frac{6}{n}$ is a natural number only for n = 1, n = 2, n = 3, n = 6, for these values of n the given fraction is a natural number.

EXAMPLE 3.10 Find the integral solutions of the equation xy = 2x – y.

SOLUTION From the given equation we find $y = \frac{2x}{x+1}$, where $x \neq – 1$.

Let us rewrite the fraction as follows :

$$y = \frac{(2x+2)-2}{x+1} = 2 - \frac{2}{x+1}$$

Hence it follows that the value of y will be an integer only if the fraction $\frac{2}{x+1}$ is an integer which is possible for an integral x when x + 1 = ± 1 or x + 1 = ± 2.

Substituting the corresponding x into the formula

$y = \frac{2x}{x+1}$, we find all the integral solutions of the given equation; $x_1 = 0$, $y_1 = 0$; $x_2 = –2$, $y_2 = 4$; $x_3 = 1$, $y_3 = 1$; $x_4 = –3$, $y_4 = 3$.

Rational numbers

Let us now consider the set of rational numbers which consists of all positive fractions, zero, and all fractions with the minus sign. We can assume that each number belonging to this set is the ratio of an integer to a natural number. We shall therefore assume that this set consists of numbers of the form $\frac{p}{q}$, where q is a natural number and p is an integer. The set of rational numbers is denoted by the letter Q.

Decimal number system

Ten symbols are introduced in this system, which are called digits, the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9 denoting the first nine natural numbers and the symbol 0 denoting zero. In this number system, the number ten is designated as 10 and each natural number p is represented as

$p = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$,

where n is a number from the set of whole numbers, a_n is one of the numbers 1, 2, 3, ..., 9.

Another notation is usually used to write the number p which is based on the principle of the place values of digits.

The notation 2705 means that the number consists of two thousands, seven hundreds, a zero tens, and five units. i.e. $2705 = 2.10^3 + 7.10^2 + 0.10 + 5$.

Terminating decimal

We can write, for instance, the fractions

$\frac{3721}{100}, \frac{21}{10000}, \frac{131}{1000}$ as 37.21, 0.0021, 0.131.

A fraction written in this form is known as a terminating or finite decimal fraction.

Any terminating decimal fraction can be easily changed to a common fraction.

For example, $0.34 = \frac{34}{100} = \frac{17}{50}$.

Non-terminating repeating decimal

A non-terminating repeating decimal fraction is a decimal in which the decimal point is followed by infinitely many digits, with one digit or an ordered set of digits, beginning with a certain place after the decimal point, recurring.

For example, 4.27131313.... is a non-terminating repeating decimal fraction with the period 13.

$4.21131313... = 4.27\overline{13}$, $0.454545... = 0.\overline{45}$.

To change a non-terminating repeating decimal fraction to a common fraction, we must subtract the number preceding the first period from the number preceeding the second period and make this difference the numerator, and write the digit 9 in the denominator as many time as there are digits in the period, and add as many zeros after the nines as there are digits between the decimal point and the first period. For example, $0.117\overline{20}$

$$= \frac{11720 - 1172}{90000} = \frac{10548}{90000} = \frac{1172.9}{9.10000}$$

$$= \frac{293.4}{4.2500} = \frac{293}{2500}$$

$$0.\overline{45} = \frac{45 - 0}{99} = \frac{5.9}{9.11} = \frac{5}{11},$$

$$4.27\overline{13} = \frac{42713 - 427}{9900} = \frac{42286}{9900}$$

$$= \frac{2.21143}{2.4950} = \frac{21143}{4950}$$

EXAMPLE 3.11 Write the infinitely repeating decimal $0.3\overline{45} = 0.345454545.....$ as the quotient of two natural numbers.

SOLUTION The trick is to obtain multiples of x = 0.345454545..... so that they have the same infinite tail, and then subtract these tails, cancelling them out. So observe that

$$10x = 3.45454545....;\ 1000x = 345.454545...$$

$$\Rightarrow 1000x - 10x = 342 \Rightarrow x = \frac{342}{990} = \frac{19}{55}$$

From the above examples, the following should be clear.

Every rational number has a terminating or a repeating decimal expansion. Conversely, a real number with a terminating or repeating decimal expansion must be a rational number.

Note Any fraction $\frac{p}{q}$, where the natural number q does not have any prime divisors other than 2 and 5 can be written as a terminating decimal fraction. In other words, a rational number has a terminating decimal expansion if and only if its denominator is of the form 2^m5^n, where m and n are natural numbers.

From the above fact we can tell, without actually carrying out the long division, that say, $\frac{1}{1024} = \frac{1}{2^{10}}$ has a terminating decimal expansion, but that, say, $\frac{1}{6}$ does not.

Irrational numbers

A need arises for introducing new numbers, distinct from rational numbers, such as, for instance, the number whose square is 2. A calculator probably gives about 9 decimal places when one tries to compute $\sqrt{2}$, say, it says $\sqrt{2} \approx 1.414213562$. What happens after the final 2 is the interesting question. Do we have a pattern or do we not ?

Theorem $\sqrt{2}$ is irrational.

Proof Assume there is $s \in Q$ such that $s^2 = 2$. We can find integers m, $n \neq 0$ such that $s = \frac{m}{n}$. The crucial part of the argument is that we can choose m, n such that this fraction be in lowest terms, and hence, m, n cannot be both even. Now, $n^2s^2 = m^2$, that is $2n^2 = m^2$ this means that m^2 is even. But then m itself must be even, since the product of two odd numbers is odd. Thus $m = 2a$ for some non-zero integer a (since $m \neq 0$). This means that $2n^2 = (2a)^2 = 4a^2 \Rightarrow n^2 = 2a^2$. This means once again that n is even. But then we have a contradiction, since m and n were not both even.

Suppose that we knew that every strictly positive natural number has a unique factorization into primes. Then if n is not a perfect square we may deduce that, in general, $\sqrt{n}$ is irrational.

Irrational numbers are numbers which cannot be expressed in p/q form and their set is denoted by Q^c or Q'. (i.e. complementary set of Q). These numbers cannot be expressed as recurring decimals. For example, $\sqrt{2}$, $1 + \sqrt{3}$, e, π, etc. are irrational numbers.

Irrational numbers are those numbers having infinite non-repeating decimal expansions. Of course, by simply " looking" at the decimal expansion of a number we can't tell whether it is irrational or rational without having more information.

We expect a number like 0.1001000010000000001....,

where there are 2, 4, 8, 16,.... zeroes between consecutive once, to be irrational, since the gaps between successive 1's keep getting longer, and so the decimal does not repeat. For the same reason, the number

0.123456789101112....,

which consists of enumerating all strictly positive natural numbers after the decimal point, is irrational. This number is known as the Champernowne-Mahler number.

EXAMPLE 3.12 Suppose that you are given a finite string of integers, say, 12345. Can you find an irrational number whose first five decimal digits after the decimal point are 12345 ?

SOLUTION Yes, there are multiple ways of doing this. An idea is to take the first five decimal digits of $\sqrt{2}$, remove them and supplant them with the given string. For example, for 12345 we proceed as follows :

$$\sqrt{2} \approx 1.414213562...$$

$$\Rightarrow \frac{\sqrt{2}}{10^6} \approx 0.000001414213562....$$

Then the number $\frac{\sqrt{2}}{10^6} + 0.12345$

is an irrational number whose first five digits after the decimal point are 12345.

Another idea is to form the number 0.12345012345001234500001234500000000012345....
where puts 2^k zeros between appearances of the string 12345.

Note

1. Integers are rational numbers, but converse need not be true.
2. Negative of an irrational number is an irrational number.
3. Sum of a rational number and an irrational number is always an irrational number, e.g.,

   e.g. $2 + \sqrt{3}$
4. The product of a non zero rational number & an irrational number will always be an irrational number.
5. If $a \in Q$ and $b \notin Q$, then ab = rational number, only if a = 0.
6. Sum, difference, product and quotient of two irrational numbers need not be an irrational number (it may be a rational number also).

EXAMPLE 3.13 Give examples, if possible, of the following.

(i) The sum of two rational numbers giving an irrational number.
(ii) the sum of two irrational giving an irrational number.
(iii) the sum of two irrationals giving a rational number
(iv) the product of a rational and an irrational giving an irrational number.
(v) the product of a rational and an irrational giving a rational number.
(vi) the product of two irrationals giving an irrational number
(vii) the product of two irrationals giving a rational number.

SOLUTION

(i) This is impossible. The rational numbers are closed under addition and multiplication.
(ii) Take both numbers to be $\sqrt{2}$. Their sum is $2\sqrt{2}$ which is also irrational.
(iii) Take one number to be $\sqrt{2}$ and the other $-\sqrt{2}$. Their sum is 0, which is rational.
(iv) Take the rational number to be 1 and the irrational to be $\sqrt{2}$. Their product is $1.\sqrt{2} = \sqrt{2}$
(v) Take the rational number to be 0 and the irrational to be $\sqrt{2}$. Their product is $0.\sqrt{2} = 0$
(vi) Take one irrational number to be $\sqrt{2}$ and the other to be $\sqrt{3}$. Their product is $\sqrt{2}.\sqrt{3} = \sqrt{6}$
(vii) Take one irrational number to be $\frac{1}{\sqrt{2}}$. Their product is $\sqrt{2}.\frac{1}{\sqrt{2}} = 1$

EXAMPLE 3.14 Prove that $\sqrt[3]{2}$ cannot be represented in the form $p + \sqrt{q}$, where p and q are rational ($q > 0$ and is not a perfect square).

SOLUTION Put $\sqrt[3]{2} = p + \sqrt{q}$.

Hence $2 = p^3 + 3pq + (3p^2 + q)\sqrt{q}$,

Since q is not a perfect square, $3p^2 + q = 0$ must be true, which is impossible.

EXAMPLE 3.15 Show that $\sqrt{2} + \sqrt{5}$ is irrational.

SOLUTION Suppose $\sqrt{2} + \sqrt{5} = x = p/q$ is a rational number with $p, q \in Z$.

Then $(x - \sqrt{2})^2 = 5$ i.e. $x^2 - 2\sqrt{2}x + 2 = 5$

Hence $x^2 - 3 = 2\sqrt{2}x$, which gives $\sqrt{2} = (x^2 - 3) / (2x)$, a rational number.

This contradicts the fact that $\sqrt{2}$ is irrational.

So $\sqrt{2} + \sqrt{5}$ is irrational.

EXAMPLE 3.16 Prove that $\log_3 5$ is irrational.

SOLUTION Let $\log_3 5$ is rational.

$\therefore \quad \log_3 5 = \dfrac{p}{q}$; where p and q are co-prime numbers

$\Rightarrow \quad 3^{p/q} = 5 \qquad \Rightarrow \qquad 3^p = 5^q.$

which is not possible, hence our assumption is wrong and $\log_3 5$ is irrational.

EXAMPLE 3.17 Find all rational x for which $\sqrt{x^2+3x+5}$ is a rational number.

SOLUTION Let $y = \sqrt{x^2+3x+5}$

Since, x and y are rational numbers, the difference between them is also a rational number

i.e. $\sqrt{x^2+3x+5} - x = q$ (a rational number)

$\Rightarrow \quad x^2 + 3x + 5 = (x + q)^2 = x^2 + 2qx + q^2$

$\Rightarrow \quad x = \dfrac{q^2-5}{3-2q}$

Hence, y is rational for every $x = \dfrac{q^2-5}{3-2q}$

where q is rational and $q \neq \dfrac{3}{2}$.

Real numbers

When we combine the set of all irrational numbers with the set of all rational numbers, we obtain the set of all real numbers.

The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus, $R = Q \cup Q^c$.

The number system is one such system, in which different number groups are related. Natural number is a subset of integers. Integers are subset of rational numbers and rational numbers are subset of real numbers. None of these sets are equal. Hence, relations are described by proper subsets.

$N \subset Z$

We can write the chain of relation among number sets:

$\Rightarrow \quad P \subset N \subset Z \subset Q \subset R$

However, irrational numbers are also subset of real numbers, but irrational numbers is not rational numbers. We represent this relation by emphasising that rational numbers is not a subset of irrational numbers or vice-versa. We depict this relation as :

Q(rational numbers) $\not\subset$ T (irrational numbers)

But irrational numbers is subset of real numbers. The real numbers comprises of only two subsets at the highest level rational and irrational. Therefore, irrational numbers is the remaining collection after deducting rational numbers from real numbers.

Following the logic, we define set of irrational numbers as :

T (irrational numbers) =

$\{x : x \in R \text{ and } x \notin Q\}$

The figure 3.22 shows a diagram of the subsets of the real numbers.

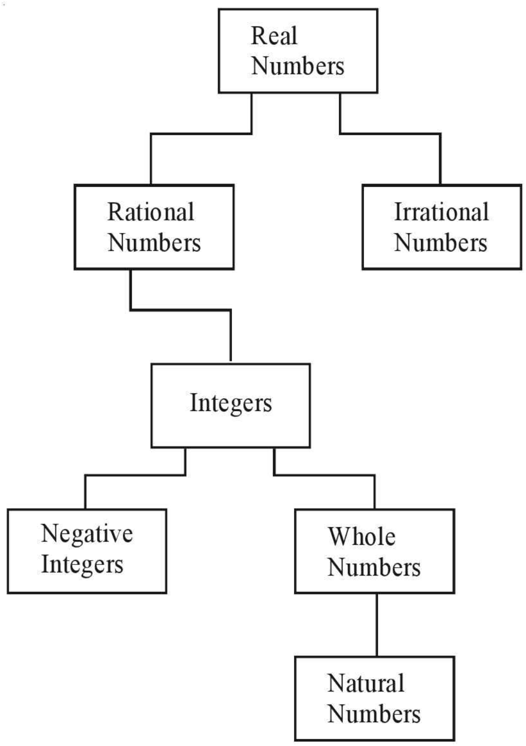


Figure 3.22

EXAMPLE 3.18 List the numbers in the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 6\frac{1}{4}, 7.3, \sqrt{81}\right\}$ that belong to each subset of the real numbers : (i) Natural numbers, (ii) Whole numbers, (iii) Integers, (iv) Rational numbers, (v) Irrational numbers, and (vi) Real numbers.

SOLUTION

(i) **Natural numbers:** The only natural number in the set is $\sqrt{81}$ since $\sqrt{81} = 9$. (9 multiplied by itself is 81.)

(ii) **Whole numbers:** The elements of the set that are whole numbers are 0 and $\sqrt{81}$.

(iii) **Integers:** The elements of the set that are integers are $\sqrt{81}$, 0, and −7,

(iv) **Rational numbers:** The rational numbers are $-7 6\frac{1}{4}\left(6\frac{1}{4}=\frac{24}{4}\right)$, $7.3\left(7.3 = 7\frac{3}{10}=\frac{73}{10}\right)$ and $\sqrt{81}\left(\sqrt{81}=\frac{9}{1}\right)$.

(v) **Irrational numbers:** The irrational numbers in the set are $\sqrt{5}(\sqrt{5}\approx 2.236)$ and π ($\pi \approx$ 3.14). Both $\sqrt{5}$ and π neither terminate nor have repeating patterns.

(vi) **Real numbers:** All the numbers in the set are real numbers.

n-th Root of a Number

Let a and b be real numbers, and let $n \geq 2$ be a positive integer. If $a = b^n$, then b is the n-th root of a. If n = 2, then the root is a square root. If n = 3, then the root is a cube root.

Some numbers have more than one n-th root. For example, both 5 and −5 are square roots of 25. The **principal n-th root** of a number is defined as follows :

Let a be a real number that has atleast one n-th root. The principal n-th root of a is the n-th root that has the same sign as a. It is denoted by a radical symbol: $\sqrt[n]{a}$

The positive integer n is the index of the radical, and the number a is the radicand. If n = 2, we omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$.

CAUTION $\sqrt{\ }$ symbol stands for the positive square root only. Let us evaluate the expressions involving radicals :

$\sqrt{49} = 7$ because $7^2 = 49$.

$\sqrt[3]{\frac{125}{64}} = \frac{5}{4}$ because $\left(\frac{5}{4}\right)^3 = \frac{5^3}{4^3} = \frac{125}{64}$.

$\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

$\sqrt[4]{-81}$ is not a real number because there is a no real number that can be raised to the fourth power to produce –81.

Here are some generalizations about the n-th roots of a real number

(i) If a is a positive real number and n is a positive even integer, then a has exactly two (real) n-th roots. We denote these roots by $\sqrt[n]{a}$ and $-\sqrt[n]{a}$.

(ii) If a is any real number and n is an odd integer, then a has only one (real) n-th root, which is denoted by $\sqrt[n]{a}$.

(iii) If a is a negative real number and n is an even integer, then a has no (real) n-th rot.

(iv) $\sqrt[n]{0} = 0$.

(v) Integers such as 1, 4, 9, 16, 25, and 49 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

(vi) If a is not a perfect n-th power, then $\sqrt[n]{a}$ is called a surd of the n-th order. Two surds of the same order are said to be like surds; otherwise unlike surds.

(vii) In an expression of the form $\dfrac{a}{\sqrt{b}+\sqrt{c}}$, the denominator can be rationalized by multiplying numerator and the denominator by $\sqrt{b}-\sqrt{c}$ which is called the conjugate of $\sqrt{b}+\sqrt{c}$.

(viii) If $x + \sqrt{y} = a + \sqrt{b}$ where x, y, a, b are rationals, then x = a and y = b.

EXAMPLE 3.19 Simplify (make the denominator rational) $\dfrac{12}{3+\sqrt{5}-2\sqrt{2}}$.

SOLUTION The expression

$$= \frac{12(3+\sqrt{5}+2\sqrt{2})}{(3+\sqrt{5})^2-(2\sqrt{2})^2}$$

$$= \frac{12(3+\sqrt{5}+2\sqrt{2})}{6+6\sqrt{5}}$$

$$= \frac{2(3+\sqrt{5}+2\sqrt{2})(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)}$$

$$= \frac{2(2+2\sqrt{5}+2\sqrt{10}-2\sqrt{2})}{4}$$

$$= 1+\sqrt{5}+\sqrt{10}-\sqrt{2}$$

EXAMPLE 3.20 Find the factor which will rationalise

$$\sqrt{3}+\sqrt[3]{5}$$

SOLUTION Let $x = 3^{1/2}$ and $y = 5^{1/3}$. The L.C.M. of the denominators of the indices 2 and 3 is 6.

Hence x^6 and y^6 are rational.

Now $x^6 + y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5)$

Hence the rationalizing factor required

$= x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$

where $x = 3^{1/2}$ and $y = 5^{1/3}$.

EXAMPLE 3.21 Find the square root of $7 + 2\sqrt{10}$

SOLUTION Let $\sqrt{7+2\sqrt{10}} = \sqrt{x}+\sqrt{y}$

Squaring, $x + y + 2\sqrt{xy} = 7+2\sqrt{10}$

Hence, x + y = 7 and xy = 10. These two relations give x = 5, y = 2

Thus, $\sqrt{7+2\sqrt{10}} = \sqrt{5}+\sqrt{2}$

EXAMPLE 3.22 If $a + b\sqrt{p} + c\sqrt{q} = 0$ where a, b, c are rationals and $\sqrt{p}$, $\sqrt{q}$ are unlike surds, then prove that $a = 0, b = 0, c = 0$.

SOLUTION By transposing and squaring we can show that $2ab\sqrt{p} = c^2q - a^2 - b^2p$.

If $ab \neq 0$, the left-hand side would be irrational and the right-hand side would be rational; which is impossible. Therefore, $ab = 0$, and consequently $a = 0$ or $b = 0$.

If $a = 0$, then $b\sqrt{p} + c\sqrt{q} = 0$; and if $b \neq 0$, then $\sqrt{p}/\sqrt{q} = -c/b$; so that $\sqrt{p}$ and $\sqrt{q}$ would be like surds; which is not the case. Therefore, $b = 0$ and $c = 0$.

If $b = 0$ then $a + c\sqrt{q} = 0$, and therefore $a = 0$ and $c = 0$.

EXAMPLE 3.23 If $a + b\sqrt[3]{p} + c\sqrt[3]{p^2} = 0$ where a, b, c, p are rationals and p is not a perfect cube, then prove that a, b, c are all zero.

SOLUTION Multiplying the given equation

by $\sqrt[3]{p}$ we have $cp + a\sqrt[3]{p} + b\sqrt[3]{p^2} = 0$,

and eliminating the terms containing

$$\sqrt[3]{p^2},\ (b^2 - ac)\sqrt[3]{p} = c^2p - ab.$$

since $\sqrt[3]{p}$ is irrational, it follows that $b^2 = ac$

and

$c^2p = ab$,

therefore $c^4p^2 = a^2b^2 = a^3c$.

if $c \neq 0$ we should have $p^2 = \left(\dfrac{a}{c}\right)^3$, so that

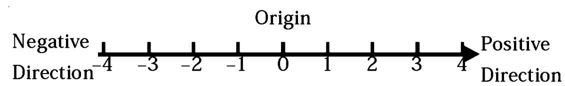
$\sqrt[3]{p^2}$ would be rational, which is not the case. Hence $c = 0$, and therefore also $a = 0$ and $b = 0$.

The Real Number Line

The real number line is a graph used to represent the set of real numbers.

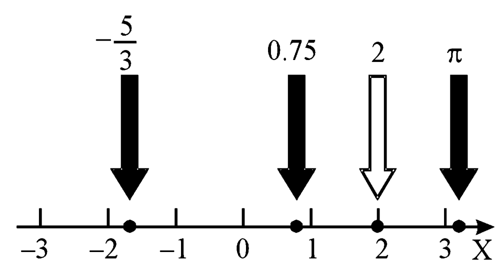
Real numbers can be represented as points of a line which is called as real number line.

An arbitrary point, called the origin, is labeled 0; units to the right of the origin are positive and units to the left of the origin are negative.



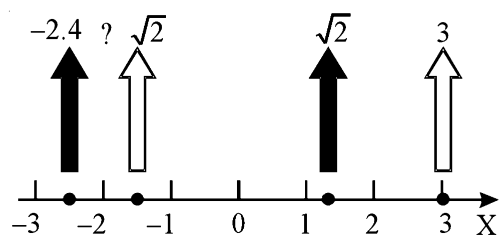
The real number line

Figure 3.12 illustrates that every real numbers corresponds to a point on the number line and every point on the number line corresponds to a real number. For this reason, we say that there is a one-to-one correspondence between all the real numbers and all points on a real number line. The real number corresponding to a particular point on the line is called the coordinate of the point. If you draw a point on the real number line corresponding to a real number, you are plotting the real number.



Every real numbers corresponds to exactly one point on the real numbers line.

Figure 3.12



Every point on the real number line corresponds to exactly one real numbers.

Figure 3.13

A one-to-one correspondence between real numbers and points on a number line

Ordering the Real Numbers

All real numbers follow the **law of trichotomy** i.e. if there are two real numbers a and b then either a = b or a < b or a > b.

The real number line is useful in demonstrating the order of real numbers. We say that the real number a is less than real number b, written a < b, if a is to the left of b on the number line. Equivalently, b is greater than a, written b > a, if b is to the right of a on the number line (Figure 3.14)

The symbols < and > are sometimes combined with an equal sign.

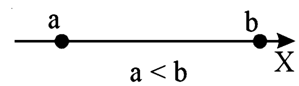


Figure 3.14

a is less than b : a is to the left of b

The symbols ≤ and ≥ can be understood with the following examples :

Symbols	*Meaning*	*Examples*
$a \le b$	a is less than or equal to b.	$3 \le 7$ (because $3 < 7$) $7 \le 7$ (because $7 = 7$)
$b \ge a$	b is greater than or equal to a.	$7 \ge 3$ (because $7 > 3$) $-5 \ge -5$ (because $-5 = -5$)

PRACTICE PROBLEMS [D]

1. Indicate which numbers in the given sets are (a) Natural numbers (b) Whole numbers (c) Integers (d) Rational numbers (e) Irrational numbers.

(i) $\left\{-10, -\sqrt{2}, -\frac{3}{4}, 0, \frac{4}{5}, \sqrt{4}, \pi, 7, \frac{18}{2}, 100\right\}$

(ii) $\left\{-\sqrt[3]{8}, \frac{0}{3}, \sqrt[3]{7}, \sqrt{\frac{4}{9}}, 1.\overline{126}\right\}$

2. Prove that if a natural number ends in the digit 7, it cannot be the square of an integer.

3. For what natural numbers n the fraction (3n + 4)/5 is an integer.

4. Determine for what natural numbers n the number $n^4 + 4$ is a composite number

5. If x and y are rational numbers such that $\sqrt{xy}$ is irrational then what is $\sqrt{x} + \sqrt{y}$?

6. Prove that the number [nm(n – m)] is even for any integers n and m.

7. Prove that the sum of four successive natural numbers cannot be a prime number.

8. Find the number equal to $3.1\overline{45}$ when expressed as a rational number in lowest terms.

9. Prove that $\log_2 7$ is an irrational number.

10. Prove that the expression $\dfrac{(2x^2+2xy)(x-1)}{x^2-x+xy-y}$ takes on even values for any integral x.

11. Express with a rational denominator $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

12. Simplify : $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} + \dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$

13. The positive integers p, q & r are all primes . If $p^2 - q^2 = r$ then the set of all possible values of p, q and r.

14. Find the integral pair(s) (x , y) whose sum is equal to their product .

15. A printer numbers the pages of a book starting with 1 and uses 3189 digits in all. How many pages does the book have ?

16. Prove that, if n > 4 , then the number 1! + 2! + 3! + ... n! is never a square.

1.5 | INTERVALS

Intervals is an alternative way to represent a subset of real numbers. Real numbers is represented by a number line having infinite membership. We can think any segment of this number line as subset or interval. Consider an interval, where "a" and "b" belongs to real numbers and a < b :

$$a < x < b$$

The value of "x" falls between "a" and "b". For example, an interval 2<x<4 is a collection of all points lying between end points 2 and 4. The important thing is that this interval does not include end points and is called "open" interval. We can represent this collection as a set in "set builder form"as :

$$\{x : x \in R \text{ and } 2 < x < 4\}$$

Alternatively, we can use pair of small brackets to represent open interval as :

$$(2, 4)$$

The two forms of representations are equivalent. The later form is obviously an easier and convenient representation of the subset of real number. We use small bracket "(" or ")" to denote interval that excludes

end point. Likewise, we use square bracket "[" or "]" to denote interval that includes end point. We can represent a "close" interval as [2,4]. This interval is equivalent to :

$$[2; 4] = 2 \le x \le 4$$

We can have combination of "open" and "close" brackets like :

$$(2, 4] = 2 < x \le 4$$

As a reminder, we should note that interval corresponding to real numbers or its subset is an infinite set as we can have infinite points on the line segment corresponding to an interval.

Graphical representation

The graphical representation uses a segment of line on the number line representing real numbers. The line segment is demarcated by a pair of two small circles – a filled circle to mean that end point is included in the interval and an unfilled circle to mean that end point is excluded from the interval.

Let us consider a, b ∈ R and a < b, then

$$(a, b) = a < x < b$$
$$[a, b) = a \le x < b$$
$$(a, b] = a < x \le b$$
$$[a, b] = a \le x \le b$$

Set of real numbers

The real numbers is represented graphically by a straight line. The question that we seek to be answer here is whether the set of integers is bounded by infinity. In other words, whether we can define interval of real numbers such as:

$$[-\infty, \infty]$$

The literal meaning of infinity is "unboundedness". Infinity is considered as a large number, which may either be positive or negative. It does not have a finite (fixed) value. Infinity, therefore, is not a part of real number system. It does not lie on the real number line. For this reason, we can not assign infinity to a real variable like (though we do generally):

$$x = \infty$$

It follows, then, that appropriate interval, representing real numbers, is open at both ends: R : $-\infty < x < \infty$ or $(-\infty, \infty)$

Interval of real numbers greater than or less than a given value

In the interval form, we can write the set of real numbers greater than a given value, "a", as:

$a < x < \infty$ or (a, ∞)

This is equivalent to :

$x > a$

The final notation "x > a" does not require to mention about infinity. It is an interval of real numbers greater than the given value 'a' appearing on the right. It is implied that it can be any large value. Similarly, the interval of real numbers less than a given value is :

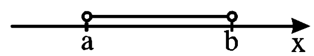
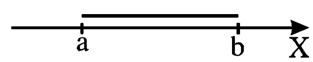
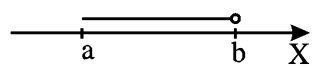
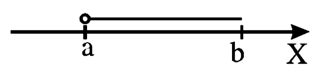
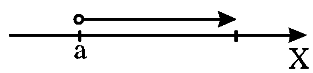
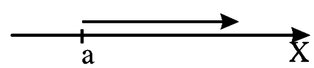
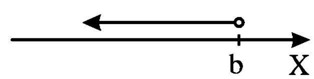
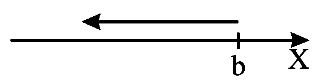
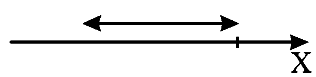
$-\infty < x < a = (-\infty, a)$

$x < a$

The following table lists the nine possible types of intervals.

Table: Intervals on the Real Number Line

Let a and b be real numbers such that a < b.

Interval Notation	*Set-Builder Notation*	*Graph*
(a, b)	$\{x \mid a < x < b\}$	
[a, b]	$\{x \mid a \le x \le b\}$	
[a, b)	$\{x \mid a \le x < b\}$	
(a, b]	$\{x \mid a < x \le b\}$	
(a, ∞)	$\{x \mid a > a\}$	
$[a, \infty)$	$\{x \mid a \ge a\}$	
$(-\infty, b)$	$\{x \mid a < b\}$	
$(-\infty, b)$	$\{x \mid a \le b\}$	
$(-\infty, \infty)$	$\{x \mid a \le b\}$	

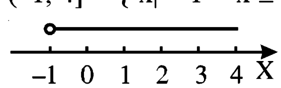
EXAMPLE 3.1 Express the intervals in terms of inequalities and graph :

(i) (–1, 4] (ii) [2.5, 4]

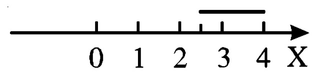
(iii) $(-4, \infty)$

SOLUTION

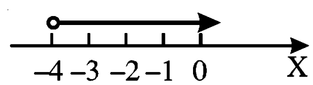
(i) $(-1, 4] = \{ x \mid -1 < x \le 4\}$



(ii) $[2, 3, 4] = \{x \mid 2.5 \le x \le 4\}$



(iii) $(-4, \infty) = \{x \mid x > -4\}$



Note

1. For some particular values of x, we use symbol { }.
   For example if x = 1, 2 we write it as $x \in \{1, 2\}$
2. If there is no value of x, then we write $x \in \phi$.

Union and Intersection of Intervals

Let $A = \{x \in I : -1 \le x \le 4\} = \{-1, 0, 1, 2, 3,\}$,

$B = \{x \in I : 1 \le \frac{x}{2} \le 7\} = \{2, 3, 4, 5, 6\}$

Then,

$A \cap B = \{x \in I : x \in A \text{ and } x \in B\}$

$= \{x \in I : -1 \le x \le 4 \text{ and } 1 \le \frac{x}{2} \le 7\} = \{2, 3\}$

Furthermore,

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

$= \{x \in I : -1 \le x \le 4 \text{ or } 1 \le \frac{x}{2} \le 7\}$

$= \{-1, 0, 1, 2, 3, 4, 5, 6\}$

EXAMPLE 3.2 Express each set in interval notation :

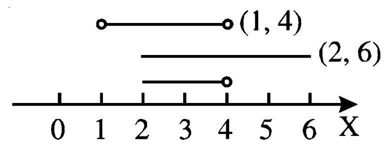
(i) $(1, 4) \cap [2, 6]$ (ii) $(-2, \infty) \cap (3, \infty)$

(iii) $(-2, -1) \cap (2, 3)$ (iv) $(1, 4) \cap [2, 6]$

(v) $(-2, \infty) \cup (3, \infty)$ (vi) $(-2, -1) \cap (2, 3)$

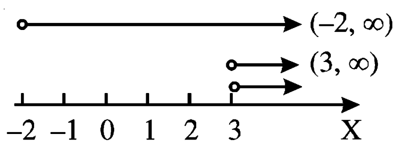
SOLUTION

(i) To find $(1, 4) \cap [2, 6]$, we graph each set above the number line and then locate the interval on the number line common to both. Figure illustrates that numbers common to both intervals begin at 2 and extend up to 4 but not including 4. Thus, $(1, 4) \cap [2, 6] = [2, 4)$.



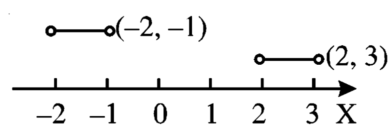
$(1, 4) \cap [2, 6] = [2, 4)$

(ii) Figure illustrates those numbers common to both intervals are all real numbers greater than 3. Thus, $(-2, \infty) \cap (3, \infty) = (3, \infty)$



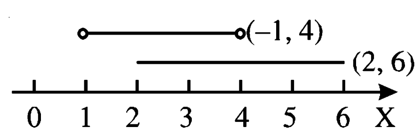
$(-2, \infty) \cap (3, \infty) = (3, \infty)$

(iii) Figure shows that there are no numbers common to the two intervals. Their intersection is a set with no elements, called the empty set or the null set, denoted by ϕ. Thus, $(-2, -1) \cap (2, 3) = \phi$)



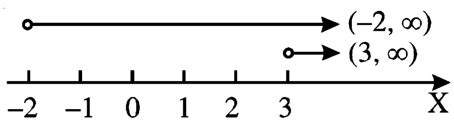
$(-2, -1) \cap (2, 3) = \phi$

(iv) To find $(1, 4) \cup [2,6]$, we graph each set above the number line and then locate the interval on the number line that unites or joins these intervals. As shown in figure. numbers in either or both intervals are all real numbers greater than 1, extending up to and including 6. Thus, $(1, 4) \cup [2, 6] = (1, 6]$



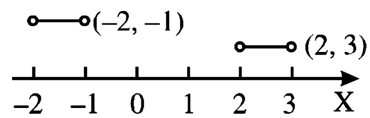
$(1, 4) \cup [2, 6] = (1, 6]$

(v) Figure illustrates that uniting both intervals shown above the number line results in all real numbers greater than –2. Thus, $(-2, \infty) \cup (3, \infty) = (-2, \infty)$.



$(-2, \infty) \cup (3, \infty) = (-2, \infty)$

(vi) Figure indicates that the union of the intervals consists of real number that are either in (–2, –1) or (2, 3). We cannot express the answer as a single interval. Using interval notation, the best we can do is to write $(-2, -1) \cup (2, 3)$



$(-2, -1) \cup (2, 3)$ is not a single interval

Using set-builder notation, we could express this union as $\{x \mid -2 < x < -1 \text{ or } 2 < x < 3\}$.

PRACTICE PROBLEMS [E]

1. Express each interval in terms of an inequality and graph the interval on a number line :
   (i) (1, 6] (ii) [–5, 2)
   (iii) [–3, 1] (iv) (2, ∞)
   (v) [–3, ∞) (vi) (–∞, 3)
2. Express each English phrase in both set-builder and interval notations :
   (i) x is less than 6.
   (ii) x is greater than or equal to –1.
   (iii) x lies between 5 and 12, excluding 5 and 12.
   (iv) x lies between 2 and 13, excluding 2 and including 13.
   (v) x is at most 6.
   (vi) x is at least 2 and at most 5.
3. Express each set in problems in interval notation.
   (i) $(1, 5) \cap [2, 7]$ (ii) $(-1, \infty) \cap (5, \infty)$
   (iii) $(-3, -1) \cap (-\infty, 5)$ (iv) $(-1, \infty) \cup (5, \infty)$
   (v) $(-\infty, 2) \cup (-\infty, 4)$ (vi) $(-3, -1) \cup [2, 4]$

3.6 | CARTESIAN PRODUCT

We have seen that set operations convey the notion of arithmetic operations. One such similar operation is product of two sets called "Cartesian product". Since sets are collections, not a single quantity, the product operation here involves combining or pairing each of the elements of one set with that of another set.

We use symbol "X" to denote product operation. The Cartesian product of two sets "A" and "B" is symbolically represented as :

$$A \times B$$

It is important to understand that we do not multiply elements as we do in arithmetic instead we pair elements together. This is the meaning of "product" for the sets. We denote one such pair within a pair of small brackets like :

$$(a, b)$$

where $a \in A$ and $b \in B$.

Note that elements from two sets are separated by comma.

Ordered pair

The order of pairing is important. The pair (a,b) and (b,a) are different. This ordering is required as there are real time situations, where order makes a difference. For example, we are required to find the integers which can be formed from two integer subsets like {1,2,3} and {3,4,5}. Clearly, "13" and "31" represent different integers. We need to distinguish them. All pairs formed from two sets should be distinct.

Keeping this restriction in mind, let us work out an example to find ordered pairs formed from elements of two sets.

A = set of first letter of the names of cities = {N,D,H}

B = set of numbers denoting flight numbers = {001, 002, 003}

All possible ordered pairs formed from two sets are :

(N, 001) , (N, 002) , (N, 003) , (D, 001) , (D, 002) , (D, 003) , (H, 001) , (H, 002) , (H, 003)

There are all together 9 ordered pairs. From this example, we can deduce a method for writing ordered pairs from two sets. We begin with the first elements of two sets. Progressively, we change the elements from the second set till it is exhausted, while keeping the elements from the first set unchanged. Then, we switch to next element from first set and start with first from the second set. Again, we change the elements from the second set progressively till it is exhausted, while keeping the elements from the first set unchanged. We continue in this manner till all elements from the first set is also exhausted.

From this discussion, it is also evident that two ordered pairs are equal if and only if the corresponding first and second elements are equal.

Let (a, b) and (c, d) be ordered pairs. Then (a, b) = (c, d) if and onlyif a = c and b = d.

EXAMPLE 3.1 If $(x^2 - 1, y + 2) = (0, 2)$, find x and y.

SOLUTION Two ordered pairs are equal. It means that corresponding elements of the ordered pairs are equal. Hence,

$x^2 - 1 = 0 \quad \Rightarrow x = 1 \text{ or } -1$ and

$y + 2 = 2 \quad \Rightarrow y = 0$

Graphical representation

The ordered pairs can be represented in the form of points of intersection of perpendicular lines. The elements of one set are represented as rows, whereas elements of other set are represented as columns. Look at the representation of ordered pairs by points in the figure 3.15 for the example given earlier.

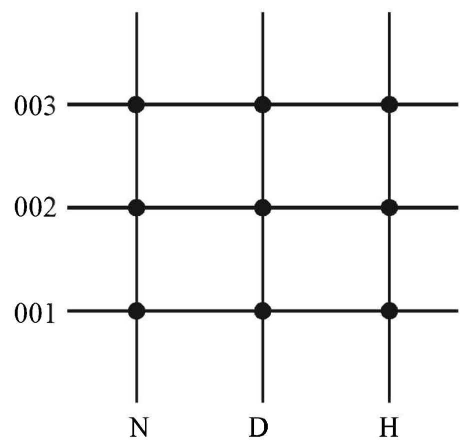


Figure 3.15

In the figure the elements of one set are represented as rows, whereas elements of other set are represented as columns. Note that there are a total of 9 intersection points, corresponding to 9 ordered pairs.

Definition The Cartesian (cross) product A × B of two non-empty sets "A" and "B" is the set of all ordered pairs of the elements from two sets. So, the operation × pairs the elements of A with the elements of B in such a way that the elements of A appear as first components, and the elements of B appear as second components.

We should emphasize the use of word "non-empty", The Cartesian product of a non-empty

set with an empty set is equal to empty set.

$$A \times \phi = \phi$$

On the other hand, if one of the sets is infinite, then resulting Cartesian product is also infinite. In general, if a finite set A has m elements and a finite set B has n elements, then A × B has m times n elements. If either A or B is empty, then A × B is empty. Lastly, if either A or B is infinite, and the other is not empty, then A × B is also infinite.

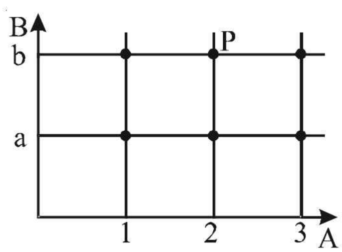
We express the Cartesian product set in set building form as :

$$A \times B = \{(x, y) : x \in A, y \in B\}$$

Consider some examples :

(i) Let A = {1, 2, 3}, and B = {a, b} then A × B = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

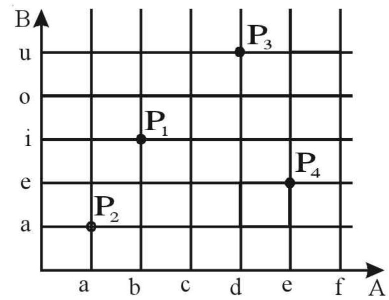
Since A and B do not contain many elements, it is possible to represent A × B by a coordinate diagram as shown in the figure 3.16. Here the vertical lines through the points of A and the horizontal lines through the points of B meet in 6 points which represents A × B clearly. The point P is the ordered pair (2, b).



(ii) Let A = R and B = R; then A × B = R × R is the set of all points in the Cartesian coordinate plane.

(iii) Let A = B = $\{x \in R : -1 \le x \le 1\}$; then A × B is the set of all points in the plane which lie inside a square of side length 2 with centre at the origin.

EXAMPLE 3.2 Let A = {a, b, c, d, e, f} and B = {a, e, i, o, u}. Determine the ordered pairs corresponding to the points P_1, P_2, P_3 and P_4 which appear in the coordinate diagram of A × B on the right.



SOLUTION The vertical line through P_1 crosses the A axis at b and the horizontal line through P_1 crosses the B axis at i ; hence P_1 corresponds to the ordered pair (b, i). Similarly, P_2 = (a, a), P_3 = (d, u) and P_4 = (e, e).

EXAMPLE 3.3 If A = {5, 6, 7, 2}, B = {3, 5, 6, 1} and C = {4, 1, 8}, then find $(A \cap B) \times (B \cap C)$.

SOLUTION In order to evaluate the given expression, we first find out the intersections given in the brackets.

$A \cap B = \{5, 6\}$ and $B \cap C = \{1\}$

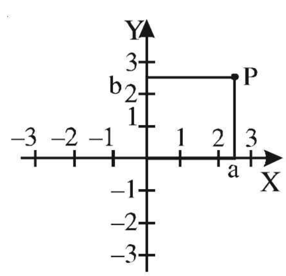
Thus,

$(A \cap B) \times (B \cap C) = \{(6, 1), (5, 1)\}$

Note that the elements in the given set are not ordered. It is purposely given this way to emphasize that order is requirement of ordered pair – not that of a set.

The product of a set with itself, say A × A, is sometimes denoted by A^2.

We are familiar with the Cartesian plane $R^2 = R \times R$ (see figure). Here each point P represents an ordered pair (a, b) of real numbers and vice versa ; the vertical line through P meets the x-axis at a, and the horizontal line through P meets the y-axis at b.



Numbers of elements

We have seen that ordered pairs are represented graphically by the points of intersection. The numbers of intersections equal to the product of numbers of rows and columns. Thus, if there are "p" elements in the set "A" and "q" elements in the set "B", then the total number of ordered pairs is "pq". In symbolic notation,

$$n(A \times B) = pq$$

Multiple products

Like other set operations, the product operation can also be applied to a series of sets in sequence. If $A_1, A_2, \ldots\ldots\ldots, A_n$ is a finite family of sets, then their Cartesian product, one after another, is symbolically represented as :

$$A_1 \times A_2 \times \ldots\ldots\ldots \times A_n$$

This product is set of group of ordered elements. Each group of ordered elements comprises of "n" elements. This is stated as :

$$A_1 \times A_2 \times \ldots\ldots \times A_n$$
$$= \{(x_1, x_2, \ldots\ldots, x_n) : x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n\}$$

Ordered triplets

The Cartesian product A × A × A is set of triplets. This product is defined as :

$$A \times A \times A = \{(x, y, z) : x, y, z \in A\}$$

We can also represent Cartesian product of a given set with itself in terms of Cartesian power. In general,

$$A^n = A \times A \times \ldots\ldots \times A$$

where "n" is the Cartesian power. If n = 2, then

$$A^2 = A \times A$$

This Cartesian product is also called Cartesian square.

EXAMPLE 3.4 If $A = \{-1, 1\}$, then find Cartesian cube of set A.

SOLUTION Following the method of writing ordered sequence of numbers, the product can be written as :

$$A \times A \times A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1),$$
$$(-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$$

The total numbers of elements are $2 \times 2 \times 2 = 8$.

Cartesian Coordinate system

The Cartesian product, consisting of ordered triplets of real numbers, represents Cartesian three dimensional space.

$$R \times R \times R = \{(x, y, z) : x, y, z \in R\}$$

Each of the elements in the ordered triplet is a coordinate along an axis and each ordered triplet denotes a point in the three dimensional coordinate space.

Commutative property of Cartesian product

The Cartesian product is a set of ordered pairs. Now, the order of elements in the ordered pair depends on the position of sets across product sign. If sets "A" and "B" are unequal and non-empty sets, then :

$$A \times B \neq B \times A$$

In general, any operation involving Cartesian product that changes the "order" in the "ordered pair" will yield different result.

However, if "A" and "B" are non-empty, but equal sets, then the significance of the order in the "ordered pair" is lost. We can use this fact to formulate a law to verify "equality of sets". Hence, if sets "A" and "B" are two non-empty sets and

$$A \times B = B \times A$$

Then, A = B

It can also be verified that this condition is true other way also. If sets "A" and "B" are

equal sets, then $A \times B = B \times A$. The two way conditional statements can be symbolically represented with the help of two ways arrow,

$$A \times B = B \times A \Leftrightarrow A = B$$

Distributive property of product operator

The distributive property of product operator holds for other set operators like union, intersection and difference operators. We write equations involving distribution of product operator for each of other operators as :

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$
$$A \times (B - C) = (A \times B) - (A \times C)$$

Here, sets "A", "B" and "C" are non-empty sets. In order to ascertain distributive property product operator over other set operators we need to check validity of the equations given above.

We can check these relations proceeding from the defining statements. For the time being, we reason that sequence of operation on either side of the equation does not affect the "order" in the "ordered pair". Hence, distributive property should hold for product operator over three named operators. Let us check this with an example:

$A = \{a, b\}$, $B = \{1, 2\}$ and $C = \{2, 3\}$

1. For distribution over union operator

LHS $= A \times (B \cup C) = \{a, b\} \times \{1, 2, 3\}$
$= \{a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Similarly,

RHS $= (A \times B) \cup (A \times C) = \{(a, 1), (a, 2), (b, 1), (b, 2)\} \cup \{(a, 2), (a, 3), (b, 2), (b, 3)\}$
$= \{(a, 1),(a, 2),(a, 3), (b, 1), (b, 2), (b, 3)\}$

Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$

2. For distribution over intersection operator

LHS $= A \times (B \cup C) = \{a, b\} \times \{2\}$
$= \{(a, 2), (b, 2)\}$

Similarly,

RHS $= (A \times B) \cap (A \times C)$
$= \{(a, 1), (a, 2), (b, 1), (b, 2)\} \cap \{(a, 2), (a, 3), (b, 2), (b, 3)\}$
$= \{(a, 2), (b, 2)\}$

Hence, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

3. For distribution over difference operator

LHS $= A \times (B - C) = \{a, b\} \times \{1\}$
$= \{(a, 1), (b, 1)\}$

Similarly,

$\Rightarrow$ RHS $= (A \times B) - (A \times C) = \{(a, 1), (a, 2), (b, 1), (b, 2)\} - \{(a, 2), (a, 3), (b, 2), (b, 3)\}$
$= \{(a, 1), (b, 1)\}$

Hence, $A \times (B - C) = (A \times B) - (A \times C)$

Analytical proof

Let us consider an arbitrary ordered pair (x, y), which belongs to Cartesian product set "$A \times (B \cup C)$". Then, $(x, y) \in A \times (B \cup C)$

By the definition of product of two sets,

$x \in A$ and $y \in (B \cup C)$

By the definition of union of two sets,

$x \in A$ and ($y \in B$ or $y \in C$)

$\Rightarrow$ ($x \in A$ and $y \in B$) or ($x \in A$ and $y \in C$)

$\Rightarrow$ $(x, y) \in A \times B$ or $(x, y) \in A \times C$

By the definition of union of two sets,

$(x, y) \in (A \times B) \cup (A \times C)$

But, we had started with "$A \times (B \cup C)$" and used definitions to show that ordered pair "(x, y)" belongs to another set. It means that the other set consists of the elements of the first set – at the least. Thus,

$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Similarly, we can start with "$(A \times B) \cup (A \times C)$" and reach the conclusion that :

$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

It sets are subsets of each other, then they are qual. Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proceeding in the same manner, we can also prove distribution of product operator over intersection and difference operators,

$A \times (B \cap C) = (A \times B) \cap (A \times C)$
$A \times (B - C) = (A \times B) - (A \times C)$

EXAMPLE 3.5 A Cartesian product "A × B" consists of 6 elements. If three of these are (1, 2), (2, 3) and (3, 3), then find Cartesian product set "B × A".

SOLUTION We need to know the two sets "A" and "B" in order to evaluate "B × A". First elements of ordered pairs of " A × B " are elements of set "A". Hence, "1", "2" and "3" are the elements of set "A". On the other hand, second elements of ordered pairs of A × B are elements of set "B". Hence, "2" and "3" are elements of set "B".

Now, it is given that there are total 6 elements in the Cartesian product, which is equal to the product of numbers of elements in two sets i.e. 3 × 2. It means that we have identified all elements of sets "A" and "B".

$A = \{1, 2, 3\}$
$B = \{2, 3\}$

Following the rule for writing Cartesian product in terms of ordered pairs, we have :

$B \times A = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$.

EXAMPLE 3.6 Two sets are given as : A = {1, 2} and B = {3, 4}. Find the total numbers of subsets of "A × B". Also write power set of A × B in roaster form.

SOLUTION The total numbers of elements in the Cartesian product "A × B" is "pq", where "p" and "q" are the numbers of elements in the individual sets "A" and "B" respectively. Then the number of all possible subsets that can be formed including empty set and the product "A × B} itself is :

$$n = 2^{pq} = 2^{2\times 2} = 2^4 = 16$$

Now, the Cartesian product is :

$$A \times B = \{(1, 3), (1, 4), (2, 3)\ (2, 4)\}$$

The corresponding power set comprises of empty set, 4 sets with elements comprising of one element plus 6 sets with elements comprising of two elements taken at a time plus 4 sets with elements comprising of three elements taken at a time plus set itself. There are total of 16 subsets. The power set is set of all subsets as its elements :

$P(A \times B) = \{\phi, \{(1, 3)\}, \{(1, 4)\}, \{(2, 3)\}, \{(2, 4)\}$
$\{(1, 3), (1, 4)\}, \{(1, 3), (2, 3)\}, \{(1, 3), (2, 4)\},$
$\{(1, 4), (2, 3)\}, \{(1, 4), (2, 4)\}, \{(2, 3), (2, 4)\},$
$\{(1, 3), (1, 4), (2, 3)\}, \{(1, 3), (1, 4), (2, 4)\},$
$\{(1, 3), (2, 3), (2, 4)\}, \{(1, 4), (2, 3), (2, 4)\},$
$\{(1, 3), (1, 4), (2, 3), (2, 4)\}\}$

It is easy to follow a scheme to write combination in which order is not relevant. We can denote each of the ordered pair with a symbol like :

$$A \times B = \{a, b, c, d\}$$

As pointed out for generating combination for ordered pair, we can start with the left element and keep changing the last element of the combination till all combinations are exhausted. Here, power set in terms of symbols is:

$P(A \times B\} = \{\phi, \{a\}, \{b\}, \{c\}, \{d\},$
$\{(a, b), \{(a, c)\}, \{(a, d)\}, \{(b, c), \{(b, d)\},$
$\{(c, d)\},$
$\{(a, b, c)\}, \{(a, b, d\}), \{(a, c, d)\}, \{(b, c, d)\},$
$\{(a, b, c, d)\}\}$

EXAMPLE 3.7 Let S = {a, b}, W = {1, 2, 3, 4, 5, 6} and V = {3, 5, 7, 9}. Find (S × W) ∩ (S × V).

SOLUTION The product set (S × W) ∩ (S × V) can be found by first computing S × W and S × V, and then computing the intersection of these sets.

On the other hand, by the preceding problem, $(S \times W) \cap (S \times V) = S \times (W \cap V)$.

Now $W \cap V = \{3, 5\}$, and so

$$(S \times W) \cap (S \times V) = S \times (W \cap V)$$
$$= \{(a, 3), (a, 5), (b, 3), (b, 5)\}$$

EXAMPLE 3.8 If "A $\subseteq$ B" and "C" is any non-empty set, then prove that :

$$A \times C \subseteq B \times C$$

SOLUTION Let us first discuss the logic of the relation here. The elements of set "A" are common to set "B". Now Cartesian product of set "A" with set "C" will yield ordered pairs, which are common with the ordered pairs of the Cartesian product "B" with "C". However, as set "B" is either larger than or equal to, but not smaller than "A", it follows that above relation should hold.

Now, we prove the relation analytically. Let an arbitrary ordered pair (x,y) belongs to Cartesian product "A × C".

$$(x, y) \in A \times C$$

According to definition of Cartesian product,

$$x \in A \text{ and } y \in C$$

But "A" is subset of "B". Hence, $x \in B$.

$$\Rightarrow \quad x \in B \text{ and } y \in C$$

Again, applying definition of Cartesian product,

$$(x, y) \in B \times C$$

This means that :

$$\Rightarrow \quad A \times C \subset B \times C$$

EXAMPLE 3.9 If $A \subset B$ and $C \subset D$, then prove that

$$A \times B \subset B \times D$$

SOLUTION Let an arbitrary ordered pair (x, y) belongs to Cartesian product "A × C".

$$(x, y) \in A \times C$$

According to definition of Cartesian product,

$$x \in A \text{ and } y \in C$$

But "A" is subset of "B". Hence, $x \in B$. Also, "C" is subset of "D". Hence, $y \in D$.

$$\Rightarrow \quad x \in B \text{ and } y \in D$$

Again, applying definition of Cartesian product,

$$(x, y) \in C \times D$$

This means that :

$$A \times C \subset B \times D$$

EXAMPLE 3.10 For any given four sets "A", "B", "C" and "D", prove that :

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

SOLUTION Let an arbitrary ordered pair (x, y) belongs to intersection set "(A × B) ∩ (C × D)". Then,

$$(x, y) \in (A \times B) \cap (C \times D)$$

Applying definition of intersection,

$$(x, y) \in A \times B \text{ and } (x, y) \in C \times D$$

Applying definition of Cartesian product,

$$(x \in A \text{ and } y \in B) \text{ and } (x \in C \text{ and } y \in D)$$

$$\Rightarrow \quad (x \in A \text{ and } x \in C) \text{ and } (y \in B \text{ and } y \in D)$$

Applying definition of intersection,

$$(x \in A \cap C) \text{ and } (y \in B \cap D)$$

Again, applying definition of Cartesian product,

$$(x, y) \in (A \cap C) \times (B \cap D)$$

This means that :

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$$

Similarly, starting from RHS, we can prove that:

$$\Rightarrow \quad (A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$$

If sets are subsets of each other, then they are equal. Hence,

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

EXAMPLE 3.11 Let $A = \{a, b\}$, $B = \{2, 3\}$ and $C = \{3, 4\}$. Find

(i) $A \times (B \cup C)$

(ii) $(A \times B) \cup (A \times C)$

(iii) $A \times (B \cap C)$

(iv) $(A \times B) \cap (A \times C)$.

SOLUTION

(i) First we compute $B \cup C = \{2, 3, 4\}$. Then
$A \times (B \cup C) = \{(a, 2), (a, 3), (a, 4), (b, 2), (b, 3), (b, 4)\}$

(ii) First we find $A \times B$ and $A \times C$;
$A \times B = \{(a, 2), (a, 3), (b, 2), (b, 3)\}$
$A \times C = \{(a, 3), (a, 4), (b, 3), (b, 4)\}$

Then we compute the union of the two sets:
$(A \times B) \cup (A \times C) = \{(a, 2), (a, 3), (b, 2), (b, 3), (a, 4), (b, 4)\}$
We observe, from (i) and (ii), that
$A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iii) First we compute $B \cap C = \{3\}$. Then
$A \times (B \cap C) = \{(a, 3), (b, 3)\}$

(iv) Now $A \times B$ and $A \times C$ were computed above. The intersection of $A \times B$ and $A \times C$ consists of those ordered pairs which belong to both sets :
$(A \times B) \cap (A \times C) = \{(a, 3), (b, 3)\}$
Observe from (iii) and (iv) that
$A \times (B \cap C) = (A \times B) \cap (A \times C)$

EXAMPLE 3.12 Let "A" and "B" be two non-empty sets. If the numbers of common elements be "n" between sets "A" and "B", then find the common elements between Cartesian products "A × B" and "B × A".

SOLUTION The common elements between sets "A" and "B" is "n". This means :
$n(A \cap B) = n$
We are required to evaluate the expression,
$n [(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
If we put C = B and D = A in this equation, then expression on the left hand side of the equation becomes what is required.
$(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$
$\Rightarrow n[(A \times B) \cap (B \times A)] = n[(A \cap B)] \times n[(B \cap A)]$
$\Rightarrow n[(A \times B) \cap (B \times A)] = n \times n = n^2$

EXAMPLE 3.13 Let "A", "B" and "C" be three sets. Then prove that :
$A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

SOLUTION From De-Morgan's law, we know that :
$(B \cap C)' = B' \cup C'$
Now, component of complement set is set itself. Applying to the LHS, we have :
$A \times (B' \cup C')' = A \times (B \cap C)$
Hence, $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

Applying distributive property of product operator over intersection operator, we have :
$A \times (B' \cap' C')' = (A \times B) \cap (A \times C)$

3.7 | RELATIONS

We encounter different types of relationship in our daily life. Besides human relationship that we are so familiar with, there are numerous other relationships.

In essence, any two elements which are paired have potential to possess relationship between them. Now think of the Cartesian product that we have defined for two sets. It consists of ordered pairs of elements – one each from the two sets. The total numbers of ordered pairs in a Cartesian product is equal to the product of numbers of elements in each set. A particular relation, however, may not comprise all ordered pairs.

Here, we shall limit our discussion to binary relations only. A binary relation is a relation as defined between two elements either from the same set or from two different sets. Consider a "get together". Divide the people in two groups comprising of males and females. A certain numbers of ordered pairs will qualify for a relation say "classmate of", not all. Similarly, a relation such as "brother of" may include some of the ordered pairs of the Cartesian product of two sets.

$M = \{A, B, C, D, E\}$
$F = \{G, H, I, J, K\}$

We can represent the relationship of "classmate of ' as shown in figure 3.7:

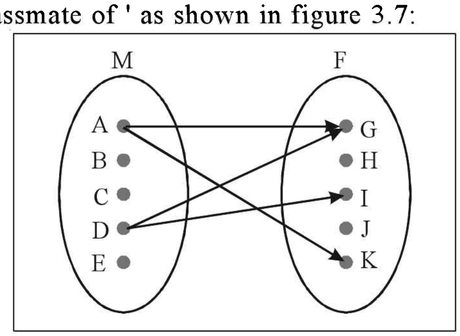


Figure 3.7

From the figure, we can write the collection of relationship "classmate of" as a set of ordered pairs of two sets,

R (classmate of) = {(A, G) , (A, K) , (D, G) , (D, I)}

In a nutshell, we can think of relation as a collection (set), which comprises of ordered pairs (instances of relation). Note that it is a specific relation. This is a relation between elements of two sets. Clearly, this relation set cannot exceed the Cartesian product of two sets under consideration. Thus, a relation set is a subset of Cartesian product set.

Definition A relation "R" from a non-empty set "A" to non-empty set "B" is a subset of the Cartesian product "A × B".

We need to note that a relation is defined in a particular order "from" to "to". It is for this reason, we denoted relation pictorially by an arrow which is directed from the elements of "from" set "A" to "to" set "B".

We shall be using the symbol, "xRy" to denote an instance of relation (ordered pair). The symbol conveys that the instance of relation denoted by the symbol is an ordered pair (x,y), which follows relation "R".

A relation R from a set A to a set B assigns to each pair (a, b) in A × B exactly one of the following statements :

(i) "a is related to b", written a R b.

(ii) "a is not related to b", written $a \not R b$.

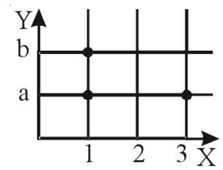
For example, perpendicularity is a relation in the set of lines in the plane. For, given any pair of lines a and b, either a is perpendicular to b or a is not perpendicular to b.

Let R be the following relation from

A = {1, 2, 3} to B = {a, b} :

R = {(1, a), (1, b), (3, a)}

Then 1 R a, $2 \not R$ b, 3 R a, and $3 \not R$ b. The relation R is shown on the coordinate diagram of A × B below.



Consider some more examples :

(i) Let A = {2, 4, 6, 8}. and define the relation R on A by (x, y) ∈ R iff x divides y. Then, R = {(2, 2), (2, 4), (2, 6), (2, 8), (4, 4,), (4, 8), (6, 6), (8, 8)}.
Observe that each number is divisor of itself.

(ii) Let A = N, and define R ⊆ A × A by xRy iff x and y have the same remainder when divided by 3.
Since A is infinite, we cannot explicitly list all elements of R; but, for example (1, 4), (1, 7), (1, 10),….., (2, 5), (2, 8), …., (0, 0), (1, 1),…. ∈ R.

(iii) Let A = R, and define the relation R on A by x Ry iff $y = x^2$. Then R consists of all points on the parabola $y = x^2$.

(iv) Let A = {1, 2, 3}, and define R on A by xRy iff x + y = 7. Since the sum of two elements of A is atmost 6, we see that xRy for no two elements of A; hence, R = ϕ.

(v) $R = \{(x, y) : x = y^2, x \in Q, y \in I\}$

(vi) $R = \{(x, y) : x = y + 1, x \in N, y \in W\}$

For small sets we can use a pictorial representation of a relation R on A. Sketch two copies of A and, if xRy then draw an arrow from the x in the left sketch to the y in the right sketch.

Let A = {a, b, c, d, e}, and consider the relation

R = {(a, a), (a, c), (c, a), (d, b), (d, c)}

Relation on A

A relation "R" from set "A" to "A" is called a "relation on A".

In certain circumstances, we are required to work with relation among the elements of the same set. For example, consider the male set

defined earlier. Some of the male members may as well be classmates and hence related to each other. Such relation is relation on one set only and is called "relation on A" or "relation on B" etc. For example:

$R = \{(x, y) : y = x + 1 \text{ and } x, y \in I\}$

An arrow representation of R is given in figure 3.18:

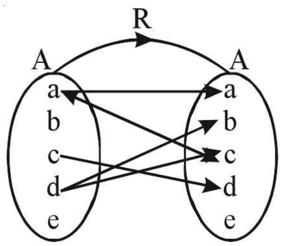


Figure 3.18

We observe that e does not appear at all in the elements of R, and that, for example, b is not the first component of any pair in R.

EXAMPLE 3.1 Let A = {1, 2, ..., 10}. Write down the relation set in roaster form, which is defined as :

$R = \{(x, y) : y = 3x \text{ and } x, y \in A\}$

SOLUTION We begin with the first element of "A" i.e. x = 1. Since other element also belongs to set "A", it is required that the value of "y" be one of the elements in the set "A".

For $x = 1, y = 3 \times 1 = 3$

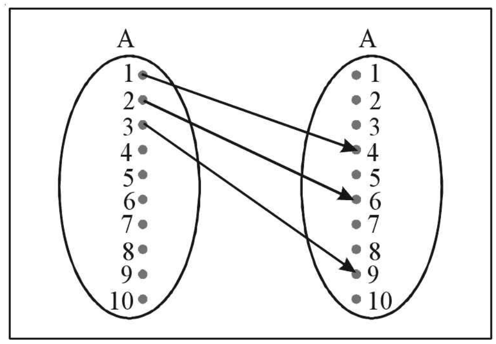
For $x = 2.\ y = 3x = 3 \times 2 = 6$

For $x = 3, y = 3x = 3 \times 3 = 9$

Thus "x" can assume values "1", "2" and "3" for which "y" can assume values "3", "6" and "9" respectively in accordance with the given relation. The relation, therefore, is a set of ordered pairs :

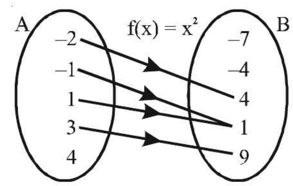
$R = \{(1, 3), (2, 6), (3, 9)\}$

We can visualize the relation pictorially as shown in the figure.



EXAMPLE 3.2 Let A = {–2, –1, 1, 3, 4} B = {–7, –4, 1, 4, 9} and $f(x) = x^2$. Draw the mapping of the relation f from A to B. Find the image of –1 and the pre-images of –4 and 9.

SOLUTION The relation $f : A \to B$ is shown below.



The image of –1 is equal to 1.
There is no pre-image of – 4.
The pre-image of 9 is equal to 3.

Domain of the relation

The domain represents the valid values of the first element of the ordered pairs in the relation. Clearly, the elements of domain of a relation belong to "from" set "A". But, elements in the domain are only those which are valid for the relation. It means that domain does not consist of all elements of "from" set "A". Thus, domain set is a subset of "from" set "A".

Definition The set of first elements of all ordered pairs in the relation "R" from set "A" to "B" is called the domain of relation "R".

We can write the domain set of relation "R" from set "A" to set "B" in set builder form as :

Domain (R) = $\{x : (x, y) \in R\}$

Consider the example given earlier. The relation set is R = $\{(1, 3), (2, 6), (3, 9)\}$

The domain according to definition is :

$$\text{Domain } (R) = \{1, 2, 3\}$$

$$\Rightarrow \quad \text{Domain } (R) \subseteq A$$

Co-domain

In a relation "R" from set "A" to "B", the set "B" is called co-domain.

Range of the relation

The range represents the valid values of the second element of the ordered pairs in the relation.

Definition The set of second elements of all ordered pairs in the relation "R" from set "A" to "B" is called the range of relation "R".

We can write the range set of relation "R" from set "A" to set "B" in set builder form as :

$$\text{Range } (R) = \{y : (x, y) \in R\}$$

Consider the relation given earlier :

$$R = \{(1, 3), (2, 6), (3, 9)\}$$

The range according to definition is :

$$\text{Range } (R) = \{3, 6, 9\}$$

Clearly, the elements of range of a relation belong to "to" set "B". But, elements in the range are only those which are valid for the relation. It means that range does not consist of all elements of "to" set "B". Thus, range set is a subset of "to" set "B".

$$\text{Range } (R) \subseteq B$$

EXAMPLE 3.3 Let A = $\{1, 2, 3, 4, 5\}$ and B = $\{1, 4, 5\}$. Let a relation from "A" to "B" is :

$$R = \{(x, y) : x < y, (x, y) \in A \times B\}$$

Find R, Domain (R) and Range (R).

SOLUTION Let us find "y" for each value "x".

For $x = 1$, $y = 4, 5$
For $x = 2$, $y = 4, 5$
For $x = 3$, $y = 4, 5$
For $x = 4$, $y = 5$

For $x = 5$, There is no value of "y" in set "B"

Hence,

R = $\{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

$$\text{Domain } (R) = \{1, 2, 3, 4\}$$

$$\text{Range } (R) = \{4, 5\}$$

Void relation

Relation is a subset of Cartesian product of two sets. We have seen that power set of Cartesian product "A × B" is a set of all possible relations among the elements of sets "A" and "B". In the case of "relation on A", the power set of Cartesian product "A × A" is a set of all possible relations among the elements of set "A".

One of the subsets of the power set is the empty set or void set. This subset without any element is called the void relation.

$$R = \phi = \{\}$$

Universal relation

Universal relation is the widest possible relation. This relation consists of all ordered pairs of the Cartesian product "A × A".

$$R = A \times A$$

Consider a set A = $\{1, 2, 3\}$. Then, universal relation set is :

$$R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

Note Both the empty relation and the universal relation are sometimes called trivial relations.

EXAMPLE 3.4 Let A be the set of all students of a boys school. Show that the relation R in A given by R = $\{(a, b)$: a is sister of b$\}$ is the empty relation and R′ = $\{(a, b)$: the difference between

heights of a and b is less than 3 metres} is the universal relation.

SOLUTION Since the school is boys school, no student of the school can be sister of any student of the school. Hence, R = ϕ, showing that R is the empty relation. It is also obvious that the difference between heights of any two students of the school has to be less than 3 metres. This shows that $R' = A \times A$ is the universal relation.

Numbers of relations

Between two sets, the Cartesian product set consists of all possible instances of relation as ordered pair. Here, we need to find the total possible relations that can be generated from these ordered pairs. We have seen that total numbers of ordered pairs in the Cartesian product of sets "A" and "B" is "pq", where "p" and "q" are the numbers of elements in two sets respectively.

Now, relation is nothing but a subset of the Cartesian product. It means that total numbers of relation is equal to total numbers of possible subsets of the Cartesian product. Recall that the set formed from all possible subsets is called power set. The numbers of subsets in the power set is given by

$$n = 2^{pq}$$

Clearly, this number "n" denotes all possible relations (subsets) that can be generated from two finite sets. We should, however, be careful in interpreting this number as it also contains the mandatory empty set, which is not a meaningful set from the point of view of relation.

Let A {1, 2, 3) and B = (4, 5) then the number of different relations from A to B is $2^{3 \times 2} = 2^6 = 64$

because A has 3 elements and B has 2 elements.

Identity relation

In an identity relation "R", every element of the set "A" is related to itself only.

Note that the conditions conveyed through words "every" and "only". The word "every" conveys that identity relation consists of ordered pairs of element with itself – all of them. The word "only" conveys that this relation does not consist of any other combination.

Consider a set A = {1, 2, 3}. Then, its identity relation is :

R = {(1, 1) , (2, 2) , (3, 3)}

It is evident that a set has only one such relation. This relation, as we can see, identifies the set - as it identifies each elements of the set, which are related to itself. By looking at the relation, we can identify the set itself. For this reason, the name of this relation is identity relation. In set builder form, we express an identity relation as

$$R = \{(x, x) : \text{for all } x \in A\}$$

The qualification of the relation is that first and second element of the ordered pair is the same element, which belongs to set A.

The followings are not an identity relation:

$R_1 = \{(1, 1), (2, 2)\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3)\}$

First one is not an identity relation as it does not include the pairing of remaining element "3". Second is not an identity relation, because there are other combinations of pairs in the relation.

Composition of Relations

Let R and S be relations on A; then $R \circ S = \{(x, z)$: there is a $y \in A$ such that xRy and ySz}. The operation $\circ$ is called the composition of R and S.

Let A = N and R defined by xRy iff x + 1 = y, S defined by ySz iff z = 2y.

Then $(x, z) \in R \circ S$ iff z = 2(x + 1):

$(x, z) \in R \circ S$

$\Leftrightarrow$ There is some $y \in A$ with xRySz

$\Leftrightarrow$ y = x + 1 and z = 2y,

$\Leftrightarrow$ z = 2(x + 1).

Inverse relation

Inverse relation of a given relation "R" from set "A" to set "B" is set of ordered pairs in which first and second elements exchanges their positions. The inverse set is defined in reference to a given relation. The inverse relation of a given relation "R" from "A" to "B" is denoted as "R^{-1}". Clearly,

$R^{-1} = \{(y, x) : (x, y) \in R\}$

where,

$R = \{(x, y) : (x, y) \in A \times B\}$

We should be careful to understand that "–1" is not a power, but a part of symbol to represent inverse relation with respect to a given relation. It is also clear that :

If $(x, y) \in R \Leftrightarrow (y, x) \in R^{-1}$

It is also clear that :

Domain (R^{-1}) = Range (R)

Range (R^{-1}) = Domain (R)

EXAMPLE 3.5 Let A = {1, 2, ….., 10}. A relation on "A" is defined as

$R = \{(x, y) : y = 3x, \text{ where } x, y \in A\}$

Find R^{-1}, Domain (R^{-1}) and Range (R^{-1}).

SOLUTION In one of the earlier examples, we determined the relation "R" as :

$R = \{(1, 3), (2, 6), (3, 9)\}$

According to the definition of inverse set, the elements of the ordered pair in the relation set are exchanged

$R^{-1} = \{(3, 1), (6, 2), (9, 3)\}$

Clearly, inverse relation can be represented in the set builder form as :

$\Rightarrow R^{-1} = \{(y, x) : y = x/3,$ where $x, y \in A\}$

Now, the domain and range of R^{-1} are :

Domain $(R^{-1}) = \{3, 6, 2\}$

Range $(R^{-1}) = (1, 2, 3)$

Consider the relation

$R = \{(1, 2), (1, 3), (2, 3)\}$ in $A = \{1, 2, 3\}$. Then

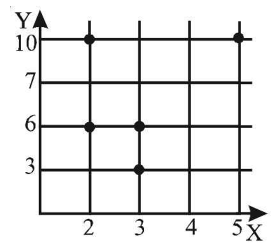
$R^{-1} = \{(2, 1), (3, 1), (3, 2)\}$

Observe that R and R^{-1} are identical, respectively, to the relations < and > in A, i.e.

$(a, b) \in R$ iff $a < b$ and $(a, b) \in R^{-1}$ iff $a > b$.

EXAMPLE 3.6 Let R be the relation from E = {2, 3, 4, 5} to F = {3, 6, 7, 10} defined by "x divides y".

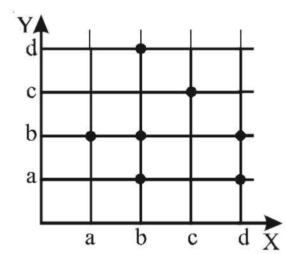
(i) Write R as a set of ordered pairs.
(ii) Plot R on a coordinate diagram of E × F.
(iii) Find the inverse relation R^{-1}.



SOLUTION

(i) Choose from the sixteen ordered pairs in E × F those in which the first element divides the second ; then
$R = \{(2, 6), (2, 10), (3, 3), (3, 6), (5, 10)\}$
(ii) R is sketched on the coordinate diagram of E × F as shown in the figure.
(iii) To find the inverse of R, write the elements of R but in reverse order :
$R^{-1} = \{(2, 6), (10, 2), (3, 3), (6, 3), (10, 5)\}$.

EXAMPLE 3.7 Let M = {a, b, c, d} and let R be the relation in M consisting of those points which are displayed on the coordinate diagram of M × M on the right. Find the inverse relation R^{-1}.



SOLUTION First write R as a set of ordered pairs, and then write the pairs in reverse order :
R = {(a, b), (b, a), (b, b), (b, d), (c, c), (d, a), (d, b)}
R^{-1} = {(b, a), (a, b), (b, b), (d, b), (c, c), (a, d), (b, d)}

3.8 | TYPES OF RELATION

1. One-one (injective)

A relation $f : A \to B$ is said to be one-one or injective if for every element in A there is not more than one corresponding element in B, and for every element in B there is not more than one corresponding element in A.

In other words, distinct elements of A either have distinct images in B or do not have any image at all, i.e.

$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ for every $x_1, x_2 \in A$.

For example, the relation $f : A \to B$ shown in the figure 3.19 is one-one.

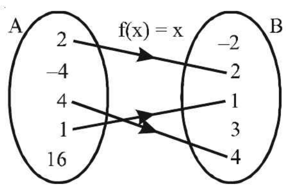


Figure 3.19

Every element in A has a distinct image in B or has no image at all and every element in B has a distinct pre-image in A or has no pre-image at all.

2. Many-One

A relation $f : A \to B$ is said to be many-one if for every element in A there is not more than one corresponding element in B but there is atleast one element in B for which there are more than one corresponding elements in A.

For example, the relation $f : A \to B$ shown in the figure 3.20 is many-one.

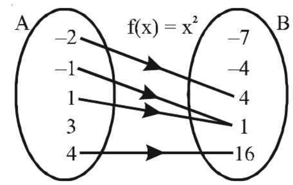


Figure 3.20

Every element in A has a unique image in B or has no image at all, but there is an element in B which has more than one corresponding pre-image in A.

3. One-Many

A relation $f : A \to B$ is said to be one-many if for every element in B there is not more than one corresponding element in A but there is atleast one element in A for which there are more than one corresponding elements in B.

For example, the relation $f : A \to B$ shown in the figure 3.21 is one-many.

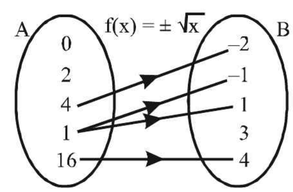


Figure 3.21

Every element in B has a distinct pre-image in A or has no pre-image at all, but there is an element in A which has more than one corresponding image in B.

4. Many-Many

A relation $f : A \to B$ is said be many-many if there is atleast one element in A for which there are more than one corresponding elements in B, and there is atleast one element in B for which there are more one corresponding elements in A.

For example, the relation $f : A \to B$ shown in the figure 3.22 is many-many.

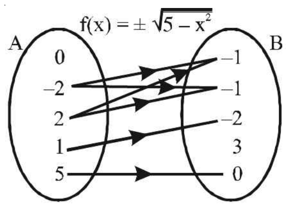


Figure 3.22

There is one element in A which has more than one corresponding image in B, and there is an element in B which has more than one corresponding pre-image in A.

Graphical Representation

From the illustrations given below, we can see that from the graph of $y = f(x)$, we can easily categorize the kind of mapping.

1. One-one (Injective)

Note that none of the lines drawn parallel to the X-axis or parallel to the Y-axis intersect the curve at more than one point.

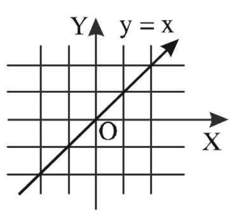


Figure 3.23

2. Many-One

Note that none of the lines drawn parallel to the Y-axis intersect the curve at more than one point but there are lines (even one line is sufficient) parallel to the X-axis which intersect the curve at more than one point.

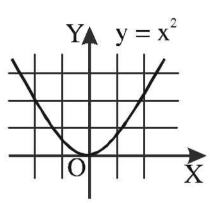


Figure 3.24

3. One-Many

Note that none of the lines drawn parallel to the X-axis intersect the curve at more than one point but there are lines (even one line is sufficient) parallel to the Y-axis which intersect the curve at more than one point.

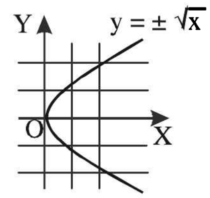


Figure 3.25

4. Many-Many

Note that there are lines (even one line is sufficient) parallel to the X-axis which intersect the curve at more than one point and there are lines parallel to the Y-axis (even one line is sufficient) which intersect the curve at more than one point.

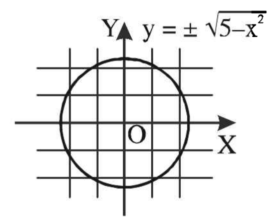


Figure 3.26

3.9 | EQUIVALENCE RELATION

Reflexive relation

In reflexive relation, "R", every element of the set "A" is related to itself.

The definition of reflexive relation is exactly same as that of identity relation except that it misses the word "only" in the end of the sentence. The implication is that this relation includes identity relation and permits other combination of paired elements as well.

Consider a set $A = \{1, 2, 3\}$. Then, one of the possible reflexive relations can be :

R = {(1, 1) , (2, 2) , (3, 3) , (1, 2) , (1, 3)}

However, following is not a reflexive relation :

R_1 = {(1, 1) , (2, 2) , (1, 2) , (1, 3)}

It is not a reflexive relation as one instance of identity relation (3,3) is absent and violates the condition that every element of the set is related to itself.

We state the condition for reflexive relation as:

R is reflexive if $(x, x) \in R$, for all $x \in A$

It is clear that identity relation is a reflexive relation. Further, universal relation consists of all combinations of ordered pairs in the Cartesian product. It means it consists of all elements of the identity relation apart from other ordered pairs. Hence, universal relation is also a reflexive relation.

Consider some examples :

(i) Let R be the relation of similarity in the set of triangles in the plane. Then R is reflexive since every triangle is similar to itself.

(ii) Let R be the relation < in any set of real numbers, i.e. $(a, b) \in R$ iff $a < b$. Then R is not reflexive since $a \not< a$ for any real number a.

Following relations are reflexive :

(i) "is equal to"
(ii) "is less than or equal to"
(iii) "is greater than or equal to"
(iv) "divides"
(v) "is subset of"

The relation "is less than" or "greater than", however, are not reflexive.

EXAMPLE 3.1 Determine whether "greater than or equal to" is a reflexive relation for natural number.

SOLUTION A relation, "R", representing "greater than or equal to" is defined as relation on natural number (N) as:

$(x, y) \in R \Leftrightarrow x \geq y$ where $x, y \in N$

We construct data for "x" and "y" in accordance with the given relation for few initial natural numbers, say 1, 2 and 3, as under :

For x = 1, y = 1, 2, 3
For x = 2, y = 2, 3
For x = 3, y = 3

Thus, the relation set is :

R = {(1, 1) , (1, 2) , (1, 3) , (2, 2) , (2, 3), (3, 3)}

Evidently, this set consists of relation of all elements of the set, which are related to itself i.e.. (1,1), (2,2) and (3,3). Thus, we conclude that "is greater than or equal to" is a reflexive relation.

EXAMPLE 3.2 Determine whether "is not equal to" is a reflexive relation for natural number?

SOLUTION A relation, "R", representing "is not equal to" is defined as relation on natural number (N) as :

$(x, y) \in R \Leftrightarrow x \neq y$ where $x, y \in N$

We construct data for "x" and "y" in accordance with the given relation for few initial natural numbers, say 1,2 and 3, as under :

For x = 1, y = 2, 3
For x = 2, y = 1, 3
For x = 3, y = 1, 2

Thus, the relation set is :

R = {(1, 2) , (1, 3) , (2, 1) , (2, 3) , (3, 1), (3, 2)}

Evidently, this set does consists of all ordered pair representing relation of an element with itself. The instances (1,1), (2,2) and (3,3) are missing. Thus, we conclude that "is not equal to" is an irreflexive relation.

Symmetric relation

In symmetric relation, the instance of relation has a mirror image. It means that if (1,3) is an instance, then (3,1) is also an instance in the relation. Clearly, an ordered pair of element with itself like (1,1) or (2,2) are themselves their mirror images. Consider some of the examples

of the symmetric relation,

$R_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$

$R_2 = \{(1, 2), (1, 3), (2, 1), (3, 1), (3, 3)\}$

We have purposely jumbled up ordered pairs to emphasize that order of elements in relation is not important. In order to decide symmetry of a relation, we need to identify mirror pairs. We state the condition of symmetric relation as :

If $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in A$

The condition of symmetric relation can also be stated as :

$xRy \Rightarrow yRx$ for all $x, y \in A$

In words, we say that if (x,y) be an instance of relation, then (y, x) will also be the instance of a symmetric relation "R".

It is clear that identity relation is a symmetric relation. Also, universal set consists of the Cartesian product of a set with itself. It means that the relation consists of instances with mirror instances. Therefore, universal relation is also symmetric relation.

Consider some examples :

(i) The relation R of similarity of triangles is symmetric. For if triangle α is similar to triangle β, then β is similar to α.

(ii) The relation $R = \{(1, 3), (2, 3), (2, 2), (3, 1)\}$ in $A = \{1, 2, 3\}$ is not symmetric since $(2, 3) \in R$ but $(3, 2) \notin R$.

Symmetric and inverse relation

An inverse relation (R^{-1}) consists of ordered pairs with exchange of positions of the elements in a given relation (R). Now let us consider a symmetric relation,

$R = \{(1, 2), (1, 3), (2, 1), (3, 1), (3, 3)\}$

By definition, its inverse relation is :

$R^{-1} = \{(2, 1), (3, 1), (1, 2), (1, 3), (3, 3)\}$

Using the fact that order does not change a set, we conclude that :

$\Rightarrow \quad R = R^{-1}$

We use this fact to identify symmetric relation. The given set is a symmetric relation, if it equals its inverse set.

Proof: Let "R" be a symmetric relation on set "A". In order to prove that $R = R^{-1}$, we consider an arbitrary instance of relation "R" :

$(x, y) \in R$

According to definition of symmetric relation,

$(y, x) \in R$

According to definition of inverse relation,

$(y, x) \in R^{-1}$

But, we had started with "R" and used definitions to show that "(x, y)" belongs to another set " R^{-1}". It

means that the "R^{-1}" set consists of the elements of set "R" at the least. Thus,

$R \subseteq R^{-1}$

Similarly, we can start with "R^{-1}" set and reach the conclusion that :

$R^{-1} \subseteq R$

If sets are subsets of each other, then they are equal. Hence,

$R = R^{-1}$

Transitive relation

If "R" be the relation on set A, then we state the condition of transitive relation as :

If f $(x, y) \in R$ and $(y, z) \in R$) then $(x, z) \in R$ for all $a, b, c \in A$

Alternatively,

xRy and $yRz \Rightarrow xRz$ for all $x; y; z \in A$

In words, we say that if (x, y) and (y, z) be the instances of a relation R such that (x, z) is also the instance of the relation, then that relation is transitive.

The identity and universal relations are transitive.

Consider some examples :

(i) The relation R of similarity of triangles is transitive since if triangle α is similar to β and β is similar to γ, then α is similar to γ.

(ii) The relation R of perpendicularity of lines in the plane is not transitive. Since

if line a is perpendicular to line b and line b is perpendicular to line c, then a is parallel and not perpendicular to c.

Some other important transitive relations are:

(i) "is equal to"
(ii) "is greater than"
(iii) "is at least as great as"
(iv) "is a subset of"
(v) "divides"

EXAMPLE 3.3 Determine whether "divides" is a transitive relation for natural number?

SOLUTION Let us consider three elements "x","y" and "z" of set N of natural numbers such that a relation "R" on N is :

$(x, y) \in R$, $(y, z) \in R$, "divides" , $x, y, z \in N$

This means that :

"x divides y" and "y divides z"

Let us now consider two natural numbers "a" and "b" such that :

$y = ax$ and $z = by$

$z = abx$

This means that "x divides z". Hence, we conclude that the relation "divides" is a transitive relation.

EXAMPLE 3.4 Let W = {1, 2, 3, 4}. Consider the following relations in W :

$R_1 = \{(1, 2), (4, 3), (2, 2), (2, 1), (3, 1)\}$

$R_2 = \{(2, 2), (2, 3), (3, 2)\}$

$R_3 = \{(1, 3)\}$

Determine whether each relation is

(i) symmetric (ii) transitive.

SOLUTION

(i) Now a relation R is not symmetric if there exists an ordered pair $(a, b) \in R$ such that $(b, a) \notin R$. Hence,

R_1 is not symmetric since $(4, 3) \in R_1$ but $(3, 4) \notin R_1$.

R_3 is not symmetric since $(1, 3) \in R_3$ but $(3, 1) \notin R_3$.

On the other hand, R_2 is symmetric.

(ii) A relation R is not transitive if there exist elements a, b and c not necessarily distinct, such that

$(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$

Hence R_1 is not transitive since

$(4, 3) \in R_1$ and $(3, 1) \in R_1$ but $(4, 1) \notin R_1$

Furthermore, R_2 is not transitive since

$(3, 2) \in R_2$ and $(2, 3) \in R_2$ but $(3, 3) \notin R_2$

On the other hand, R_3 is transitive.

Equivalence relation

A relation is equivalence relation if it is reflexive, symmetric and transitive at the same time. In order to check whether a relation is equivalent or not, we need to check all three characterizations.

By the preceding examples, the relation R of similarity of triangles is an equivalence relation since it is reflexive, symmetric and transitive.

Parallelism between lines is an equivalence relation if we agree that a line is parallel to itself.

EXAMPLE 3.5 Consider the relation R = {(1, 1), (2, 3), (3, 2)} in X = {1, 2, 3}. Determine whether or not R is (i) reflexive, (ii) symmetric, (iii) transitive

SOLUTION

(i) R is not reflexive since $2 \in X$ but $(2, 2) \notin R$.
(ii) R is symmetric since $R^{-1} = \{(1, 1), (3, 2), (2, 3)\} = R$.
(iii) R is not transitive since $(3, 2) \in R$ and $(2, 3) \in R$ but $(3, 3) \notin R$.

EXAMPLE 3.6 Let T be the set of all triangles in a plane with R a relation in T given by $R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

SOLUTION R is reflexive, since every triangle is congruent to itself. Further, $\{(T_1, T_2) \in R$

$\Rightarrow$ T_1 is congruent to T_2

$\Rightarrow$ T_2 is congruent to T_1

$\Rightarrow$ $\{(T_2, T_1) \in R$. Hence, R is symmetric. Moreover, $(T_1, T_2), (T_2, T_3) \in R$

$\Rightarrow$ T_1 is congruent to T_2 and T_2 is congruent to T_3

$\Rightarrow$ T_1 is congruent to T_3

$\Rightarrow$ $(T_1, T_3) \in R$. Hence, R is transitive.

Therefore, R is an equivalence relation.

EXAMPLE 3.7 Let L be the set of all lines in a plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$. Show that R is symmetric but neither reflexive nor transitive.

SOLUTION R is not reflexive, as a line L_1 cannot be perpendicular to itself, i.e., $(L_1, L_1) \notin R$.

R is symmetric as $(L_1, L_2) \in R$.

$\Rightarrow L_1$ is perpendicular to L_2

$\Rightarrow L_2$ is perpendicular to L_1

$\Rightarrow (L_2, L_2) \in R$.

R is not transitive. Indeed, if L_1 is perpendicular to L_2 and L_2 is perpendicular to L_3, then L_1 can never be perpendicular to L_3. In fact, L_1 is parallel to L_3, i.e., $(L_1, L_2) \in R$, $(L_2, L_3) \in R$ but $(L_1, L_3) \notin R$.

EXAMPLE 3.8 Let $N = \{1, 2, 3, ...\}$, and let R be the relation $\cong$ in $N \times N$ defined by $(a, b) \cong (c, d)$ iff $ad = bc$. Prove that R is an equivalence relation.

SOLUTION For every $(a, b) \in N \times N$, $(a, b) \cong (a, b)$ since $ab = ba$; hence R is reflexive.

Suppose $(a, b) \cong (c, d)$. Then $ad = bc$, which implies that $cb = da$. Hence $(c, d) \cong (a, b)$ and so R is symmetric.

Now suppose $(a, b) \cong (c, d)$ and $(c, d) \cong (e, f)$. Then, $ad = bc$ and $ef = dc$. Thus,

$$(ad)\ (cf) = (bc)\ (de)$$

and, by cancelling from both sides, $af = be$. Accordingly, $(a, b) \cong (e, f)$ and so R is transitive. Since R is reflexive, symmetric and transitive, R is an equivalence relation.

Observe that if the ordered pair (a, b) is written as a fraction $\frac{a}{b}$, then the above relation R is, in fact, the usual definition of equality between fractions, i.e.

$$\frac{a}{b} = \frac{c}{d} \text{ iff } ad = bc.$$

EXAMPLE 3.9 Show that the number of equivalence relations in the set $\{1,2,3\}$ containing (1,2) and (2,1) is two.

SOLUTION The smallest equivalence relation R_1 containing (1,2) and (2,1) is $\{(1,1), (2,2), (3,3), (1,2), (2,1)\}$. Now we are left with only 4 pairs namely (2,3), (3,2), (1,3) and (3,1). If we add any one, say (2,3) to R_1, then for symmetry we must add (3,2) also and now for transitivity we are forced to add (1,3) and (3,1). Thus, the only equivalence relation bigger than R_1 is the universal relation. This shows that the total number of equivalence relations containing (1,2) and (2,1) is two.

EXAMPLE 3.10 Let $A = \{1, 2, 3\}$. Then show that the number of relations containing (1, 2) and (2, 3) which are reflexive and transitive but not symmetric is three.

SOLUTION The smallest relation R_1 containing (1,2) and (2,3) which is reflexive and transitive but not symmetric is $\{(1, 1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$. Now, if we add the pair (2,1) to R_1 to get R_2, then the relation R_2 will be reflexive, transitive but not symmetric.

Similarly, we can obtain R_3 by adding (3,2) to R_1 to get the desired relation. However, we can not add two pairs (2,1), (3,2) or single pair (3,1) to R_1 at a time, as by doing so, we will be forced to add the remaining pair in order to maintain transitivity and in the process, the relation will become symmetric also which is not required. Thus, the total number of desired relations is three.

EXAMPLE 3.11 Show that the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$ is reflexive but neither symmetric nor transitive.

SOLUTION R is reflexive, since (1,1), (2,2) and (3,3) lie in R. Also, R is not symmetric, as $(1, 2) \in R$ but $(2, 1) \notin R$. Similarly, R is not transitive, as $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

EXAMPLE 3.12 Show that the relation R in the set I of integers given by $R = \{(a, b) : 2 \text{ divides } a - b\}$ is an equivalence relation.

SOLUTION R is reflexive, as 2 divides $(a - a)$ for all $a \in I$. Further, if $(a, b) \in R$, then 2 divides $a - b$. Therefore, 2 divides $b - a$. Hence $(b, a) \in R$, which show that R is symmetric. Similarly, if $(a, b) \in R$ and $(b, c) \in R$, then $a - b$ and $b - c$ are divisible by 2. Now, $a - c = (a - b) + (b - c)$ is even. So, $(a - c)$ is divisible by 2. This shows that R is transitive. Thus, R is an equivalence relation in I.

In the above example, note that all even integers are related to zero, as $(0, \pm 2)$, $(0, \pm 4)$, etc., lie in R and no odd integer is related to 0, as $(0, \pm 1)$, $(0, \pm 3)$ etc., do not lie in R. Similarly, all odd integers are related to one and no even integer is related to one. Therefore, the set E of all even integers and the set O of all odd integers are subsets of I satisfying the following conditions :

(i) All elements of E are related to each other and all elements of O are related to each other.

(ii) No element of E is related to any element of O and vice-versa.

(iii) E and O are disjoint and $I = E \cup O$.

The subset E is called the **equivalence class** containing zero and is denoted by [0]. Similarly, O is the equivalence class containing 1 and is denoted by [1]. Note that $[0] \neq [1]$, $[0] = [2r]$ and $[1] = [2r + 1]$, $r \in I$. In fact, what we have seen above is a true relation R in an arbitrary set X, R divides X into mutually disjoint subsets A_i called partitions or subdivisions of X satisfying :

(i) All elements of A_i are related to each other, for all i.

(ii) No element of A_i is related to any element of A_j, $i \neq j$.

(iii) $\cup A_i = X$ and $A_i \cap A_i = \phi$, $i \neq j$.

The subsets A_i are called equivalence classes. The interesting part of the situation is that we can go reverse also. For example, consider a subdivision of the set I given by three mutually disjoint subsets A_1, A_2 and A_3 whose union is I with

$$A_1 = \{x \in I : x \text{ is a multiple of } 3\} = \{..., -6, -3, 0, 3, 6, ...\}$$
$$A_2 = \{x \in I : x - 1 \text{ is a multiple of } 3\} = \{..., -5, -2, 1, 4, 7, ...\}$$
$$A_3 = \{x \in I : x - 2 \text{ is a multiple of } 3\} = \{..., -4, -1, 2, 5, 8, ...\}$$

Define a relation R in I given by $R = \{(a, b) : 3 \text{ divides } a - b\}$. Following the arguments similar to those used in the previous example, we can show that R is an equivalence relation. Also, A_1 coincides with the set of all integers in I which are related to zero, A_2 coincides with the set of all integers which are related to 1 and A_3 coincides with the set of all integers in I which are related to 2. Thus, $A_1 = [0]$, $A_2 = [1]$ and $A_3 = [2]$. In fact, $A_1 = [3r]$, $A_2 = [3r + 1]$ and $A_3 = [3r + 2]$, for all $r \in I$.

Equivalence Classes or Equivalence Sets

Let A be a non-empty set and let R be an equivalence relation in A. Further, let a be an arbitrary element of A. The elements $x \in A$ satisfying xRa constitute a subset A_a of A, called an equivalence class of a with respect to R. We shall denote this equivalence class by A_a or by [a] or by $\bar{a}$. Thus symbolically,

A_a or [a] or $\bar{a} = \{x : x \in A \text{ and } (x, a) \in R$ i.e., $xRa\}$

Let A be the set of all triangles in a plane and let R be an equivalence relation in A defined by "x is congruent to y", $x \in A$, $y \in A$. When $a \in A$ we shall mean by the equivalence class [a] the set of all triangles of A congruent to the triangle a. Similarly, when $b \in A$ we shall mean by the equivalence class [b] the set of all triangles of A congruent to the triangle b.

Properties of Equivalence Classes

Let A be a non-empty set and let R be an equivalence relation in A. Let a and b be arbitrary elements in A. Then

(i) $a \in [a]$.

(ii) If $b \in [a]$, then $[b] = [a]$.

(iii) $[a] = [b]$ iff $(a, b) \in R$ i.e., iff aRb.

(iv) Either $[a] = [b]$ or $[a] \cap [b] = \phi$ i.e., two equivalence classes are either disjoint or identical.

PRACTICE PROBLEMS [F]

1. Suppose $(x + y, 1) = (3, x - y)$. Find x and y.
2. Let $A = \{2, 4, 6\}$ and $B = \{4, 8, 12\}$; find $A \times B$ and $B \times A$.
3. Let $A = B \cap C$. Determine if either statement is true :
   (i) $A \times A = (B \times B) \cap (C \times C)$
   (ii) $A \times A = (B \times C) \cap (C \times B)$.
4. Given $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, find $A \cup (B \cup C)$ and $(A \times B) \cap (B \times C)$.
5. Let n be a fixed positive integer. Define a relation R on I as follows : a R b if $n \mid (a - b)$ i.e. $(a - b)$ is divisible by n. Show that R is an equivalence relation on I.
6. The following three relations are defined on the set of natural numbers N :
   $R = \{(x,y) : x < y, x \in N, y \in N\}$
   $S = \{(x,y) : x + y = 10, x \in N, y \in N\}$
   $T = \{(x,y) : x = y \text{ or } x - y = 1, x \in N, y \in N\}$
   Explain clearly which of the above relations are
   (A) Reflexive (B) Symmetric
   (C) Transitive (D) None of these
7. Let R be the relation defined on the set of natural number N as
   $R = \{(x,y) : x \in N, y \in N, 2x + y = 41\}$
   Find the domain and range of this relation R. Also verify whether R is
   (i) Reflexive (ii) Symmetric
   (iii) Transitive
8. Let $S = \{1,2,3,4,5\}$ and let $A = S \times S$ define the relation R on A as follows "(a,b)R(c,d) if and only if ad = cb". Show that R is an equivalence relation. Compute the collection of all equivalence classes.
9. Determine whether each of the following relations are reflexive, symmetric and transitive :
   (i) Relation R in the set $A = \{1, 2, 3, ..., 13, 14\}$ defined as $R = \{(x, y) : 3x - y = 0\}$
   (ii) Relation R in the set N of natural numbers defined as $R = \{(x,y) : y = x + 5 \text{ and } x < 4\}$
   (iii) Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ as
   $R = \{(x,y) : y \text{ is divisible by } x\}$
   (iv) Relation R in the set I of all integers defined as
   $R = \{(x,y) : x - y \text{ is an integer}\}$
10. Determine whether each of the following relations are reflexive, symmetric and transitive : Relation R in the set A of human beings in a town at a particular time given by
    (a) $R = \{(x,y) : x \text{ and } y \text{ work at the same place}\}$

(b) R = {(x,y) : x and y live in the same locality}
(c) R = {(x,y) : x is exactly 7 cm taller than y}
(d) R = {(x,y) : x is wife of y}
(e) R = {(x,y) : x is father of y}

(a) Reflexive, symmetric and transitive
(b) Reflexive, symmetric and transitive
(c) Neither reflexive nor symmetric nor transitive.
(d) Neither reflexive nor symmetric but transitive
(e) Neither reflexive nor symmetric nor transitive.

11. Show that the relation R in the set R of real numbers, defined as $R = \{(a, b) : a \le b^2\}$ is neither reflexive nor symmetric nor transitive.

12. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as $R = \{(a, b) : b = a + 1\}$ is reflexive , symmetric or transitive.

13. Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2,1)} is symmetric but neither reflexive nor transitive.

14. Show that the relation R in the set A = {1, 2, 3, 4, 5} given by $R = \{(a,b) : |a - b| \text{ is even}\}$, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

15. Show that each of the relation R in the set $A = \{x \in I : 0 \le x \le 12\}$, given by
(i) $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$
(ii) $R = \{(a, b) : a = b\}$
is an equivalence relation. Find the set of all elements related to 1 in each case.

3.10 | FUNCTIONS

Function is a special relation. It is also conceived as a "rule", because function is a relation between elements of two sets, following certain rule. Every element of a set (say A) is related to exactly one element of other set (say B). This relationship is described as mapping of all elements of one set to elements of another set. In order to emphasise, we need to enumerate the way "function" relation is special :

1. Every element of set "A" is related to elements in set "B".
2. An element of set "A" is related to exactly one element of "B".

It can be deduced from the above characterisation of a function that the element of set "B" may be paired with none or one or more elements of set "A".

In order to illustrate function relation, let us consider an example. Let "A" and "B" be two sets as given here :`

$A = \{-1, 0, 1, 2, 3\}$
$B = \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

The two sets are related by the relation :
$R = \{(x, y) : y = x^2 - 1, x \in A, y \in B\}$

The values of "y" for given values of "x" are :

For $x = -1$, $y = 1 - 1 = 0$
For $x = 0$, $y = 0 - 1 = -1$
For $x = 1$, $y = 1 - 1 = 0$
For $x = 2$, $y = 4 - 1 = 3$
For $x = 3$, $y = 9 - 1 = 8$

The relation between two sets is pictorially shown with arrow diagram in figure 3.27. We note that all elements of "A" are mapped. Further, elements in "A" are uniquely mapped i.e. they are paired to exactly one element of set "B". It is,

however, possible that few of the elements in set "A" are related to same element in "B" like "–1" and "1" in set "A" are related to "0" in set "B".

In the nutshell, we see that this relation meets both properties as enumerated for a function relation and hence is a function relation.

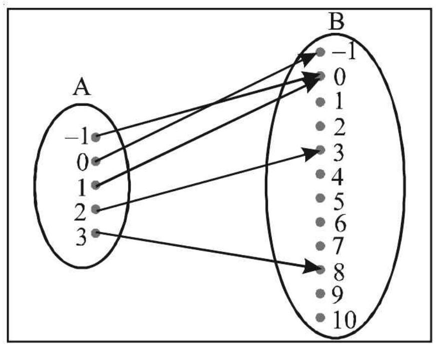


Figure 3.27

Looking in reverse direction, we see that elements in "B" may be paired with no element (1,2,4,5,6,7,9,10) or with one element (–1,3,8) or with more than one element (0) in "A".

We generally drop word "relation" from "function relation" and call it simply as "function".

The function is denoted by a small letter like "f". To elaborate the direction of function, we expand the symbol as :

$$f : A \rightarrow B$$

This means that function is mapped from "A" to "B". Now, in order to define the function, we need to understand the concept of "image" and "pre-image" elements. We call first element "x" of set "A" in the ordered pair (x,y) of the function as the "pre-image" of second element "y" of set "B".

The second element "y" of set "B" is called the "image" of the first element "x" of set "A".

The image is also denoted as "f(x)". We read "f(x)" as image of "x" under rule "f".

For a particular value of x = a, "f(a)" is a particular instance of image : f (a) = b

Definition A relation "f" is a function, if every element in set "A" has one and only one image in set "B".

Consider the following relations in

A = (1, 2, 3) :

f = {(1, 3), (2, 3), (3, 1)}

g = {(1, 2), (3, 1)}

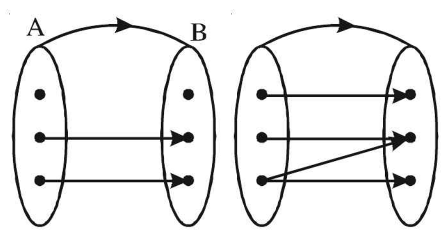
h = {(1, 3), (2, 1), (1, 2), (3, 1)}

f is a function from A into A since each member of A appears as the first coordinate in exactly one ordered pair in f ; here f(1) = 3, f(2) = 3 and f(3) = 1.

g is not a function from A into A since 2 ∈ A is not the first coordinate of any pair in g and so g does not assign any image to 2.

Also h is not a function from A into A since 1 ∈ A appears as the first coordinate of two distinct ordered pairs in h, (1, 3) and (1, 2). If h is to be a function it cannot assign both 3 and 2 to the element 1 ∈ A.

It must be emphasised that the exhaustion of the elements of the domain is crucial in the definition of a function. For example, the first diagram in figure below does not represent a function, as some elements of the domain are not assigned. Also important in the definition of a function is the fact that the output must be unique. For example, the second diagram below does not represent a function, since the last element of the domain is assigned to two outputs (Figure 3.28).



Not a function Not a function

Figurre 3.28

EXAMPLE 3.1 Let X = {1, 2, 3, 4}. Determine whether or not each relation is a function from X into X.

(i) f = {(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)}

(ii) g = {(3, 1), (4, 2), (1, 1)}

(iii) h = {(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)}

SOLUTION Recall that a subset f of X × X is a function f : X → X if and only if each a ∈ X appears as the first coordinate in exactly one ordered pair in f.

(i) No. Two different ordered pairs (2, 3) and (2, 1) in f have the same number 2 as their first coordinate.

(ii) No. The element 2 ∈ X does not appear as the first coordinate in any ordered pair in g.

(iii) Yes. Although 2 ∈ X appears as the first coordinate in two ordered pairs in h, these two ordered pairs are equal.

Sometimes the assignment rule of a function varies through various subsets of its domain. We call any such function a **piecewise-defined** function.

For example, consider the function f : [–5, 4] → R defined by

$$f(x) = \begin{cases} 1 & \text{if } \ 2x \in [-5, 1) \\ 2 & \text{if } \ x = 1 \\ x+1 & \text{if } \ x \in [1, 4) \end{cases}$$

Determine f(–3), f(1), f(4) and f(5).

f(–3) = 2(–3) = –6, f(1) = 2, f(4) = 4 + 1 = 5, and f(5) is undefined.

Domain, range and co-domain of function

As all the elements of set "A" are involved, it emerges that the set of first elements in the ordered pairs i.e. domain set is same as set "A". We can not say the same for set "B". The set "B" may have other elements than those, which have been mapped with the elements of "A". The range is simply the set of images of the function. However, as defined earlier, the set "B" is co-domain of the relation and hence that of function in this special case. It is clear that range is a subset of co-domain "B".

Domain of "f" = A

Co-domain of "f" = B

Range of "f" = Set of images = {f (x) : x ∈ A}

Real function

If the range of a function is a set of real numbers, then the function is called "real valued function". In other words, if the range of a function is either the set "R" or its subset, then it is a real valued function. We should emphasize here that "R" denotes set of real number and it is not the symbol for relation, which is also denoted as "R".

Further, we distinguish "real valued function" from "real function". The very terminology is indicative of the difference. The term "real valued function" means that the value of function i.e. image is real. It does not say anything about "pre-image". Now, there can be a function, which accepts non-real complex numbers, but maps to a real value.

On the other hand, a real function has both image and pre-image as real numbers. It follows then that the domain of a "real function" is also either a set or subset of real numbers.

Definition A function is a real function, if its domain and range are either "R" or subset of "R".

Note Our discussion from this point onwards in the course relates to real function only unless otherwise stated.

Interpretation of function

It is intuitive to find similarity of an algebraic equation to the "rule" of a function. Consider an equation,

$$y = x^2 + 1$$

This equation is valid for all real values of "x". The set of real values of "x", belongs to set "R". The set of values of "y" also belongs to set of "R". On the other hand, the equation itself is the rule that maps two sets comprising of values of "x" and "y".

Alternatively, we can write the rule also as:

$f(x) = x^2 + 1$

In terms of rule, we define function, saying that :

$f : R \rightarrow R$ by $f(x) = x^2 + 1$

We read it as : "f" is a function from "R" to "R" by the rule given by $f(x) = x^2 + 1$.

From this description, we think a function as a relation, which is governed by a specified rule. The rule relates two sets known as domain and co-domain, which are sets of real numbers. One of the quantities "x" is independent of the other quantity "y". The other quantity "y" is dependent on the quantity "x". In plain words, one of the interpretations is that function relates dependent and independent variables. As a matter of fact, we would attach additional meanings to the concept of function as we proceed to study it in details.

EXAMPLE 3.2 Let "A" be the set of first three natural numbers. A real function is defined as :

$f : A \rightarrow N$ by $f(x) = x^2 + 1$

Find (i) domain of "f" (ii) range of "f" (iii) co-domain of "f" (iv) f(3) and (v) pre-images of 2 and 4.

SOLUTION Here set "A" is the domain of "f". Hence,

Domain of "f" = A = {1, 2, 3}

For determining the range, we need to find images for the each element of the domain as :

For $x = 1$, $f(x) = x^2 + 1 = 1^2 + 1 = 2$

For $x = 2$, $f(x) = x^2 + 1 = 2^2 + 1 = 5$

For $x = 3$, $f(x) = x^2 + 1 = 3^2 + 1 = 10$

Hence, range of function is given as :

Range of "f" = A = {2, 5, 10}

Co-domain of the function is the second set on which the elements of first set are mapped. It is given that "f" is a function from set "A" to set N. Hence, co-domain of "f" = N

The image of set for x = 3 has been already calculated. It is :

$f(3) = x^2 + 1 = 3^2 + 1 = 10$

For pre-image of f(x) = 2, we have :

$f(x) = 2 = x^2 + 1 \quad \Rightarrow x^2 = 1 \quad \Rightarrow x = 1. -1$

But, only "1" is an element of domain set "A" –not "–1". Hence, "pre-image" of "2" is "1".

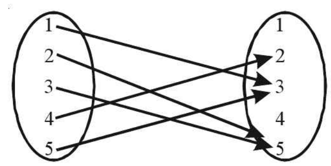
For pre-image of f(x) = 4, we have :

$f(x) = 4 = x^2 + 1 \ \Rightarrow x^2 = 3$

$x = \sqrt{3}, -\sqrt{3}$

But it is given that domain is first three natural numbers only. Thus, we conclude that "4" has no pre-image.

EXAMPLE 3.3 Let A = {1, 2, 3, 4, 5} and let $f : A \rightarrow A$ be the function defined in the diagram



(i) Find the range of f.

(ii) Write f as a set of ordered pairs.

SOLUTION

(i) The range f[A] of the function f consists of all the image points. Now only 2, 3 and 5 appear as the image of any elements of A ; hence f[A] = {2, 3, 5}.

(ii) Now f(1) = 3, f(2) = 5, f(3) = 5, f(4) = 2 and f(5) = 3 ; hence f = {(1, 3), (2, 5), (3, 5), (4, 2), (5, 3)}.

Numbers of functions

We can find out maximum numbers or total possible numbers of functions that can be generated by the rule from given domain and co-domain sets, provided these sets are finite sets. We have noted that the total numbers of relations generated from Cartesian product of two sets "A" and "B" is given by :

$N = 2^{pq}$

where "p" and "q" are the finite numbers of elements in sets "A" and "B".

However, function is a special relation, in which each element of set "A" is related to exactly one element of set "B" - unlike in the case of power set in which we count all possible combinations. Hence, number of possible relations is not same as the numbers of possible functions.

For determining total numbers of functions from two given sets, let us consider that "p" and "q" denote the numbers of elements in domain "A" and co-domain "B", respectively. Then, an element of domain can combine with one of the "n" elements in "B". Another element of domain can also combine with one of the "n" elements in "B". Hence, the total number of functions is :

$$N_f = n \cdot n \ldots\ldots n = n^n.$$

Finite and infinite functions

The numbers of ordered pairs in the function set is equal to the numbers of elements in the domain set. This follows from the fact that every element of domain set "A" is related to an unique element in "B". Thus, if domain is a finite set, then the resulting function is finite. Consider the earlier example, when A = {1,2,3} and function is defined as :

$$f : A \rightarrow N \text{ by } f(x) = x^2 + 1$$

The function set is a finite set :

$$f = \{(1, 2), (2, 5), (3, 10)\}$$

On the other hand, if we expand this function by defining the relation from the infinite set of natural numbers, "N" to "N", then the resulting set of ordered pair is an infinite set and so is the function :

$$g : N \rightarrow N \text{ by } f(x) = x^2 + 1$$

The resulting function set, in the set builder form, is given as :

$$g = \{(x, y) : y = x^2 + 1, \text{ where } x, y \in N\}$$

Graphs of functions

Here, we shall introduce an alternative way to represent a function. We should be aware that we can define a function even with a graph. Graphical representation of function is intuitive and revealing about their characteristics.

Function is a set of ordered pairs between "x" and "y" from domain and co-domain sets respectively in accordance with certain rule. If we look closely at the function set, then it is easy to realise that the elements of ordered pair (x,y) can be considered to be coordinates "x" and "y" of a planar coordinate system.

We represent independent variable, "x" i.e. the element of domain set "A" as abscissa along x-axis and dependent variable, "y", i.e. the element of co-domain "B" as ordinate along y-axis. A point on the plot represented by coordinate (x,y) is an instance of or value of the function for a given value of "x". Compositely, the graph itself is the collection of all such points, which form part of the function set.

For example, we draw a graph, which is defined as :

$$f : N \rightarrow N \text{ by } f(x) = x, \text{ where } x \in N$$

In order to plot the function, we evaluate function values for values of "x" :

For $x = 1$, $y = 1$
For $x = 2$, $y = 2$
....
....
For $x = n$, $y = n$

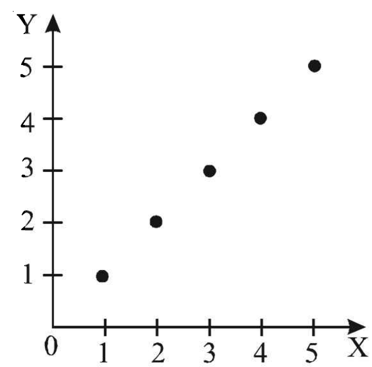


Figure 3.29

Note that plot of the function is a collection of discrete points only.

For the plot to be continuous, it is clear that the domain and co-domain of the function should be set of real numbers. In that case, we can define the function as :

$g : R \rightarrow R$ by $f(x) = x$, where $x \in R$.

The corresponding plot is a continuous straight line bisecting the first quadrant and passing through the origin, as shown in the figure here :

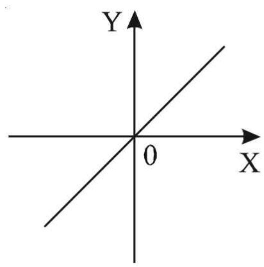
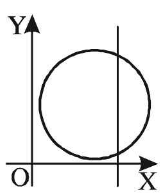


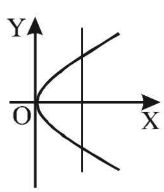
Figure 3.30

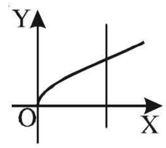
Testing a graph for function

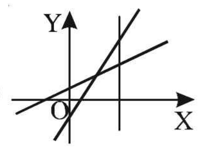
The graph of $y = f(x)$ where f is a function is such that lines drawn parallel to the Y-axis do not intersect the curve at more than one point. This is called the **vertical line test** for f to be a function.

EXAMPLE 3.4 Which of the following graphs represents a function.

(a) 

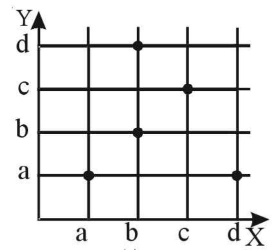
(b) 

(c) 

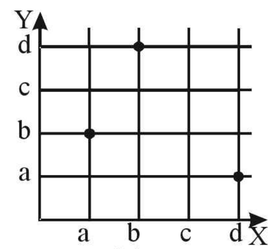
(d) 

SOLUTION Using the vertical line test, we can see that only (c) represents a function.

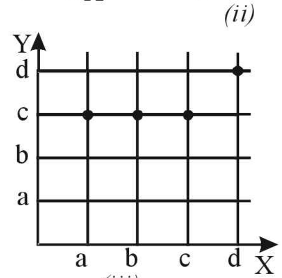
EXAMPLE 3.5 Let $W = \{a, b, c, d\}$. Determine whether the set of points in each coordinate diagram of $W \times W$ is a function from W into W.



(i)



(ii)



(iii)

SOLUTION

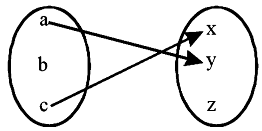
(i) No. The vertical line through b contains two points of the set, i.e. two different ordered pairs (b, b) and (b, d) contain the same first element b.

(ii) No. The vertical line through c contains no point of the set, i.e. $c \in W$ does not appear as the first element in any ordered pair.

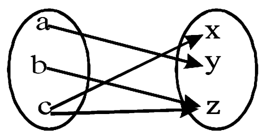
(iii) Yes. Each vertical line contains exactly one point of the set.

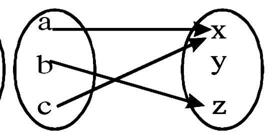
PRACTICE PROBLEMS [G]

1. State whether or not each diagram defines a function from A = {a, b, c} into B = (x, y, z).



(i)





(ii) (iii)

2. Let W = {a, b, c, d}. Determine whether each set of ordered pairs is a function from W into W.

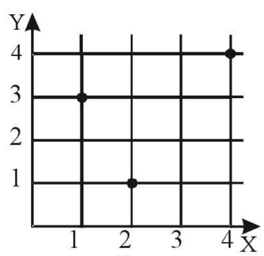
(i) {(b, a), (c, d), (d, a), (c, d), (a, d)}

(ii) {(d, d), (c, a), (a, b), (d, b)}

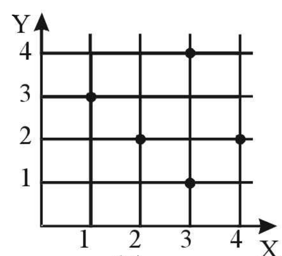
(iii) {(a, b), (b, b), (c, b), (d, b)}

(iv) {(a, a), (b, a), (a, b), (c, d)}

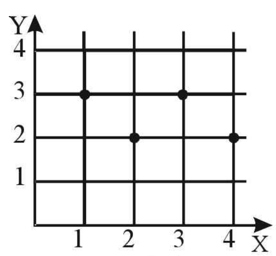
3. Let V = {1, 2, 3, 4}. Determine whether the set of points in each coordinate diagram of V × V is a function from V into V.



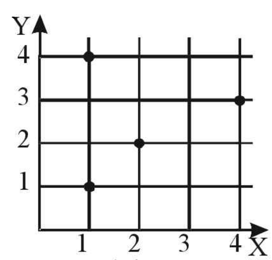
(i)



(ii)



(iii)



(iv)

4. Let $g : R \to R$ be defined by

$$g(x) = \begin{cases} x^2 - 3x \text{ if } x \geq 2 \\ x + 2 \text{ if } x < 2 \end{cases}.$$

Find (i) g(5), (ii) g(0), (iii) g(–2).

5. Is g = {(1, 1), (2, 3), (3, 5), (4, 7)} a function? If this is described by the formula $g(x) = \alpha x + \beta$, then what values should be assigned to α and β?

6. Given A = {–1, 0, 2, 5, 6, 11} and B = {–2, –1, 0, 18, 108}, define $f : A \to B$ as $f(x) = x^2 - x - 2$. Find f(A). Is f(A) = B?

7. Let A = {–1, 0, 1, 2}, B = {–4, –2, 0, 2} and f, g : A → B be functions defined by $f(x) = x^2 - x$, $x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in A$. Are f and g equal?

Things to REMEMBER

1. Identity law
$A \cap U = A$
2. Commutative law
$A \cap B = B \cap A$
3. Associative law
$(A \cap B) \cap C = A \cap (B \cap C)$
4. Distributive law
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5. Difference of sets
$A - B = \{x : x \in A \text{ and } x \notin B\}$
6. Symmetric Difference of two sets
$A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
7. Complement of a set
A' or $\overline{A} = \{x : x \in U \text{ and } x \notin A\}$
8. De-Morgan's laws
$A' \cap B' = (A \cup B)'$
$A' \cup B' = (A \cap B)'$
9. Number of elements in the union of two sets
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
10. Distributive property of product operator
$A \times (B \cup C) = (A \times B) \cup (A \times C)$
$A \times (B \cap C) = (A \times B) \cap (A \times C)$
$A \times (B - C) = (A \times B) - (A \times C)$
11. Numbers of relations
Between two sets A and B, where p and q are the numbers of elements in two sets respectively, the number of possible relations that can be generated is given by 2^{pq}.
12. Reflexive relation
In reflexive relation, "R", every element of the set "A" is related to itself.
R is reflexive if $(x, x) \in R$, for all $x \in A$
13. Symmetric relation
The condition of symmetric relation is:
$xRy \Rightarrow yRx$ for all $x, y \in A$
If (x,y) be an instance of relation, then (y, x) will also be the instance of a symmetric relation "R".
14. Transitive relation
The condition of transitive relation is :
xRy and $yRz \Rightarrow xRz$ for all $x; y; z \in A$
We say that if (x, y) and (y, z) be the instances of a relation R such that (x, z) is also the instance of the relation, then that relation is transitive.
15. Equivalence relation
A relation is equivalence relation if it is reflexive, symmetric and transitive at the same time.

OBJECTIVE EXERCISES

SINGLE CORRECT ANSWER TYPE

1. Let R = {(1,3), (2,2), (3, 2)} and S = {(2, 1), (3, 2), (2, 3)} be two relations on set A = {1, 2, 3}. Then RoS is equal to

(A) {(1,3), (2,2), (3, 2)} (B) {(1,3), (2,2), (3, 2), (2, 1), (2, 3)}

(C) {(3, 2), (1, 3)} (D) {(2, 3), (3, 2)}

2. If X and Y are two sets, then $X \cap (Y \cap X)'$ equals

(A) X (B) Y

(C) ϕ (D) None

3. A survey shows that 63% of the Americans like cheese whereas 76% like applies. If x% of the Americans like both cheese and apples, then

(A) x = 39 (B) x = 63

(C) $39 \le x \le 63$ (D) None of these

4. For real numbers x and y, we write $x \, R \, y \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is

(A) Reflexive (B) Symmetric

(C) Transitive (D) Equivalence

5. Which one of the following is not true?

(A) $A - B \subseteq A$ (B) $B' - A' \subseteq A$

(C) $A \subseteq A - B$ (D) $A \cap B' \subseteq A$

6. Let R be the relation in the set {1, 2, 3, 4,} given by R = {(1,2), (2,2), (1,1), (4,4), (1,3), (3,3), (3,2)}. Choose the correct answer :

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric

(C) R is symmetric and transitive but not reflexive

(D) R is an equivalence relation.

7. Let R and S be two non-void relations on a set A. Which of the following statements is false?

(A) R and S are transitive $\Rightarrow R \cup S$ is transitive

(B) R and S are transitive $\Rightarrow R \cap S$ is transitive

(C) R and S are symmetric $\Rightarrow R \cup S$ is symmetric

(D) R and S are reflexive $\Rightarrow R \cup S$ is reflexive

8. If A = {1, 2, 3} and B {3, 8}, then $(A \cup B) \times (A \cap B)$ is :

(A) {(3,1), (3,2), (3,3), (3,8)}

(B) {(1,3), (2,3), (3,3), (8,3)}

(C) {(1,2), (2,2), (3,3), (8,8)}

(D) {(8,3), (8,2), (8,1), (8,8)}

9. Let A = {p, q, r}, which of the following is not an equivalence relation on A?

(A) R_1 = {(p,q), (q,r), (p,r), (p,p)}

(B) $R_2 = \{(r,q), (r,p), (r,r), (q,q)\}$
(C) $R_3 = \{(p,p), (q,q), (r,r), (p,q)\}$
(D) None of these

10. Which of the following is the empty set?
(A) $\{x : x$ is a real number and $x^2 - 1 = 0\}$
(B) $\{x : x$ is a real number and $x^2 + 1 = 0\}$
(C) $\{x : x$ is a real number and $x^2 - 9 = 0\}$
(D) $\{x : x$ is a real number and $x^2 = x + 2\}$

11. Let $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$. Which of the following relations from A to B is not a function?
(A) $R_1 = \{(p,1), (q,2), (r,1), (s,2)\}$
(B) $R_2 = \{(p,1), (q,2), (r,1), (s,1)\}$
(C) $R_3 = \{(p,1), (q,2), (r,2)\}$
(D) $R_4 = \{(p,2), (q,3), (r,2), (s,2)\}$

12. Two finite sets A and B are having m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. The value of m and n are
(A) 7, 6 (B) 6, 3
(C) 5, 1 (D) 8, 7

13. Let A be a set containing 10 distinct elements, then the total number of distinct function from A to A is :
(A) 10 ! (B) 10^{10}
(C) 2^{10} (D) $2^{10}-1$

14. Let S be a set containing n elements. Then the total number of binary operations on S is
(A) n^n (B) 2^{n^2}
(C) n^{n^2} (D) n^2

15. If sets A and B are defined as
$A = \{(x,y) : y = e^x, x \in R\}$
$A = \{(x,y) : y = x, x \in R\}$
(A) $B \subset A$ (B) $A \subset B$
(C) $A \cap B = \phi$ (D) $A \cup B$

16. Let R be the relation in the set N given by $R = \{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer.
(A) $(2, 4) \in R$ (B) $(3, 8) \in R$
(C) $(6, 8) \in R$ (D) $(8, 7) \in R$

17. Let $A = \{1, 2, 3\}$. Then number of relations containing (1,2) and (1,3) which are reflexive and symmetric but not transitive is.
(A) 1 (B) 2
(C) 3 (D) 4

18. Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing (1,2) is
(A) 1 (B) 2
(C) 3 (D) 4

19. f and h are functions from $A \to B$ where $A = \{a,b,c,d\}$, $B = \{s, t, u\}$ defined as follows :
$f(a) = t$, $f(b) = s$, $f(c) = s$

f(d) = u, h(a) = s, h(b) = t
h(c) = s, h(a) = u, h(d) = u
which one of the following statement is true
(A) f and h are function (B) f is a function and h is not a function
(C) f and h are not function (D) None of these

20. Let A = {x : x is a multiple of 3} and
B = {x : x is a multiple of 5} then $A \cap B$ is given by
(A) {3, 6, 9...} (B) {5, 10, 15, 20.....}
(C) {15, 30, 45,.....} (D) None of these

21. If A = {2, 4} and B = {3, 4, 5}, then $(A \cap B) \times (A \cup B)$ is
(A) {(2,2), (3,4), (4,2), (5,4)
(B) (2,3), (4,3), (4,5)
(C) {(2,4), (3,4), (4,4), (4,5)
(D) {(4,2), (4,3), (4,4), (4,5)

22. In the set X = {a,b,c,d}, which of the following is a function in X?
(A) R_1 = {(b,a), (a,b), (c,d), (a,c)}
(B) R_2 = {(a,d), (d,c), (b,b), (c,c)}
(C) R_3 = {(a,b), (b,c), (c,d), (b,d)}
(D) R_4 = {(a,a), (b,b), (c,c), (a,d)}

23. If R is the relation "less than" from
A = {1, 2, 3, 4, 5} to B = {1, 4, 5}, then the inverse of R is
(A) R^{-1} = {(4, 1) (5, 1) (2, 3) (4, 3)}
(B) R^{-1} = {(4, 1) (5, 1) (4, 2) (5, 2) (4, 3) (5, 3) (5, 4)}
(C) R^{-1} = {(3, 5) (5, 4) (4, 5)
(D) None of these

24. In a town of 10,000 families it was found that 40% families buy newspaper A, 20% buy newspaper B and 10% families buy newspaper C, 5% families buy A and B, 3% buy B and C, 4% buy A and C. If 2% families buy all the three newspapers, then number of families which buy A only is
(A) 3100 (B) 3300
(C) 2900 (D) 1400

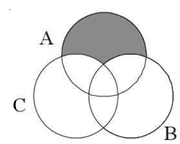
25. Let U be the universal set and $A \cup B \cup C = U$, then $\{(A - B) \cup (B - C) \cup B\,(C - A)\}'$ is
(A) $A \cup B \cup C$ (B) $A \cup (B \cap C)$
(C) $A \cap B \cap C$ (D) $A \cap (B \cup C)$

26. For real numbers x and y, we write $x\ R\ y \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is
(A) Reflexive (B) Symmetric
(C) Transitive (D) None of these

27. Let $A = \{(x,y) : y = e^x, x \in R\}$. $B = \{(x,y) : y = e^{-x}, x \in R\}$. Then
(A) $A \cap B = \phi$ (B) $A \cap B \neq \phi$
(C) $A \cup B = R^2$ (D) None of these

28. If A = {1, 2, 4}, B = {2, 4, 5}, C = {2, 5}, then (A – B) × (B – C) is
(A) {(1, 2), (1, 5), (2, 5)} (B) {(1, 4)}
(C) {(1, 4)} (D) None of these

29. The shaded region in the given figure is



(A) $A \cap (B \cup C)$ (B) $A \cup (B \cup C)$
(C) $A \cap (B - C)$ (D) $A - (B \cup C)$

30. If A and B are two sets $(A - B) \cup (B - A) \cup (A \cap B)$ is equal to
(A) $A \cup B$ (B) $A \cap B$
(C) A (D) B'

31. The number of elements in the set
$\{(a,b) : 2a^2 + 3b^2 = 35,\ a, b \in I\}$, where I is the set of all integers, is
(A) 2 (B) 4
(C) 8 (D) 12

32. A set contains 2n + 1 elements. The number of subsets of this set containing more than n elements is equal to
(A) 2^{n-1} (B) 2^n
(C) 2^{n+1} (D) 2^{2n}

33. A relation R is defined from {2, 3, 4, 5} to {3, 6, 7, 10} by $x R y \Leftrightarrow$ x is relatively prime to y. Then domain of R is
(A) {2, 3, 5} (B) {3, 5}
(C) {2, 3, 4} (D) {2, 3, 4, 5}

34. The relation R = {(1, 1), (2, 2), (3,3), (1,2), (2,3), (1,3)} on set A = {1, 2, 3} is
(A) Reflexive buy not symmetric
(B) Reflexive but not transitive
(C) Symmetric and transitive
(D) Neither symmetric nor transitive

35. If A is the set of even natural numbers less than 8 and B is the set of prime numbers less than 7, then the number of relations from A to B is
(A) 2^9 (B) 9^2
(C) 3^2 (D) 2^{9-1}

36. Let ρ be the relation on the set R of all real numbers defined by setting a ρ b iff $|a - b| \leq \frac{1}{2}$.
Then, ρ is
(A) Reflexive and symmetric but not transitive
(B) Symmetric and transitive but not reflexive
(C) Transitive but neither reflexive not symmetric
(D) None of these

37. If A = {1, 3, 5, 7, 9, 11, 13, 15, 17}, B = {2, 4...., 18} and N is the universal set, then $A' \cup ((A \cup B) \cap B')$ is

(A) A (B) N
(C) B (D) None

38. If $X = \{8^n - 7n - 1/n \in N\}$ and $Y = \{49(n - 1)/n \in N\}$ then
(A) $X \subset Y$ (B) $Y \subset X$
(C) $X = Y$ (D) None

39. Let n be a fixed positive integer. Let a relation R be defined on I (the set of all integers) as follows : a R b iff n/(a – b), that is iff a – b is divisible by n. Then, the relation R is
(A) Reflexive only (B) Symmetric only
(C) Transitive only (D) An equivalence relation

40. Consider the set of all determinants of order 3 with entries 0 or 1 only. Let B be subset of A consisting of all determinants with value 1. Set C be the subset of the set of all determinants with value – 1. Then
(A) C is empty (B) B has as many elements as C
(C) $A = B \cup C$ (D) B has twice as many elements as C.

41. In a certain town 25% families own a phone and 15 % own a car, 65% families own neither a phone nor a car. 2000 families own both a car and a phone. Consider the following statements is this regard :
1. 10% families own both a car and a phone.
2. 35% families own either a car or a phone.
3. 40,000 families live in the town.

Which of the above statements are correct?
(A) 1 and 2 (B) 1 and 3
(C) 2 and 3 (D) 1, 2 and 3

42. Let A = {1, 2, 3, 4}, and let R = {(2, 2), (3, 3), (4, 4), (1, 2)} be a relation on A. Then R is
(A) Reflexive (B) Symmetric
(C) Transitive (D) None of these

43. The void relation on a set A is
(A) Reflexive (B) Symmetric and transitive
(C) Reflexive and symmetric (D) Reflexive and transitive

44. Let R be a relation on the set N of natural numbers defined by $nRm \Leftrightarrow n$ is a factor of m (i.e. $n \mid m$). Then R is
(A) Reflexive and symmetric (B) Transitive and symmetric
(C) Equivalence (D) Reflexive, transitive but not symmetric

45. For real numbers x and y, we write $x\ R\ y \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is
(A) Reflexive (B) Symmetric
(C) Transitive (D) None of these

46. Let X = {1, 2, 3, 4, 5} and Y = {1, 3, 5, 7, 9}. Which of the following is not a relation from X to Y
(A) $R_1 = \{(x, y) \mid y = 2 + x, x \in X, y \in Y\}$
(B) $R_2 = \{(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)\}$
(C) $R_3 = \{(1, 1), (1, 3), (3, 5), (3, 7), (5, 7)\}$
(D) $R_4 = \{(1, 3), (2, 5), (2, 4), (7, 9)\}$

47. Let R be a relation defined in the set of real numbers by a R b $\Leftrightarrow$ 1 + ab > 0. Then R is
(A) Equivalence relation (B) Transitive
(C) Symmetric (D) Anti-symmetric

48. Which one of the following relations on R is equivalence relation
(A) $x R_1 y \Leftrightarrow |x| = |y|$ (B) $x R_2 y \Leftrightarrow x \geq y$
(C) $x R_3 y \Leftrightarrow x \mid y$ (D) $x R_4 y \Leftrightarrow x < y$

49. Let R be a relation in N defined by $R = \{(1+x, 1+x^2) : x \leq 5, x \in N\}$. Which of the following is false
(A) R = {(2, 2), (3, 5), (4, 10), (5, 17), (6, 25)}
(B) Domain of R = {2, 3, 4, 5, 6}
(C) Range of R = {2, 5, 10, 17, 26}
(D) None of these

50. The relation R defined in A = {1, 2, 3} by aRb if $|a^2 - b^2| \leq 5$. Which of the following is false
(A) R = {(1, 1), (2, 2), (3, 3), (2, 1), (1, 2),(2, 3), (3, 2)}
(B) $R^{-1} = R$
(C) Domain of R = {1, 2, 3}
(D) Range of R = {5}

51. The relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)} on the set A = {1, 2, 3} is
(A) Reflexive but not symmetric (B) Reflexive but not transitive
(C) Symmetric and transitive (D) Neither symmetric nor transitive

52. Let a relation R in the set N of natural numbers be defined as (x, y) $\in$ R if and only if $x^2 - 4xy + 3y^2 = 0$ for all x, y $\in$ N. The relation R is
(A) Reflexive (B) Symmetric
(C) Transitive (D) An equivalence relation

53. Let A = {2, 3, 4, 5} and let R = {(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)} be a relation in A. Then R is
(A) Reflexive and transitive (B) Reflexive and symmetric
(C) Reflexive and antisymmetric (D) None of these

54. If A = {2, 3} and B = {1, 2}, then A $\times$ B is equal to
(A) {(2, 1), (2, 2), (3, 1), (3, 2)} (B) {(1, 2), (1, 3), (2, 2), (2, 3)}
(C) {(2, 1), (3, 2)} (D) {(1, 2), (2, 3)}

55. Let R {(3, 3), (6, 6), (9, 9), (12, 12) (6, 12) (3, 9) (3, 12), (3, 6)} be a relation on the set A = {3, 6, 9, 12}. The relation is
(A) An equivalence relation (B) Reflexive and symmetric only
(C) Reflexive and transitive only (D) Reflexive only

56. Let R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)} be a relation on the set A = {1, 2, 3, 4}. The relation R is
(A) Reflexive (B) Transitive
(C) Not symmetric (D) A function

57. Let N denote the set of all natural numbers and R be the relation on N$\times$N defined by (a, b) R (c, d) if ad (b + c) = bc (a + d), then R is-
(A) Symmetric only (B) Reflexive only
(C) Transitive only (D) An equivalence relation

58. If $A = \{1, 2, 3\}$, $B = \{1, 4, 6, 9\}$ and R is a relation from A to B defined by 'x is greater than y'. The range of R is
(A) $\{1, 4, 6, 9\}$ (B) $\{4, 6, 9\}$
(C) $\{1\}$ (D) None of these

59. Let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a releation on the set $A = \{1, 2, 3, 4\}$. The relation R is **[AIEEE-2004]**
(A) transitive (B) not symmetric
(C) reflexive (D) a function

60. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$, be relation on the set $A = \{3, 6, 9, 12\}$. The relation is **[AIEEE-2005]**
(A) reflexive and transitive only
(B) reflexive only
(C) an equilvalence relation
(D) reflexive and symmetric only

61. Let W denote the words in the English dictionary. Define the relation R by $R = \{(x, y) \in W \times W \mid$ the words x and y have at least one letter in common$\}$. Then R is **[AIEEE-2006]**
(A) reflexive, symmetric and not transitive
(B) reflexive, symmetric and transitive
(C) reflexive, not symmetric and transitive
(D) not reflexive, symmetric and transitive

62. Let R be the real line. Consider the following subsets of the plane $R \times R$:
$S = \{(x, y): y = x + 1 \text{ and } 0 < x < 2\}$
$T = \{(x, y) : x - y \text{ is an integer}\}$.
Which one of the following is true ? **[AIEEE-2008]**
(A) Both S and T are equivalence relations on R
(B) S is an equivalence relation on R but T is not
(C) T is an equivalence relation on R but S is not
(D) Neither S nor T is an equivalence relation on R

63. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then **[AIEEE-2009]**
(A) $A = B$ (B) $A = C$
(C) $B = C$ (D) $A \cap B = \phi$

64. Let S be a non-empty subset of R. Consider the following statement **[AIEEE 2010]**
P : There is a rational number $x \in S$ such that $x > 0$.
Which of the following statements is the negation of the statement P ?
(A) There is no rational number $x \in S$ such than $x \le 0$
(B) Every rational number $x \in S$ satisfies $x \le 0$
(C) $x \in S$ and $x \le 0 \Rightarrow x$ is not rational
(D) There is a rational number $x \in S$ such that $x \le 0$

65. Consider the following relations **[AIEEE-2010]**
$R = \{(x,y) \mid x,y$ are real numbers and $x = wy$ for some rational number $w\}$
$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right) \mid m,n,p \text{ and } q \text{ are integers such that } n, q \neq 0 \text{ and } qm = pm\right\}$. Then
(A) Neither R nor S is an equivalence relation

(B) S is an equivalence relation but R is not an equivalence relation
(C) R and S both are equivalence relations
(D) R is an equivalence relation but S is not an equivalene relation

REVIEW EXERCISES

1. Write the set $\{x \in R : x^2 - x - 6 \le 0\} \cap \left\{x \in R : \dfrac{1-x}{x+3} \ge 1\right\}$ in interval notation.
2. List the elements of the set $\left\{x \in Z : \min\left(x+2, 4-\dfrac{x}{3}\right) \ge 1\right\}$
3. Find the largest integer in the set {1, 2, 3,....., 2008} that does not belong to S.
4. Write the infinitely repeating decimal $0.\overline{123} = 0.123123123....$ as the quotient of two positive integers.
5. Find a rational number between the irrational numbers $\sqrt{2}$ and $\sqrt{3}$
6. Find an irrational number between the irrational numbers $\sqrt{2}$ and $\sqrt{3}$, There are infinitely many answers, One may take the average $\dfrac{\sqrt{2}+\sqrt{3}}{2}$,
7. Find an irrational number between the rational number $\dfrac{1}{10}$ and $\dfrac{1}{9}$
8. Find all real solutions to the equation
$(x + y)^2 = (x - 1)(y + 1)$.
9. Write the infinite repeating decimal $0.10101010... = 0.\overline{10}$ as a quotient of two integers.
10. Consider the decimal
0.10100100001000000001.....
where between successive 1′s there are 2^0, 2^1, 2^2, 2^3, etc. 0′s. Argue why this number must be irrational.
11. Express the set $\{x \in R : x^2 - x - 6 \le 0\}$ as an interval, or as a union of intervals. Draw a graph of the set on the line.
12. If $A = \left\{x : \cos x > -\dfrac{1}{2} \text{ and } 0 \le x \le \pi\right\}$

$B = \left\{x : \sin x > \dfrac{1}{2} \text{ and } \dfrac{\pi}{3} \le x \le \pi\right\}$

Find $A \cap B$ and $A \cup B$.
13. Let A = {a, b, c, d, e, f} and B = {a, e, i, o, u}. Find $A \cup B$, $A \cap B$, $A - B$ and $B - A$.
14. A survey of 500 television watchers produced the following information : 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games. How many watch all the three games? How many watch exactly one of the three games?

15. In a survey of 100 students, the numbers studying various languages were found to be : Spanish, 28 ; German, 30 ; French, 42 ; Spanish and German, 8; Spanish and French, 10 ; German and French, 5; all three languages, 3.
(A) How many students were studying no language?
(B) How many students had French as their only language ?
(C) How many students studied German if and only if they studied French ?

16. Derive the following results :
(A) $A = (A \cap B) \cup (A \cap \overline{B})$.
(B) $A \cup B = (A \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap B)$
(C) $A \cap (A \cup B) = A$
(D) $A \cup (\overline{A} \cap B) = A \cup B$.

17. In a chemistry class there are 20 students, and in a psychology class there are 30 students. Find the number in either the psychology class or the chemistry class if
(a) The two classes meet at the same hour.
(b) The two classes meet at different hours and 10 students are enrolled in both courses.

18. Assume that the incidence of lung cancer is 16 per 100,000, and that it is estimated that 75 percent of those with lung cancer smoke and 60 percent of those without lung cancer smoke. Estimate the fraction of smokers with lung cancer, and the fraction of nonsmokers with lung cancer.

19. Let N denote the set of all natureal numbers and R be the relation on N x N defined by (a,b) R (c,d) if ad(b+c) = bc (a+d) Show that R is an equivalence relation .

20. Let R be a relation in the set of natural numbers N, defined by,
$R = \{(x, y) , x \in N , y \in N \text{ and } 5x^2 - 6xy + y^2 = 0\}$
Show that the relation R is reflexive, not symmetric and not transitive .

21. If R is a relation in N x N defined by (a, b) R (c, d) if an only if a + d = b + c, then show that R is an equivalence relation .

22. A relation R on the set of non-zero complex numbers is defined by $z_1 R z_2$ if an only if $\frac{z_1 - z_2}{z_1 + z_2}$ is real . Show that R is an equivalence relation .

23. In the set of real numbers, a relation is defined by a R b if an only if 1 + ab > 0 . Show that R is reflexive and symmetric but not transitive .

24. Let R be a relation in N x N defined by, $R = \{(x, y) : x \in N , y \in N , x - y \text{ is divisible by } 3\}$ Prove that R is an equivalence relation .

25. Given a relation R = {(1, 2) , (2, 3)} on the set of natural numbers, write the minimum number of ordered pairs so that the enlarged relation is symmetric, transitive & reflexive.

TARGET EXERCISES

1. The report of one survey of 100 students stated that the numbers studing the various languages were : Sanskrit , hindi and tamil, 5 ; hindi and sanskrit , 10 ; tamil and sanskrit, 8 ; hindi and tamil ,20 ; sanskrit, 30 ; hindi, 23 ; tamil, 50. The surveyor who prepared this report was fired. Why ?

2. A class has 175 students. The following table shows the number of students studying one or more of the following subjects in this class :

Maths	100
Maths & Physics	30
Maths Physics Chemistry	18
Physics	70
Maths & Chemistry	28
Chemistry	46
Physics & Chemistry	23

How many students are enrolled in Maths alone, Physics alone and Chemistry alone ? Are there students who have not offered any of these three subjects .

3. In a certain city only two news papers A & B are published. It is known that 25% of the city population read A and 20% read B while 8% read both A & B. It is also known that 30% of those who read A but not B look into advertisement & 40% of those who read B but not A look into advertisements while 50% of those who read both A & B look into advertisement. What percentage of the population read an advertisement .

4. The following three relations are defined on the set of natural numbers N .

$R = \{(x,y)/\ x < y,\ x \in N,\ y \in N\}$

$S = \{(x,y)/\ x + y = 10,\ x \in N,\ y \in N\}$

$T = \{(x,y)/\ x = y \text{ or } x\text{-}y=1,\ x \in N,\ y \in N\}$.

Explain clearly which of the above relations are

(i) Reflexive

(ii) Symmetric

(iii) Transitive

5. In a pollution study of 1500 Indian rivers the following data were reported . 520 were polluted by Sulphur compounds, 335 were polluted by phosphates, 425 were polluted by crude oil, 100 were polluted by both crude oil and sulphur compounds, 180 were polluted by both sulphur compounds and phosphates, 150 were polluted by both phosphates and crude oil and 28 were polluted by sulphur compounds, phosphates and crude oil. How many of the rivers were polluted by at least one of the three impurities ? How many of the rivers were polluted by exactly one of the three impurities ?

6. Let R be the relation defined on the set of natural numbers N as ;

$R = \{\ (x,y) \mid\ x \in N,\ y \in N,\ 2x + y = 41\ \}$.

Find the domain & range of this relation R . Also verify whether R is :

(i) Reflexive

(ii) Symmetric

(iii) Transitive

7. Let $S = \{1,2,3,4,5\}$ and let $A = S \times S$. Define the relation R on A as follows : "(a,b) R (c,d) if and only if ad = cb ". Show that R is an equivalence relation. Compute the collection of all equivalence classes .

8. A survey of 500 TV watches produced the following informations : 285 watch football, 195 watch hockey, 115 watch basket ball, 45 watch football and basket ball, 70 watch football and

hockey, 50 watch hockey and basketball , 50 do not watch any of the three games . How many watch all the three games ? How many watch exactly one of the three games .

9. Suppose that in a survey concerning the reading habits of students it is found that :
60 Percent read magazine A ; 50 percent read magazine B ; 50 percent read magazine C ; 30 percent read magazines A and B ; 20 percent read magazines B and C ; 30 percent read magazines A and C ; 10 percent read all three magazines.
   (a) What percent read exactly two magazines ?
   (b) What percent do not read any of the magazines?
   (c) The survey has been enlarged to include a fourth magazine D. It was found that no one who reads either magazine A or magazine B reads magazine D. However, 10 percent of the people read magazine D and 5 per cent read both C and D. What per cent of the people do not read any magazine ?

10. In a survey of population of 450 people, it is found that 205 can speak English, 210 can speak Hindi and 120 people can speak Urdu. If 100 people can speak both Hindi and English; 80 people can speak both English and Urdu, 35 people can speak Hindi and Urdu and 20 people can speak all the three languages, find the number of people who can speak English but not Hindi or Urdu. Find also the number of people who can speak neither English nor Hindi nor Urdu.

11. On a transcontinental airliner, there are 9 boys, 5 American children, 9 men, 7 foreign boys, 14 Americans, 6 American males, and 7 foreign females. What is the number of people on the plane?

12. A college awarded 38 varsity letters in football, 15 in basketball, and 20 in baseball. If these letters went to a total of 58 men and only three of these men lettered in all three sports, how many men received letters in exactly two of the three sports?

13. Let U be a finite set. For any two sets A and B define the "distance" from A to B to be d (A, B) $= n(A \cap \overline{B}) + n(\overline{A} \cap B)$.
   (a) Show that $d(A, B) \geq 0$. When is $d(A, B) = 0$?
   (b) If A, B and C are nonintersecting sets, show that $d(A, C) \leq d(A, B) + d(B, C)$
   (c) Show that for any three sets A, B and C $d(A, C) \leq d(A, B) + d(B, C)$

14. Let the symbol $n_r(k)$ stand for "the number of elements that are in k or more of the r sets A_1, A_2, ..., ..., A_r".
   (i) Show that $n_3^{(1)} = n(A_1 \cup A_2 \cup A_3)$.
   (ii) Show that
   $n_3^{(2)} = n((A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3))$
   $= n(A_1 \cap A_2) + n(A_1 \cap A_3) + n(A_2 \cap A_3) - 2n(A_1 \cap A_2 \cap A_3)$
   by using the inclusion-exclusion formula. Also develop an independent argument for the last formula.
   (iii) Find the number of letters that appear two or more times in the three words TABLE, BASIN, and CLASP.
   (iv) Give an interpretation for $n_3^{(1)} - n_3^{(2)}$.
   (v) Find the number of letters that occur exactly once in the three words of (iii).
   (vi) Develop a general argument like that in (i) to show that

$n_4^{(2)} = n(A_1 \cap A_2) + n(A_1 \cap A_3) + n(A_1 \cap A_4) + n(A_2 \cap A_3) + n(A_2 \cap A_4) + n(A_3 \cap A_4)$
$- 2[n(A_1 \cap A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_4 + n(A_1 \cap A_3 \cap A_4) + (A_2 \cap A_3 \cap A_4)]$
$+ 3n(A_1 \cap A_2 \cap A_3 \cap A_4)$

(vii) Find the number of letters used two or more times in the four words of (iii) .

15. Let W = {1, 2, 3, 4}. Consider the following relations in W :

$R_1 = \{(1, 1), (1, 2)\}$
$R_2 = \{(1, 1), (2, 3), (4, 1)\}$
$R_3 = \{(1, 3), (2, 4)\}$
$R_4 = \{(1, 1), (2, 2), (3, 3)\}$
$R_5 = W \times W$
$R_6 = \Phi$

(i) Determine whether or not each relation is reflexive.
(ii) Determine whether or not each relation is symmetric.
(iii) Determine whether or not each relation in is transitive.

16. Let N = {1, 2, 3, ...} and let $\cong$ be the relation in N × N defined by $(a, b) \cong (c, d)$ iff $a + d = b + c$

(i) Prove that $\cong$ is an equivalence relation.
(ii) Find the equivalence class of (2, 5), i.e. [(2, 5)].

17. Prove : if R and S are equivalence relations in a set X, then $R \cap S$ is also an equivalence relation in X.

18.

(i) Show that if relations R and S are each reflexive and symmetric, then $R \cup S$ is also reflexive and symmetric.
(ii) Give an example of transitive relations R and S for which $R \cup S$ is not transitive.

19. The set S is formed according to the following rules :

(i) 2 belongs to S;
(ii) if n is in S then n + 5 is also in S;
(iii) If n is in S then 3n is also in S.

20. Let N denote the set of all natural numbers and R be the relation on N × N defined by $(a,b)R(c,d) \Leftrightarrow ad(b + c) = bc(a + d)$. Check whether R is an equivalence relation.

11. Let R = {(a,a,), (b,c), (a,b)} be a relation on a set A = {a,b,c}. Then find the minimum number of ordered pairs which when added to R makes it transitive.

22. Define a relation '$\equiv$ (mod 5)' on I by $m \equiv n \pmod 5$ if $m - n$ is divisible by 5.
Prove that this is an equivalence relation.

23. Let A = N × N. Define ~ on A by
$(m, n) \sim (m', n')$ if $m + n' = m' + n$
Prove that this is an equivalence relation.

24. Define $(m, n) \sim (m', n')$ on N × N if $mn' = m'n$. Show that ~ is an equivalence relation.
Prove that this is not an equivalence relation on I × I.

25. If R be a relation < from A = {1, 2, 3, 4} to B = {1, 3, 5} i.e, $(a, b) \in R$ iff $a < b$. Then find RoR^{-1}.

26. Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

27. The relation R is defined on N × N as follows:
(a, b) R (c, d) $\Leftrightarrow$ ad = bc
Prove that R is an equivalence relation.

ANSWERS

Practice Problems [A]

1. $x : x = \dfrac{n}{n+1}$, where n is a natural number and $1 \le n \le 6$.

2. (i) A = {17, 26, 35, 44, 53, 62, 71, 80} (ii) B = {2, 3, 5}
(iii) C = {T, R, I, G, O, N, M, E, Y}

3. (i) $\{x : x = 2^n, n \in N \text{ and } 1 \le n \le 5\}$ (ii) $\{x : x = n^2, n \in N \text{ and } 1 \le n \le 10\}$

4. {–9, –6, –3, 0, 3, 6, 9}

5. $x \in \{13, 14, 15, 16, 17\}$

6. A = {..., –2, –1, 0, 1, 2, 3, 4}.B = {1, 2, 3, 4, 6, 8, 12, 24}

7. $A = \{x^2 : x \in N, 1 \le x \le 4\}$.
$B = \{2x : x \in W, 0 \le x \le 4\}$.
$C = \{2x + 1 : x \in W\}$.
$D = \{x \in N : 1 \le x \le 15$ and x is prime$\}$.
These descriptions are not unique; here are other possibilities:
$A = \{x^2 : x \in Z, 1 \le x \le 4\}$.
$B = \{\frac{x}{2} : x \in \{0, 4, 8, 16, 32\}\}$.
$C = \{2x - 1 : x \in N\}$.
$D = \{x \in N : 2 \le x \le 13$ and x is prime$\}$.

8. (ii)

9. (i) infinite (ii) finite
(iii) infinite

10. (i) finite, (ii) finite,
(iii) infinite

11. (i) finite, (ii) finite,
(iii) infinite

Practice Problems [B]

1. (i) C and E (ii) D and E
(iii) A, B and D (iv) None

2. (i) $\not\subset$ (ii) $\not\subset$

3. (i) No (ii) Yes

5. $\{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, , \{a, c, d\}, \{a, b, c, d\}\}$.

It has 16 elements.

6. sets $\{2, 3\}$, $\{2\}$ and $\{5, 6\}$.

Practice Problems [C]

1. $A' = \{x : x \in R \text{ and } x \geq 2\}$

$B' = \{x : x \in R \text{ and } x > 3\}$

$A \cup B = \{x : x \in R \text{ and } 0 < x \leq 3\}$

$A \cap B = \{x : x \in R \text{ and } 1 < x < 2\}$

$A - B = \{x : x \in R \text{ and } 0 < x \leq 1\}$

2. (i) $\{1, 2, 3, 4, 5, 6\}$ (ii) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

(iii) $\{3, 4, 5, 6, 7, 8, 9, 10\}$

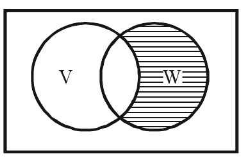
3. (i) $\{7, 9, 11\}$ (ii) ϕ

(iii) $\{7, 9, 11\}$ (iv) $\{7, 9, 11\}$

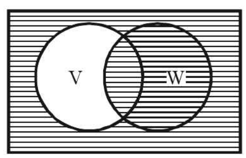
4. (i) $\{a, c\}$ (ii) $\{f, g\}$

(iii) $\{b, d\}$

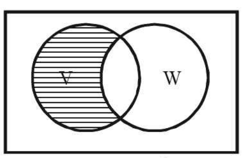
5.

(i) 

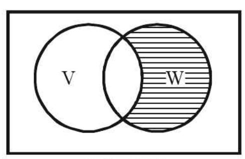
$W - V$

(ii) 

$V^c \cup W$

(iii) 

$V \cap W^c$

(iv) 

$V^c - W^c$

6. (i) $\{d, e, f, g, h\}$ (ii) $\{a, b, c, h\}$

8. A' is the set of all equilateral triangles.

9. False

12. (i) $S \cup W$ (ii) $T' \cup W'$

(iii) $(M \cup T \cup S)'$

13. 25, 35

14. (i) 52 (ii) 30

15. 48, 1

16. 5, 2, 12, 60

Practice Problems [D]

1. $x = 2, y = 1$.
2. $A \times B = \{(2, 4), (2, 8), (2, 12), (4, 4), (4, 8), (4,12), (6,4), (6, 8), (6, 12)\}$
   $B \times A = \{ (4, 2), (4, 4), (4, 6), (8, 2), (8, 4), (8, 6), (12, 2), (12, 4), (12, 6)\}$
3. Both are true
4. $\{1, 2, 3, 4, 5, 6\}, \{3, 4\}$
6. R is not reflexive, not symmetric, transitive
   S is not reflexive, symmetric, not transitive
   T is reflexive, not symmetric, not transitive
7. Domain of R = $\{1, 2, 3, 4,....20\}$ and Range of R = $\{39, 37, 35, 33, ..., 5, 3, 1\}$.
8. All equivalence classes are as follows :
   $\{(1,1), (2,2), (3,3), (4,4,), (5,5)\}$; $\{(1,2), (2,4)\}$; $\{(2,1), (4,2)\}$;$\{(1,3), (1,4)\}$; $\{(1,5)\}$; $\{(2,3\}$; $\{(3,1)\}$, $\{(3,2)\}$; $\{(3,4)\}$; $\{(3,5\}$; $\{(4,1), (4,3)\}$; $\{(4,5)\}$; $\{(5,1\}$; $\{(5,2)\}$, $\{(5,3)\}$; $\{(5,4)\}$.
9. (i) Neither reflexive nor symmetric nor transitive
   (ii) Neither reflexive nor symmetric but transitive
   (iii) Reflexive and transitive but not symmetric
   (iv) Reflexive, symmetric and transitive
12. Neither reflexive nor symmetric nor transitive
15. (i) $\{1, 5, 9\}$, (ii) $\{1\}$

Practice Problems [E]

1. (i) No. There is nothing assigned to the element $b \in A$. (ii) No. Two elements, x and z, are assigned to $c \in A$. (iii) Yes.
2. (i) Yes, (ii) No,
   (iii) Yes (iv) No.
3. (i) No (ii) No
   (iii) Yes (iv) No
4. (i) $g(5) = 10$, (ii) $g(0) = 2$,
   (iii) $g(-2) = 0$.
5. $g(x) = 2x - 1$
6. $f(A) = \{-2, 0, 18, 28, 108\} \neq B$.
7. Yes

Objective Exercises

1. A	**2.** C	**3.** C	**4.** A	**5.** C
6. B	**7.** A	**8.** B	**9.** D	**10.** B
11. C	**12.** B	**13.** B	**14.** C	**15.** B
16. C	**17.** A	**18.** B	**19.** B	**20.** C
21. B	**22.** B	**23.** B	**24.** B	**25.** C
26. A	**27.** B	**28.** B	**29.** D	**30.** A
31. C	**32.** D	**33.** D	**34.** A	**35.** A
36. A	**37.** B	**38.** A	**39.** D	**40.** B
41. C	**42.** C	**43.** B	**44.** D	**45.** A
46. D	**47.** C	**48.** A	**49.** A	**50.** D
51. A	**52.** A	**53.** B	**54.** A	**55.** C
56. C	**57.** D	**58.** C	**59.** D	**60.** A
61. A	**62.** C	**63.** C	**64.** B	**65.** B

Review Exercises

1. $[-2, -1]$

2. $\{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

3. 2005.

4. $\dfrac{41}{333}$

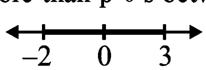
5. There are infinitely many answers. Since $\sqrt{2} < 1.5$ and $1.7 < \sqrt{3}$. We may take , say 1.6 of course, using the mentioned inequalities we may take also 1.61, 1.601, 1.52 etc.

7. Since $\dfrac{1}{10} = 0.1$ and $0.111 < \dfrac{1}{9}$, we may take say 0.1101001000010000000001... where there are 2^k, k = 1, 2,....0's between consecutive 1's. Another approach can be taking $\sqrt{2} - 1.314$, since $\sqrt{2} - 1.314 < 0.1003$.

8. No solution

9. $\dfrac{10}{99}$

10. If this number were rational, then eventually, it would be a repeating decimal, say, with period p. But we can then take a sufficiently large power of 2, larger than p, and so eventually, there would be more than p 0′s between 1′s, a contradiction.

11. $x \in [-2, 3]$, 

12. $A \cap B = \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$, $A \cup B = \left[0, \frac{5\pi}{6}\right)$

13. $A \cup B = \{a, b, c, d, e, f, i, o, u\}$, $A \cap B = \{a, e\}$.
$A - B = \{b, c, d, f\}$, $B - A = \{i, o, u\}$

14. 20, 325

15. (a) 20 (b) 30
(c) 38

17. (a) 50 (b) 40

18. 20 and 10 per 100,000

25. [(1, 1) ; (1, 2) ; (2, 1) ; (2, 3) ; (3, 2) ; (1, 3) ; (3, 1) ; (2, 2) ; (3, 3)]

TARGET EXERCISES

2. 60 ; 35 ; 13 and 22

3. 13.9%

4. R is only reflexive ; S is only symmetric and T is only transitive .

5. 878,504

6. R is neither reflexive nor symmetric nor transitive .
$D = \{x \mid x \in N, x \le 20\}$
$R = \{1, 3, 5, \ldots\ldots, 39\}$

8. 20, 325

9. (a) 50 (b) 10
(c) 5 per cent

10. 45, 110

11. 33

12. 9

15. (i) R_5 is the only reflexive relation.
(ii) R_4, R_5 and R_6 are the only symmetric relations.
(iii) All the relations are transitive.

16. (ii) $[(2, 5)] = \{(a, b) : a + 5 = b + 2,\ a, b \in N\}$
$= \{(a, a + 3) : a \in N\} = \{(1, 4), (2, 5), (3, 6), (4, 7), \ldots\}$

18. (ii) $R = \{(1, 2)\}$ and $S = \{(2, 3)\}$.

21. $\{(b,b), (c,c), (c,b), (b,a)\}$

25. $RoR^{-1} = \{(3, 3)\ (3,5).\ (5,3), (5,5)\}$.

26. T_1 is related to T_3.

CHAPTER 4

FUNCTIONS

4.1 | INTRODUCTION

Let A and B be two non-empty sets. Then a function 'f ' from set A to set B denoted as

$f : A \to B$

$y = f(x), \quad x \in A, y \in B$

is a rule which associates each element of set A to a unique (single) element of set B.

Note that $f : A \to B$ is read as f is a function from A to B or f maps A to B.

The symbol x, which represents an element in set A is called the independent variable, and the symbol y which represents the element corresponding to x is called the dependent variable. This nomenclature is based on the fact that the value of x can be arbitrarily chosen. Then y has a value, which depends upon the chosen value of x. y is called the value of the function at x or the image of x under f, and is denoted by f(x), read as 'f of x'.

The value f(x) of a function f at x is also called the output and x is called the input or argument.

If a member 'a' of set A is associated to the member 'b' of set B, then 'b' is called the image of 'a' and we write b = f(a). Further 'a' is called a pre-image of 'b'.

Mathematically, the function is a special relation with two conditions:

(i) For every $x \in A$ there is some $y \in B$ such that $(x, y) \in f$.

(ii) If $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.

Thus, every function is a relation but every relation is not a function. Terms such as "map" (or mapping), "correspondence" are also used for functions.

The symbol $A \xrightarrow{f} B$ can also be used to denote the function.

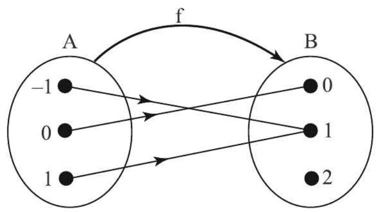
Consider $A \equiv \{-1, 0, 1\}$ and $B \equiv \{0, 1, 2\}$.

Then $A \times B = \{(-1, 0), (-1, 1), (-1, 2), (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$

Now, "$f : A \to B$ defined by f(x) = x2" is the function such that $f = \{(-1, 1), (0, 0), (1, 1)\}$.

f can also be represented diagrammatically as follows:

We represent the two sets A and B by the interiors of two circles. The function $f : A \to B$ is represented by means of a collection of arrows joining the points which represent the elements of A to points representing the corresponding elements of B.



Consider another example:
Let A = {a, b, c, d} and B = {t, x, y, z}.
Let $f : A \to B$ be defined by the correspondence $f(a) = y$, $f(b) = x$, $f(c) = z$ and $f(d) = y$.

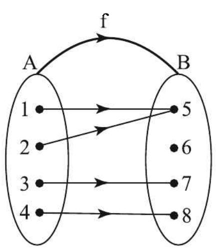
Note

By the definition of function, it is obvious that

(i) every point in A is joined to some point in B by an arrow;

(ii) a point in A cannot be joined to two or more distinct points in B;

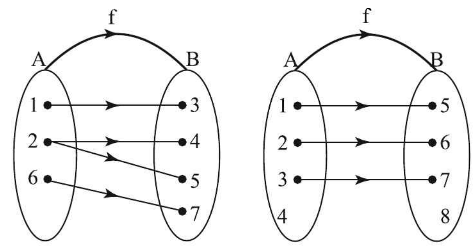
(iii) two or more points in A may be joined to the same point in B;



(iv) there may be some points in B which are not joined to any point in A.

(v) $f : A \to B$ is not a function if any element of A is associated with more than one element of B, or any element of A is not associated with any of the elements of B.

For example, the following relations are not functions:



EXAMPLE 4.1 If X = {a, b, c, d, e} and Y = {p, q, r, s, t} then which of the following subset(s) of X × Y is/are a function from X to Y.

(i) {(a, r) (b, r) (b, s) (d, t) (e, q) (c, q)}

(ii) {(a, r) (b, p) (c, t) (d, q)}

(iii) {(a, p) (b, t) (c, r) (d, s) (e, q)}

(iv) {(a, r) (b, r) (c, r) (d, r) (e, r)}

SOLUTION We check for the two conditions of the function.

(i) Since b has two outputs (images) namely r and s, it is not a function.

(ii) Since e, an element of X, does not have any image, it is also not a function.

(iii) Since every element of X has one and only one output, it is a function.

(iv) Even if every element's output is r, it is a function.

In a relation from A to B an element of A may be related to more than one element in B. Also there may be some elements of A which may not be related to any element in B. But in a function from A to B each element of A must be associated to one and only one element of B.

Let A = {1, 2, 3, 4}, B = {a, b, c}.
Let R = {(1, a), (2, a), (3, b), (4, b)}.
Then R is a function from A to B. Obviously R is also a relation from A to B. But consider the subset S of A × B given by

S = (1, a), (2, b), (3, b), (4, b)}.

Here S is a relation from A to B. But S is not a function from A to B. The obvious reason is that

the element $1 \in A$ is associated to two different elements a and $c \in B$.

Here, set 'A' is called domain of f and set 'B' is called co-domain of f. If R is a relation from A to B then domain of R may be a subset of A. But if f is a function from A to B, then domain of f is equal to A.

Domain of a function $y = f(x)$ is the set of all values of x for which the function f is defined.

If both the inputs and outputs of a function are real numbers, we shall call the function, a real function of a real variable or simply a real valued function.

The domain of a real function is a subset of R such that f(x) is real.

The domain of f is denoted as D_f or Dom(f).

Co-domain is a set of values of y which are likely to be the output values of the function $y = f(x)$ under the given domain.

It covers all output values of the function, though it may also contain some elements which are not involved in the mapping.

Range is the set of all values of y which are the actual output values of the function $y = f(x)$ under the given domain. It is denoted by f(A) or Ran(f), where A is the domain of the function.

Mathematically, the set $\{f(a) : \forall\, a \in A\}$, which is the set of all images of the function, is called the range of f.

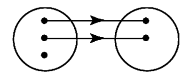
Note that range is a subset of co-domain i.e. $f(A) \subseteq B$.

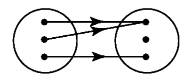
Note

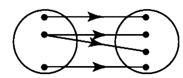
Let $f : X \to Y$ be a function and let $A \subseteq X$, $B \subseteq X$. Then

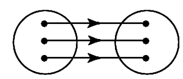
1. $A \subseteq B \Rightarrow f(A) \subseteq f(B)$
2. $f(A \cup B) = f(A) \cup f(B)$
3. $f(A \cap B) \subseteq f(A) \cap f(B)$
4. $f(A) - f(B) \subseteq f(A - B)$

EXAMPLE 4.2 Which of the following diagrams represents a function?

(i) 

(ii) 

(iii) 

(iv) 

SOLUTION

(ii) and (iv). In (i) one element of domain has no image, while in (iii) one element of domain has two images in co-domain.

EXAMPLE 4.3 Which of the following correspondences can be called a function ?

(i) $f : \{-1, 0, 1\} \to \{0, 1, 2, 3\}, f(x) = x^3$

(ii) $f : \{0, 1, 2\} \to \{-2, -1, 0, 1, 2\}, f(x) = \pm\sqrt{x}$

(iii) $f : \{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}, f(x) = \sqrt{x}$

(iv) $f : \{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}, f(x) = -\sqrt{x}$.

SOLUTION f(x) in (iii) and (iv) are functions as definition of function is satisfied.

In case of (i) the given correspondence is not a function, as $f(-1) \notin$ co-domain. Hence definition of function is not satisfied.

In case of (ii), the given relation is not a function, as $f(1) = \pm 1$ and $f(4) = \pm 2$ i.e. element 1 as well as 4 in domain are related with two elements of co-domain. Hence, the definition of function is not satisfied.

EXAMPLE 4.4 Decide whether or not the following are functions from A to B where $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e\}$. If they are functions, give the range of each. If they are not, explain why ?

(i) $f = \{(1, a), (2, b), (3, b), (5, e)\}$.

(ii) $g = \{(1, e), (5, d), (3, a), (2, b), (1, d), (4, a)\}$.

(iii) $h = \{(5, a), (1, e), (4, b), (3, c), (2, d)\}$.

SOLUTION (i) Since the element $4 \in A$ is not associated to any element $\in B$, therefore f is not a function from A to B.

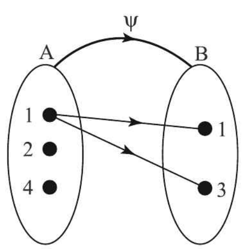
(ii) The element $t \in A$ is associated to two different elements e and $d \in B$. Therefore g is not a function from A to B.

(iii) Each element of A is associated to a unique element of B. Therefore h is a function from $A \to B$. The range of h is the set of the h-images of all elements of A. So range of h is $h(A) = \{a, e, b, c, d\} = B$.

EXAMPLE 4.5 Let $A = \{1, 2, 4\}$ and $B = \{1, 3\}$. Let $\psi = \{(x, y): x + y$ is an even number, $x \in A$, $y \in B\}$. Does ψ represent a function ?

SOLUTION We find all possible ordered pairs such that the sum of abscissa and ordinate is even.

$\psi = \{(1, 1), (1, 3)\}$



From the mapping it is clear that ψ is a relation but not a function.

Note

Any complete description of a function must include a description of its domain. Sometimes, a function is stated compactly as "$y = f(x)$" without stating its domain and co-domain.

In such a case, the domain of f is the set of all real values of x for which the expression f (x) produces a finite real number ; and the co-domain is considered as R.

For example, the domain of the function f defined by

$$f(x) = \frac{1}{x^2 - x - 2} = \frac{1}{(x+1)(x-2)}$$

is assumed to be the entire set of real numbers with the exception of – 1 and 2.

The domain of a function should be sought for using the basic properties of the elementary functions.

(i) Expressions like 1/0 and 0/0 are inadmissible. These have no meaning.

If a function has the rational form $p(x)/q(x)$, then denominator $q(x) \neq 0$.

An expression like $\frac{1}{x-1}$ or $\frac{x^2-1}{x-1}$ fails to give a definite value when $x = 1$.

(ii) An even root $\sqrt[2n]{x}$ ($n = 1, 2, ...$) is only defined for non-negative values of x. Hence, an expression under even root should be non-negative. For example, the function $\sqrt{(x^2+3x-2)}$ is real when $x^2 + 3x - 2 \geq 0$.

(iii) The expression under even root in the denominator of a function should be a positive number. For example the function $\frac{1}{\sqrt{(x^2+3x-2)}}$ is real when $x^2 + 3x - 2 > 0$. Note that zero value of expression is not permitted in the denominator.

(iv) Logarithm $\log_a x$ is only defined for positive values of x and for bases $a > 0$, $a \neq 1$.

(v) The inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ are only defined for $-1 \leq x \leq 1$. $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$ are defined for $x \leq -1$ and $x \geq 1$.

EXAMPLE 4.6 Find the domain and range of $f(x) = 2 + \sqrt{x-1}$.

SOLUTION For $2 + \sqrt{x-1}$ to be meaningful, the square root of $x - 1$ must make sense; that is, $x - 1$ must not be negative. Thus the domain consists of all numbers x such that $x - 1 \geq 0$ or, equivalently, $x \geq 1$.

That is, the domain is the interval $[1, \infty)$.

As x varies from 1 to larger numbers, f(x)

increases from $f(1) = 2 + \sqrt{1-1}$ to arbitrarily large values. Thus the range of f consists of all numbers greater than or equal to 2, that is, the interval $[2, \infty)$.

Graph of Function

In the case of function whose domain is a subset of the set of real numbers and the range is the set of real numbers, we can represent it by a graph in R^2, the Cartesian product of R with itself, which is often called the Euclidean plane. Indeed, as the function is a collection of ordered pairs, the collection of $\{(x, y) \in f : x, y \in R\}$ is a set of points in the plane R^2.

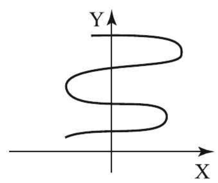
We recall that once we fix the coordinate axes, a point in the plane can be represented by an ordered pair of real numbers and conversely, every ordered pair of real numbers represents a point on the plane. Thus, if we have a function with domain and range as subsets of the set of real numbers, then the function corresponds to a set of points in the plane, the domain of the function being a subset of the abscissa and the range a subset of the ordinate. This collection of points $\{(x, y) \in f : x, y \in R\}$, which is simply the relation defining f, is called the graph of the function f.

By the definition of the graph of a function, the x-axis contains the set of inputs and y-axis has the set of outputs. The domain of a functional curve is the "shadow" of the graph on the x-axis, and the range is the "shadow" of the graph on the y-axis.

CAUTION One should notice that the two different functions:
$f : R \to R$, $f(x) = x^2$ and $g : R \to [0, \infty)$, $g(x) = x^2$ possess the same graph. It is then difficult to recover all the information about a function from its graph, in particular, it is impossible to recover its co-domain.

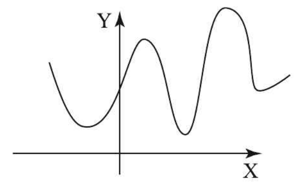
Not every curve is the graph of a function. For instance, the curve in Figure 1 is not the graph of a function. The reason is that a function assigns to a given input, a single number as the output. A line parallel to the y axis therefore meets the graph of a function in atmost one point. This observation provides a visual test for deciding whether a curve in a plane is the graph of a function $y = f(x)$.

If some line parallel to the y axis meets the curve more than once, then the curve is not the graph of a function. Otherwise it is the graph of a function. We will call this the vertical line test for a function. The curve in Figure 2 is the graph of a function.



Not the graph of a function

Figure 1



Graph of a function

Figure 2

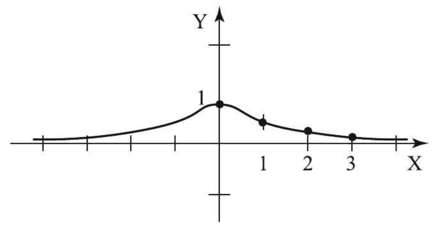
EXAMPLE 4.7 Graph the function
$f(x) = 1/(1 + x^2)$.

SOLUTION Since $1 + x^2$ is never 0, the domain of the function consists of the entire x axis. Let us pick a few convenient inputs x and calculate the corresponding outputs, as shown in the table :

x	0	1	2	3
$f(x) = \dfrac{1}{1+x^2}$	1	$\dfrac{1}{2}$	$\dfrac{1}{5}$	$\dfrac{1}{10}$

For any x, $x^2 \geq 0$, so $1 + x^2 \geq 1$ and $\dfrac{1}{1+x^2} \leq 1$.

Since x appears only to an even power, the graph is symmetric with respect to the y axis; there is no need to evaluate the function for negative x. Plotting the four calculated points suggests the general shape of the graph. (See Fig.). When |x| is large, $y = 1/(1 + x^2)$ is small. This means that for large |x| the graph approaches the x axis.



Note that the range of the function consists of all positive numbers less than or equal to 1.

EXAMPLE 4.8 The function f(x) is defined as follows : on each of the intervals $n \le x < n + 1$, where n is a positive integer, f(x) varies linearly and $f(n) = -1$,

$f\left(n+\frac{1}{2}\right) = 0$. Draw the graph of the function.

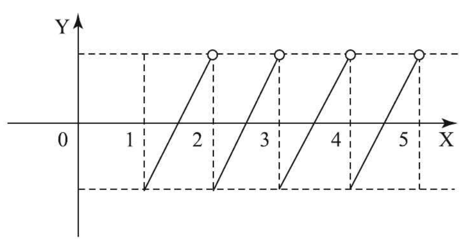
SOLUTION $f(x) = ax + b,\ 1 \le x < 2$

$$f(n) = a.n + b$$
$$f(1) = 2 \times 1 + b$$
$$-1 = a.n + b$$
$$-1 = 2 + b$$
$$b = -3 \qquad \ldots(1)$$
$$f(n + 1/2) = a(n + 1/2) + b \qquad \ldots(2)$$
$$0 = a(1 + 1/2) + b$$
$$0 = a(3/2) - 3 \qquad \text{[using (1)]}$$
$$a = 2.$$

Hence, $f(x) = 2x - 3,\ 1 \le x < 2$

Similarly, we can define f in other intervals.

The graph of f is shown below:



EXAMPLE 4.9 Let $f: R \to R, f(x) = \frac{x-1}{x^2+1}$ Find $f(0) + f(1) + f(2)$ and $f(0 + 1 + 2)$?

Is there a real solution to the equation $f(x) = \frac{1}{x}$?

Is there a real solutiton to the equation $f(x) = x$?

SOLUTION We have

$$f(0) + f(1) + f(2) = \frac{0-1}{0^2+1} + \frac{1-1}{1^2+1}$$
$$= -1 + 0 + \frac{1}{5} = \frac{4}{5}$$

Also, $\quad f(0 + 1 + 2)$

$$f(3) = \frac{3-1}{3^2+1} = \frac{1}{5}$$

Clearly then $f(0) + f(1) + f(2) \ne f(0 + 1 + 2)$

Now, $\quad f(x) = \frac{1}{x} \Rightarrow x^2 - x = x^2 + 1 \Rightarrow x = -1$

Also, $\quad f(x) = x \Rightarrow x - 1 = x^3 + x \Rightarrow x = -1.$

EXAMPLE 4.10 If $f(x) = \frac{1+x}{1-x}$, show that

$$\frac{f(x).f(x^2)}{1+[f(x)]^2} = \frac{1}{2}$$

SOLUTION $f(x^2) = \frac{1+x^2}{1-x^2} = \frac{1+x^2}{(1+x)(1-x)}$

Therefore,

$$\frac{f(x).f(x^2)}{1+[f(x)]^2} = \frac{\left(\frac{1+x}{1-x}\right)\frac{(1+x^2)}{(1+x)(1-x)}}{1+\frac{(1+x)^2}{(1-x)^2}}$$

$$= \frac{\frac{1+x^2}{(1-x)^2}}{\frac{(1-x)^2+(1+x)^2}{(1-x)^2}}$$

$$= \frac{\frac{1+x^2}{(1-x)^2}}{\frac{(1-x)^2+(1+x)^2}{(1-x)^2}}$$

$$= \frac{1+x^2}{2+2x^2} = \frac{1}{2}.$$

EXAMPLE 4.11 If $f(x) = \cos\{\log(x)\}$, then prove that

$$f(xy) + f\left(\frac{x}{y}\right) = 2f(x)f(y).$$

SOLUTION Here, we evaluate each term of the left hand side of the equation separately and then combine the result.

$f(xy) = \cos\{\log_e(xy)\} = \cos(\log_e x + \log_e y)$

$f\left(\frac{x}{y}\right) = \cos\{\log_e\left(\frac{x}{y}\right)\} = \cos(\log_e x - \log_e y)$

Substituting in the LHS expression, we have :

$$\{f(xy) + f\left(\frac{x}{y}\right)\} = \cos(\log_e x + \log_e y) + \cos(\log_e x - \log_e y)$$

We know that :

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Now, $\{f(xy) + f\left(\frac{x}{y}\right)\}$

$$= 2\cos\left(\frac{\log_e x + \log_e y + \log_e x - \log_e y}{2}\right)\cos\left(\frac{\log_e x + \log_e y - \log_e x + \log_e y}{2}\right)$$

$= 2\cos(\log_3 x)\cos(\log_e y)$

Hence, $\{f(xy) + f\left(\frac{x}{y}\right)\}$

$= 2\cos(\log_e x)\cos(\log_e y) = 2f(x)f(y).$

EXAMPLE 4.12 Let $f : \text{Dom}(f) \to R$ be a function f is said to have a fixed point at $t \in \text{Dom}(f)$ if $f(t) = t$.
Let $f : [0, \infty) \to R$, $f(x) = x^5 - 2x^3 + 2x$. Find all fixed points of f.

SOLUTION We must look for all $x \in \text{Dom}(f)$ such that $f(x) = x$. Thus

$$\begin{aligned} f(x) = x \quad &\Rightarrow x^5 - 2x^3 + 2x = x \\ &\Rightarrow x^5 - 2x^3 + x = 0 \\ &\Rightarrow x(x^4 - 2x^2 + 1) = 0 \\ &\Rightarrow x(x^2 - 1)^2 = 0 \\ &\Rightarrow x(x+1)^2(x-1)^2 = 0 \end{aligned}$$

The solutions to this last equation are $\{-1, 0, 1\}$. Since $-1 \notin \text{Dom}(f)$, the only fixed points of x are $x = 0$ and $x = 1$.

EXAMPLE 4.13 $f : R \to R$ is a function satisfying

$f(3) = 2$ and $f(x + 3) = f(3)f(x)$. Find $f(-3)$.

SOLUTION Since we are interested in $f(-3)$, we first put $x = -3$ in the relation, obtaining $f(0) = f(3)f(-3)$.

Thus we must also know $f(0)$ in order to find $f(-3)$.

Letting $x = 0$ in the relation,

$f(3) = f(3)f(0)$

$\Rightarrow f(3) = f(3)f(3)f(-3)$

$\Rightarrow 2 = 4f(-3) \Rightarrow f(-3) = \frac{1}{2}.$

CAUTION A common error is to assume that $f(2 + 3)$ is somehow related to $f(2) + f(3)$. For most functions there is no relation between the two numbers. In the case of the function $f(x) = x^2$, $f(2) + f(3) = 2^2 + 3^2 = 4 + 9 = 13$, but, $f(2 + 3) = f(5) = 25$.

EXAMPLE 4.14 Find $f(x)$, if $f(x - 1) = (x^2 - 1)$.

SOLUTION We have to evaluate the function for "x".

We need to replace "x – 1" in the given equation to find "f(x)". The right hand side expression, however, does not contain the term "x – 1". We, therefore, need to find the term, which will replace "x". Clearly if "x" replaces "x – 1", then "x + 1" will replace "x – 1 + 1 = x"

Thus, we need to replace "x" by "x + 1"

$\Rightarrow f(x + 1 - 1) = (x + 1)^2 - 1 = x^2 + 2x + 1 - 1$

$\therefore f(x) = x^2 + 2x.$

EXAMPLE 4.15 Let f be a function such that

$f\left(\frac{8}{\sqrt{1+\sqrt{x}}}\right) = x$ for all $x \geq 0$. Find the value of $f(4)$.

SOLUTION We are looking for x such that

$$4 = \frac{8}{\sqrt{1+\sqrt{x}}}. \text{ Then } f(4) = x.$$

$$4 = \frac{8}{\sqrt{1+\sqrt{x}}} \quad \Rightarrow \quad \sqrt{1+\sqrt{x}} = 2$$

$$\Rightarrow 1 + \sqrt{x} = 4 \quad \Rightarrow \quad \sqrt{x} = 3 \quad \Rightarrow \quad x = 9$$

Therefore f (4) = 9.

EXAMPLE 4.16 Consider the function g(x) defined as

$$g(x)\cdot\left(x^{(2^{2008}-1)}-1\right) = (x+1)(x^2+1)(x^4+1) \ldots\ldots \left(x^{2^{2007}}+1\right) - 1.$$

Find the value of g (2).

SOLUTION R.H.S.

$$= \frac{(x-1)(x+1)(x^2+1)(x^4+1)\ldots\ldots(2^{2^{2007}}+1)}{(x-1)} - 1$$

$$= \frac{(x^2-1)(x^2+1)(x^4+1)\ldots\ldots(2^{2^{2007}}+1)}{(x-1)} - 1$$

$$= \frac{(x^{2^2}-1)(x^{2^2}+1)\ldots\ldots(2^{2^{2007}}+1)}{(x-1)} - 1$$

$$= \frac{(x^{2^{2008}}-1)}{(x-1)} - 1.$$

Hence, $g(x)\cdot\left(x^{(2^{2008}-1)}-1\right) = \dfrac{(x^{2^{2008}}-1)}{(x-1)} - 1$

$$\Rightarrow \quad g(2)\cdot\left(2^{(2^{2008}-1)}-1\right) = (2^{2^{2008}}-1) - 1$$

$$g(2)\cdot\frac{(2^{2^{2008}}-2)}{2} = (2^{2^{2008}}-2)$$

$$\Rightarrow \quad g(2) = 2.$$

EXAMPLE 4.17 A function f is defined for all positive integers and satisfies $f(1) = 2005$ and $f(1) + f(2) + \ldots. + f(n) = n^2 f(n)$ for all $n > 1$. Find the value of f(2004) .

SOLUTION We have 4f(2) = f(1) + f(2)

$$\Rightarrow \quad f(2) = \frac{f(1)}{3} = \frac{f(1)}{1+2}.$$

Now, 9f(3) = f(1) + f(2) + f(3)

$$8f(3) = \frac{4f(1)}{3}$$

$$\Rightarrow \quad f(3) = \frac{f(1)}{6} = \frac{f(1)}{1+2+3} \text{ and so on}$$

$$\therefore \quad f(2004) = \frac{f(1)}{1+2+3+\ldots+2004}$$

$$= \frac{(2005)\times 2}{2005\times 2004} = \frac{1}{1002}.$$

EXAMPLE 4.18 Let f satisfy f(n + 1) $= (-1)^{n+1} n - 2f(n)$, $n \geq 1$. If f(1) = f(1001) find f(1) + f(2) + f(3) + ... + f(1000).

SOLUTION We have

$$f(2) = (-1)^2 1 - 2f(1) = 1 - 2f(1)$$
$$f(3) = (-1)^3 2 - 2f(2) = -2 - 2f(2)$$
$$f(4) = (-1)^4 3 - 2f(3) = 3 - 2f(3)$$
$$f(5) = (-1)^5 4 - 2f(4) = -4 - 2f(4)$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$f(999) = (-1)^{999} 998 - 2f(998) = -998 - 2f(998)$$
$$f(1000) = (-1)^{1000} 999 - 2f(999) = 999 - 2f(999)$$
$$f(1001) = (-1)^{1001} 1000 - 2f(1000) = -1000 - 2f(1000)$$

Adding columnwise,

$$f(2) + f(3) + \ldots + f(1001) = 1 - 2 + 3 - \ldots + 999 - 1000 - 2[(f(1) + f(2) + \ldots + f(1000)].$$

This gives

$$2f(1) + 3[(f(2) + f(3) + \ldots. + f(1000) + f(1001)] = -500$$

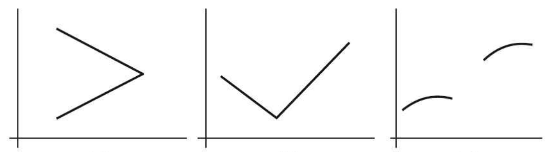
Since f(1) = f(1001) we have 2f(1) + f(1001) = 3f(1), therefore

$$f(1) + f(2) + \ldots + f(1000) = -\frac{500}{3}.$$

CONCEPT PROBLEMS [A]

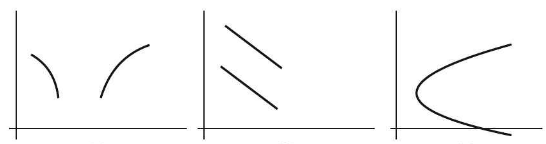
1. If A = {1, 2, 3} and B = {a, b, c}, then does
   (i) {(1, a), (2, b), (2, c), (3, c)} and (ii) {(2, b), (3, b)}
   represent a function from A to B ?

2. Which curves are the graphs of functions and which are not ?



(a) (b) (c)

3. Which curves are the graphs of functions and which are not ?



(a) (b) (c)

4. Represent all possible functions defined from {α, β} to {1, 2}.

5. Find functions from the following function(s) whose domain and range are same ?
   (i) $f(x) = \sqrt{1-x^2}$, (ii) $g(x) = \dfrac{1}{x}$,
   (iii) $h(x) = \sqrt{x}$, (iv) $j(x) = \sqrt{4-x}$

6. If $f(x) = \sqrt{1+x^2}$ then show that $f(xy) \le f(x).\ f(y)$.

7. Consider the function $f : R-\{0\} \to R$, $f(x) = \dfrac{1}{x}$. Which of the following statements are always true?
   (i) $f\left(\dfrac{a}{b}\right) = \dfrac{f(a)}{f(b)}$ (ii) $f(a + b) = f(a) + f(b)$
   (iii) $f(a^2) = (f(a))^2$.

8. Which of the following are functions :
   (i) $f = \{(x, y) : y = x + 1, x \in N, y \in W\}$
   (ii) $f = \{(x, y) : y = x - 2, x \in N, y \in W\}$
   (iii) $f = \{(x, y) : y^3 = x + 1, x \in N, y \in R\}$

9. Let $A = \{x : 1 \le x \le 4\}$ and $B = \{y : 1 \le y \le 3\}$
   Define $\psi = \{(x, y) : y \ge x + 1, x \in A, y \in B\}$.
   Plot ψ on the x-y plane and determine whether it represent a function ?

PRACTICE PROBLEMS [A]

10. If $f : R \to R$ satisfying the conditions
$f(0) = 1$, $f(1) = 2$ and $f(x + 2) = 2f(x) + f(x + 1)$, then find f(6).

11. f is a function for which f(1)=1 and f(n)=n + f(n– 1) for each natural number $n \geq 2$. Find the value of f(100).

12. The function f(x) defined on the real numbers has the property that
$f(f(x)) \cdot (1 + f(x)) - f(x) = 0$ for all x in the domain of f. If the number 2 is in the domain and range of f, then find the value of f (2).

13. For which of the following functions is f(a + b) equal to f(a) + f(b) for all positive numbers a and b ?
(a) $f(x) = x^2$
(b) $f(x) = 3x$
(c) $f(x) = -4x$
(d) $f(x) = \sqrt{x}$
(e) $f(x) = 2x + 1$

14. For which of the following functions is f(ab) equal to f(a)f(b) for all positive numbers a and b ?
(a) $f(x) = 3x$
(b) $f(x) = x^3$
(c) $f(x) = 1/x$
(d) $f(x) = \sqrt{x}$
(e) $f(x) = x + 1$

15. Let f have as its domain the set of all integers. Assume that f(x + y) = f(x) + f(y) for all integers x and y, and f(1) = 3.
(a) Show that f(2) = 6
(b) Show that f(0) = 0
(c) What can you say about f(–1) ?
(d) Find a possible formula for f.

16. In each of these give three examples of numerical functions f that meet the given condition.
(i) $f(-x) = 1/f(x)$ for all numbers x
(ii) $f(x + 1) = 2f(x)$ for all numbers x
(iii) $f(xy) = f(x) f(y)$ for all numbers x and y
(iv) $f(xy) = f(x) + f(y)$ for all positive numbers x and y.

17. A is a point on the circumference of a circle. AB and AC divide the area of the circle into three equal parts. If the angle BAC is the root of the equation, f (x) = 0 then find f (x).

4.2 | REPRESENTATION OF FUNCTIONS

1. Tabular representation

A function can also be thought of as a table consisting of two rows. one row for inputs and one row for outputs There may be repetitions in the outputs but not in the inputs.

For example, the following table represents y as a function of x.

x	1	2	3	4
y	1	0	4	6

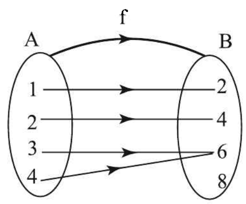
2. Ordered pair representation

A function from A to B can be represented as a collection of ordered pairs.

For example, let A = {1, 3, 5} and B = {2, 4, 8} and f = {(1, 4), (3, 2), (5, 8)}.

3. Mapping

We are already familiar with the diagrammatic representation of functions. For example, the following diagram represents a function.



4. Analytical representation

If a function can be represented by a formula, we call it an analytical representation.

$f : A \to B,\ y = f(x)$

For example, $f : R \to R$, $f(x) = x + \sin x$.

The advantage of analytical representation is the compactness and the possibility of evaluating f(x) for any value of x in the domain. This is the most effective way of representation of a function which allows application of techniques of calculus.

Most of the functions encountered at a basic level are defined by means of a single equation.

For example, $h(x) = x^3 + 3$.

It is a mistake, however, to assume that this is always true. The function f given by f(x)

$$= \begin{cases} x^2 + 1 & \text{if } x \ge 0, \\ -\dfrac{x}{2} & \text{if } x < 0, \end{cases}$$

requires two equations for its definition.

Hence, the analytical representation may use a single (uniform) definition or piece-wise definition.

For example, $f(x) = \dfrac{x}{x^2+1}$

has a single formula for the complete domain.

A **piece-wise defined function** has different analytical expressions (formulae) on different parts of its domain.

For example, $f : (0, 5) \to R$

$$f(x) = \begin{cases} x^2 - 1, & 0 \le x < 1 \\ x + 1, & 1 \le x \le 5 \end{cases}$$

is a piece-wise defined function.

In a piece-wise defined function, the domain is union of the subintervals. To evaluate the function at a particular value of x, we choose the appropriate formula.

For instance, in the above function

$f(0) = 0^2 - 1 = -1$

$f(2) = 2 + 7 = 3$

The variable used in an analytical representation may be continuous or discrete. The previous example use a continuous variable.

Now consider the factorial function whose variable is discrete. For every positive integer n, we define

$f(n) = n! = 1.2.3 \ldots\ldots n$ and for $n = 0$, $n! = 1$.

Here, the domain of f is the set of whole numbers.

Analytical representation may use either explicit form of representation or implicit form of representation.

Generally we deal with functions in which the dependent variable y is explicitly related to independent variable x. Such functions are called explicit function.

A function is an explicit function, if its dependent variable is expressed in terms of the independent variable only. For example,

$y = x^2 + 1$, $\quad y = x^2 + \cos x$.

Here, y is expressed directly in terms of x.

On the other hand, there are functions in which y is not explicitly related to x. Such functions are called implicit functions.

A function is an implicit function, if it is defined by an equation not solved for the dependent variable. For example,

$x + y^3 = x^2$, $xy = \sin(x + y)$.

Here, y is not directly expressed in terms of x. However, in some cases, an implicit function can be converted into an explicit function.

For example, $x + y^3 = x^2$ can be expressed as

$$y = \sqrt[3]{x^2 - x}.$$

Sometimes, the conversion can change the domain. But we must consider the domain of the original expression.

For example, when we change

$$\log_2 x + \log_2 y = 1 \text{ as follows} \quad \ldots(1)$$

$\Rightarrow \log_2 xy = 1$

$\Rightarrow \quad xy = 2 \quad \Rightarrow \quad y = \dfrac{2}{x}, \quad \ldots(2)$

we perceive that the domain is $x \in R - \{0\}$.

But from (1), we notice that x must be positive.

Hence, when we use (2) we consider the domain as

$x \in (0, \infty)$.

Some of the implicit equations donot represent functions. For example, the equation $x^2 + y^2 = 1$ represents a circle, since when it is solved for y, we get

$$y = \pm \sqrt{1 - x^2},$$

which gives two values of y for a given value of x.

Now consider, $X = \{x : x \geq 0\}$, $Y = \{y : y \in R\}$ and the rule $y^2 = x$. Here also, the rule gives two values of y for each $x \geq 0$, i.e. $y = \sqrt{x}$ and $y = -\sqrt{x}$

However, we can define two functions here :

$$f_1 = \{(x,y) : y = \sqrt{x}, x \geq 0,\}$$

and $\quad f_2 = \{(x,y) : y = -\sqrt{x}, x \geq 0,\}$

The union of these two sets, i.e.

$$\{(x,y) : y = \pm\sqrt{x}, x \geq 0,\}$$

is not a function but a relation since it gives more than one value of y corresponding to any x. Such a relation is sometimes called a multi-valued function.

EXAMPLE 4.1 Represent $e^y - e^{-y} = 2x$ as an explicit function of x.

SOLUTION $e^y - e^{-y} = 2x$ or $e^y - \dfrac{1}{e^y} = 2x$

or $\quad e^{2y} - 2xe^y - 1 = 0$

$$\therefore \quad e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

or $\quad e^y = x \pm \sqrt{x^2 - 1}$

$\therefore \quad e^y$ is a positive quantity,

$$e^y = x + \sqrt{x^2 + 1}$$

$$y \ln e = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\therefore \quad y = \ln\left(x + \sqrt{x^2 + 1}\right).$$

EXAMPLE 4.2 An open-top box is made from a 6×8 rectangular piece of cardboard by cutting out a square of side x from each of the four corners, and then folding the sides up. Find a formula for the volume V of the box as a function of x, and find the domain of this function.

SOLUTION Here the box we get will have height x and rectangular base of dimensions

$6 - 2x$ by $8 - 2x$. Thus,

$V = f(x) = x(6 - 2x)(8 - 2x)$.

Here V is the variable that depends on x, i.e., V is playing the role of y.

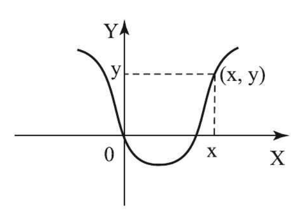
This formula makes mathematical sense for any x, but in the story problem the domain is much less. In the first place, x must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is

$$0 < x < \frac{1}{2} \text{ (minimum of 6 and 8).}$$

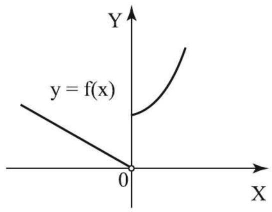
In interval notation, the domain is the interval (0, 3).

5. Graphical representation

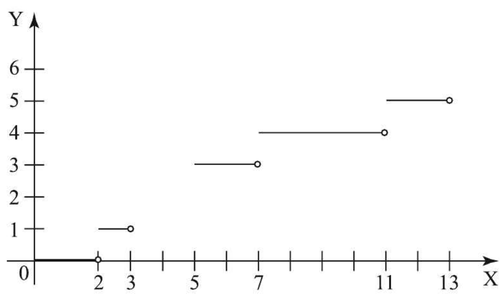
A function $y = f(x)$ can be represented in the form of a graph, where each point (x, y) on the graph represents an input x and the corresponding output y of the function. The use of graph gives a visual representation and shows the character of variation of the function.



The graph of $f(x) = \begin{cases} x^2+1 & \text{if } x \geq 0, \\ -\dfrac{x}{2} & \text{if } x < 0, \end{cases}$ is shown in the figure.



Consider the prime-number function. For any $x > 0$, let $\pi(x)$ be the number of primes less than or equal to x. The domain of π is the set of positive real numbers. Its range is the set of nonnegative integers {0, 1, 2, ...}.
We do not have an analytical form of representation for this function. A portion of the graph of $\pi(x)$ is shown below. It helps us in understanding the function.



As x increases, the function $\pi(x)$ remains constant until x reaches a prime, at which point the function value jumps by 1. Therefore the graph of π consists of horizontal line segments. This is an example of a class of functions called step functions.
Another function, which is so wild that it is impossible to draw its graph, is the following :

$$g(x) = \begin{cases} x & \text{if } x \text{ is rational}, \\ 1 & \text{if } x \text{ is irrational} \end{cases}.$$

6. Parametric representation

If a functional relationship between two variables is specified so that each variable is determined separately as a function of one auxiliary variable we say that this functional relationship is represented parametrically, and call the auxiliary variable a parameter.
Given two equations :

$$\left.\begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned}\right\} \qquad ...(1)$$

where t assumes values that lie in the interval $[T_1, T_2]$. To each value of t there correspond values of x and y (the functions φ and ψ are assumed to be single-valued). If one regards the values of x and y as coordinates of a point in a coordinate xy-plane, then to each value of t there will correspond a definite point in the plane. And when t varies from T_1 to T_2, this will describe a certain curve.

Equations (1) are called parametric equations of this curve, t is the parameter, and the curve is represented parametrically by equations (1).

The determination of the direct relationship between the variables x and y (not involving the parameter t) from the equations (1), is referred to as the elimination of the parameter. This results in an equation connecting x and y which specifies one of the variables as a function of the other.
Let us assume that the function $x = \varphi(t)$ has an inverse, $t = \sigma(x)$. Then, obviously, y is a function of x,

$$y = \psi[(\sigma(x)] \qquad ...(2)$$

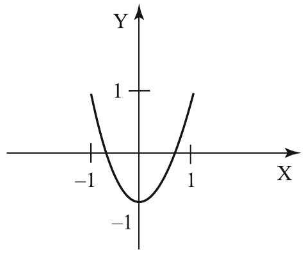
Thus, equations (1) define y as a function of x, and we say that the function y of x is represented parametrically.

For example, $\begin{cases} x = t-1 \\ y = 2t+1 \end{cases}, \qquad t \in R$

represents the straight line $y = 2(x + 1) + 1$, i.e. $y = 2x + 3$.
Let us plot the function $y = f(x)$ where

$x = \cos\theta,$
$y = \cos 2\theta,\ 0 \leq \theta \leq 2\pi.$

We have $y = 2\cos^2\theta - 1$
$\Rightarrow y = 2x^2 - 1, \quad -1 \leq x \leq 1$



A parametric specification of a function (or of a curve) is sometimes more convenient than other ways of representing the function because parametric equations expressing a complicated relationship between x and y may be considerably simpler than the original equation.

It should be noted that if at least one of the functions in system is constant it is impossible to eliminate the parameter t to obtain a function $y = f(x)$. For instance, the equations $x = 1$, $y = \sin t$ donot provide a functional relationship between x and y. Here the points with coordinates x and y corresponding to different values of t lie on the line segment $x = 1$ joining the points $(1, -1)$ and $(1, 1)$.

Let us check that the equations

$$y = a\frac{2t}{1+t^2},\ x = a\frac{1-t^2}{1+t^2}$$

specify parametrically the circle $x^2 + y^2 = a^2$.

Squaring both equations and adding them together we receive $x^2 + y^2 = a^2$.

The geometric significance of the parameter t in the above equations is as follows.

When the parameter t ranges from 0 to ∞ the variable point traverses the upper semicircle from the point (a, 0) to the point (–a, 0), and if t varies from 0 to – ∞ it describes the lower semicircle. Hence, if t continuously varies from – ∞ to ∞ the moving point makes one circuit over the circle traversing it counterclockwise, the initial point being (a, 0).

Parametric representation of curves is widely used in mechanics. If in the xy-plane there is a certain object in motion and if we know the laws of motion of the projections of this point on the coordinate axes, then

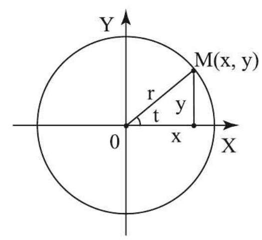
$$\left.\begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned}\right\} \quad \text{...(3)}$$

where the parameter t is the time. Then equations (3) are parametric equations of the trajectory of the moving object. Eliminating from these equations the parameter t, we get the equation of the trajectory in the form $y = f(x)$ or $F(x, y) = 0$.

Consider some examples of curves in parametric form.

(i) Given a circle with centre at the origin and radius r.

Denote by t the angle formed by the x–axis and the radius at some point M (x,y) of the circle.



Then the coordinates of point M on the circle is expressed in terms of the parameter t as follows :

$$\left.\begin{aligned} x &= r\cos t \\ y &= r\sin t \end{aligned}\right\},\ 0 \le t \le 2\pi$$

These are the parametric equations of the circle. If we eliminate the parameter t from these equations, we will have an equation of the circle containing only x and y. Squaring the parametric equations and adding, we get

$$x^2 + y^2 = r^2(\cos^2 t + \sin^2 t) \quad \text{or} \quad x^2 + y^2 = r^2.$$

(ii) Given the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(1)}$$

Set $x = a \cos t$

Putting this expression into equation (1) and performing the necessary manipulations, we get

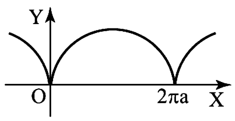
$y = b \sin t$

The equations $\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}\right\}\ 0 \le t \le 2\pi$

are the parametric equations of the ellipse.

(iii) The equations

$$\left.\begin{aligned} x &= a(t - \sin t) \\ y &= a(1 - \cos t) \end{aligned}\right\}, \ 0 \le t \le 2\pi$$



are the parametric equations of the cycloid. As t varies between 0 and 2π, the point will describe one arch of the cycloid.

Eliminating the parameter t from the latter equations, we get x as a function of y directly.

In the interval $0 \le t \le \pi$, the function $y = a(1 - \cos t)$ has an inverse:

$$t = \cos^{-1}\frac{a - y}{a}$$

Substituting the expression for t into the first of equations, we get

$$x = a\cos^{-1}\frac{a - y}{a} - a\sin\left(\cos^{-1}\frac{a - y}{a}\right)$$

or $x = \cos^{-1}\dfrac{a - y}{a} - \sqrt{2ay - y^2}$ when $0 \le x \le \pi a$.

Examining the figure we note that when $\pi a \le x \le 2\pi a$

$$x = 2\pi a - \left(a\cos^{-1}\frac{a - y}{a} - \sqrt{2ay - y^2}\right).$$

It is noted that the function

$x = a(t - \sin t)$

has an inverse, but it is not expressible in terms of elementary functions. And so the function $y = f(x)$ is not expressible in terms of elementary functions either.

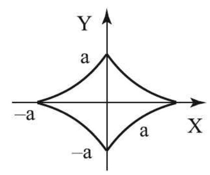
Note

The cycloid clearly shows that in certain cases it is more convenient to use the parametric equations for studying functions and curves than the direct relationship of y and x (y as a function of x or x as a function of y).

(iv) The astroid is a curve represented by the following parametric equations :

$$\left.\begin{aligned} x &= a\cos^3 t \\ y &= a\sin^3 t \end{aligned}\right\}, \ 0 \le t \le 2\pi$$

Raising the terms of both equations to the power 2/3 and adding, we get the following relationship between x and y : $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}(\cos^2 t + \sin^2 t)$



or $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

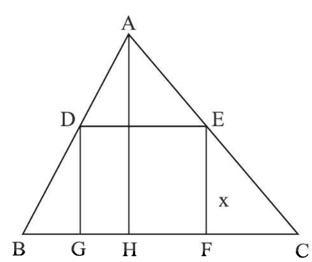
PRACTICE PROBLEMS [B]

1. Which of the following represent functions?

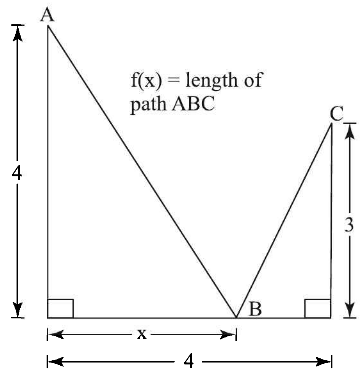
   (i) $y = \sqrt{x}$ (ii) $x^2 + 4y^2 = 4$

   (iii) $x^2 + y^2 = 1$ and $y \ge 0$ (iv) $y = \begin{cases} 2x + 3 & x \ge 0 \\ \dfrac{x^2}{2}, & x < 0 \end{cases}$

2. In the figure, BC = b and AH = h where AH ⊥ BC. If EF = x where EF ⊥ BC then express the area and perimeter of the rectangle DEFG as functions of x.



3. Give an algebraic formula for the following function For $0 \le x \le 4$, f(x) is the length of the path from A to B to C.



4. Draw the graph of the following functions:

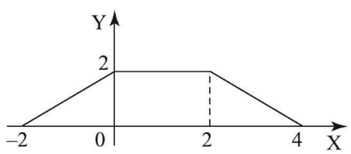
(i) $f(x) = \begin{cases} x^2+1 & \text{if } |x| > 0, \\ 2 & \text{if } x = 0, \end{cases}$ (ii) $g(x) = \begin{cases} x^2+1 & \text{if } x \geq 0, \\ -x^2-1 & \text{if } x < 0, \end{cases}$

5. Draw the graph of the following functions:

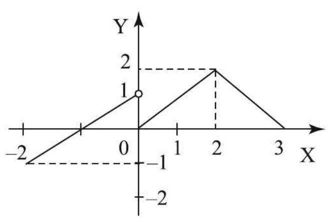
(i) $f(x) = \begin{cases} -x-1 & x < -1 \\ x^2-1 & -1 \leq x \leq 1 \\ 2x-1 & x > 1 \end{cases}$ (ii) $f(x) = \begin{cases} x^2-1, & x < -1 \\ x^3, & -1 \leq x < 1 \\ 2-x^2, & 1 \leq x \leq 2 \end{cases}$

6. Define the following functions analytically:

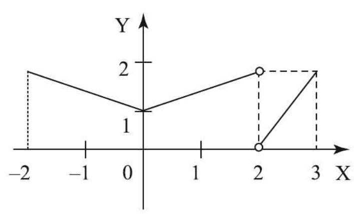
(a)



(b)



(c)



7. Which of the following equations represent a function ?

(a) $|x| + |y| = 1$ (b) $|x + y| = 1$

(c) $|y| = x^2 + \sin x$ (d) $y = |x|^2 - x$

8. Find the explicit form of the following functions :

(a) $\log_{10} x + \log_{10} (y + 1) = 4$

(b) $2^{x+y} (x^2 - 2) = x^3 - 8$

(c) $y^2 = 16x - 8y$. Also find their domains.

9. Consider the functions defined by the following equations. In each case obtain y as an explicit function of x, and state for what values of x it is defined.
   (a) $y^2 - 2y - x^2 = 0, y \geq 1$ (b) $y^2 - 2y + x^2 = 0, y \leq 1$.
10. Give an example of an algebraic function which cannot be expressed in an explicit algebraical form.

4.3 | ALGEBRA OF FUNCTIONS

Equal/ Identical functions

Two functions are equal, if each ordered pair in one of the two functions is present in the other function.

The very concept of "equal" or "identical" functions indicates that there is more than one way to represent a function. In other words, the question of equality of two functions arises when two function forms yield same values. There are few such occurrences in mathematics. This arises primarily because we have alternate ways to represent a mathematical entity. Consider, for example, modulus function. There are two equivalent expressions :

$f(x) = |x|$

$g(x) = \sqrt{x^2}$

These two function forms yield the same values for all real values of x. As such, these two functions f(x) and g(x) are equal functions. On the other hand, there are equivalent forms, which represent equal values but not for all values of x in the domains. Consider, for example,

$f(x) = 2\log_e x$

$g(x) = \log_e x^2$

The logarithmic function f(x) is defined for $x > 0$. This means its domain is $(0, \infty)$. For logarithmic function, g(x), $x^2 > 0$.

This inequality is true for all values of x except $x = 0$. It means domain of g(x) is $R-\{0\}$. Clearly, the domains of two the functions are not equal. For a value $x = -1$, g(x) yields a value while f(x) is not defined for this value of x. The two equations, therefore, are not equal. However, the two fnctions are equal if we limit our consideration for domain, limited to the intersection of two domains. Hence,

$f(x) = g(x)$ when $x \in (0, \infty)$

There is yet another possibility. Two equivalents forms have same domains, but yield different set of values. In such case also, the two functions are not equal. Consider the example given below.

EXAMPLE 4.1 Determine whether f(x) and g(x) are identical functions ?

$f(x) = x, \quad g(x) = \sqrt{x^2}$

SOLUTION Here, f(x) is defined for all values of x and its domain is R. On the other hand, domain of g(x) is also R as square of x is always non-negative. However, square root of a number is non-negative. Therefore, two function forms are not equivalent as f(x) is any real number, whereas is g(x) is non-negative and is a subset of R. Thus, range of f(x) is R and range of g(x) is $(0, \infty)$. Clearly, the two given functions are not equal. In the nutshell, two equivalent function forms are equal if their domain and function values are equal.

Definition

Two functions f(x) and g(x) are equal functions, if :

(i) Domain of f (x) = Domain of g(x) = X.

(ii) $f(x) = g(x)$ for all $x \in X$.

Since the second condition means that values of functions are equal for every x in the domain, it is guaranteed that the range of the two functions are equal.

Range of f (x) = Range of g(x) = Y.

The graphs of f and g are found to be completely overlapping each other.

For example, $f(x) = \ln x^2$ and $g(x) = 2 \ln |x|$ are identical functions.

Further, we can visualize equality of two functions in a negative context. If there exists some x such that $f(x) \neq g(x)$, then the two functions are not equal. We state this symbolically as :

If $f(x) \neq g(x)$; then $f \neq g$.

It is clear that equality of functions, however, do not require that co-domains be equal.

EXAMPLE 4.2 Determine whether f(x) and g(x) are identical functions.

$$f(x) = \frac{x}{x^2}, \quad g(x) = \frac{1}{x}.$$

SOLUTION The two functions are identical as f(x) is reduced to g(x) on simplification. Now, the expression of f(x) is defined for all values of x except x = 0. Thus, domain of f(x) is $R-\{0\}$. On the other hand, domain of reciprocal function g(x) is also $R-\{0\}$. Clearly, two given functions are equal.

EXAMPLE 4.3 Determine whether f(x) and g(x) are identical functions.

$$f(x) = \log_e x - \log_e(x^2 + 1)$$

$$g(x) = \log_e\left(\frac{x}{1+x^2}\right)$$

SOLUTION The two functions are identical as f(x) is changed to g(x) and vice versa on simplifi-cation. Now, f(x) is defined for

$x > 0$ and $x^2 + 1 > 0$

But x^2 is always positive. Hence, the domain of f(x) is $(0, \infty)$. On the other hand, g(x) is defined for :

$$\frac{x}{1+x^2} > 0 \Rightarrow x > 0$$

Thus, domain of g(x) is also $(0, \infty)$. Hence, the two functions are identical.

EXAMPLE 4.4 Examine whether the following pair of functions are identical or not :

(i) $f(x) = \frac{x^2}{x}(\sin^2x + \cos^2x)$ and

$g(x) = x(\sec^2x - \tan^2x)$.

(ii) $A = \{1, 2\}, B = \{10, 13\}, f : A \to B,$

$f(x) = x^2 + 9$ and $g : A \to B, g(x) = 3x + 7$.

SOLUTION

(i) No, as domains are not same.

Domain of f(x) is $R- \{0\}$ while that of g(x) is

$$R - \left\{(2n+1)\frac{\pi}{2}; n \in I\right\}$$

(ii) f = g because domains of both f and g are same and f(1) = 10 = g(1) and f(2) = 13 = g(2).

EXAMPLE 4.5 Over what domain

$f(x) = \sin^{-1}x$ and

$g(x) = -\cos^{-1}\sqrt{1-x^2}$ are identical ?

SOLUTION Note that the domain of both the functions are $-1 \le x \le 1$

$$\text{We have } f(x) = \sin^{-1}x = \begin{cases} \cos^{-1}\sqrt{1-x^2}, & 0 < x \le 1 \\ -\cos^{-1}\sqrt{1-x^2}, & -1 \le x \le 0 \end{cases}$$

Hence, f and g are not identical in general.

but they are identical when the domain is $[-1, 0]$.

Function operations

Each arithmetic operation provides a method of constructing a new function from the two given functions f and g. If f and g are two real valued functions, we can perform the usual arithmetic operations of addition, subtraction, multiplication, and division.

Thus, for the functions $f(x) = x^3 - 1$ and $g(x) = \frac{x+1}{x-1}$, we have

$$f(x) + g(x) = x^3 - 1 + \frac{x+1}{x-1},$$

$$f(x) - g(x) = x^3 - 1 - \frac{x+1}{x-1},$$

$$f(x)g(x) = (x^3 - 1)\frac{x+1}{x-1}$$

$$= (x^2 + x + 1)(x + 1), \text{ if } x \neq 1,$$

$$f(x)/g(x) = \frac{\frac{x^3 - 1}{x + 1}}{x - 1} = \frac{(x^3 - 1)(x - 1)}{x + 1}, \text{ if } x \neq 1,$$

Function operations are defined for real functions. The result of function operation is itself a real function. The new function resulting from function operation is defined in new interval(s) of real numbers as determined by the nature of operation involved.

Domain of resulting function

The function operations, like addition, involve more than one function. Each function has its domain in which it yields real values. The resulting domain will depend on the way the domain intervals of two or more functions interact. In order to understand the process, let us consider two functions y_1 and y_2 as given below.

$$y_1 = \sqrt{(x^2 - 3x + 2)}$$

$$y_2 = \frac{1}{\sqrt{(x^2 - 3x - 4)}}$$

Let "D_1" and "D_2" be their respective domains. Now, the expressions in the square roots need to be non-negative. For the first function :

$x^2 - 3x + 2 \geq 0$

$\Rightarrow (x - 1)(x - 2) \geq 0$

$\Rightarrow x \leq 1, x \geq 2$

$\Rightarrow D_1 : x \in (-\infty, 1] \in [2, \infty)$

In the case of second function, square root expression is in the denominator. So, we exclude end points corresponding to roots of the equation. Thus,

$x^2 - 3x - 4 > 0$

$\Rightarrow (x + 1)(x - 4) > 0$

$\Rightarrow D^2 : x \in (-\infty, -1) \in (4, \infty)$

For a new function defined by addition operation, values of x should be such that they simultaneously be in the domains of the two functions. Consider for example, x = 0.75. This falls in the domain of first function but not in the domain of second function. It is, therefore, clear that domain of new function is intersection of the domains of individual functions.

$\Rightarrow D = D_1 \cap D_2 : x \in (-\infty, -1) \cup (4, \infty)$.

We, now, define valid operations for real functions along with the domain.

Let us consider two real functions "f" and "g" with domains "D_1" and "D_2" respectively. Clearly, these domains are real number set R or subsets of R.

Addition

The addition of two real functions is denoted as "f + g". It is defined as:

$f + g : D_1 \cap D_2 \rightarrow R$ such that :

$(f + g)(x) = f(x) + g(x)$ for all $x \in D_1 \cap D_2$.

EXAMPLE 4.6 Let $f(x) = \ln\sqrt{x^2 - 4}$ and $g(x) = \cos\sqrt{x}$. Define $(f + g)(x)$ and write its domain.

SOLUTION Domain of f :

$\sqrt{x^2 - 4} > 0$ and $x^2 - 4 \geq 0$

$D_1 : x \in (-\infty, -2) \in (2, \infty)$.

Domain of g :

$D_2 : x \geq 0$

$D_1 \cap D_2 = (2, \infty)$.

Hence, $(f + g)(x) = \ln\sqrt{x^2 - 4} + \cos\sqrt{x}, x \in (2, \infty)$.

Subtraction

The subtraction of two real functions is denoted as "f – g". It is defined as :

$f - g : D_1 \cap D_2 \rightarrow R$ such that :

$(f - g)(x) = f(x) - g(x)$ for all $x \in D_1 \cap D_2$.

Scalar Multiplication

Scalar, here, means a real number constant, say "a". The scalar multiplication of a real function with a constant is denoted as "af". It is defined as :

$af : D_1 \rightarrow R$ such that:

$(af)(x) = af(x)$ for all $x \in D_1$.

Note that Supp (f) means points in the domain of f where $f \neq 0$.

Multiplication

The product of two real functions is denoted as "fg". It is defined as :

$fg : D_1 \cap D_2 \to R$ such that :

$(fg)(x) = f(x)\, g(x)$ for all $x \in D_1 \cap D_2$.

Quotient

The quotient of two real functions is denoted as "f/g", which is not defined for $g(x) = 0$. We need to exclude value of "x" for which g(x) is zero. Hence, it is defined as :

$\frac{f}{g} : D_1 \cap D_2 - \{x \mid g(x) \neq 0\} \to R$ such that :

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for all $x \in D_1 \cap D_2 - \{x|g(x) \neq 0\}$.

EXAMPLE 4.7 Let $f(x) = \frac{x^2-1}{x}$ and $g(x) = \frac{x+2}{x-3}$. Define $\frac{f}{g}$ and write its domain.

SOLUTION The domain of f is $x \neq 0$.

The domain of f is $x \neq 3$.

$g(x) \neq 0$ for $x \neq -2$

Hence,

$$\left(\frac{f}{g}\right)(x) = \frac{(x^2-1)(x-3)}{x(x+2)},\ x \in R - \{-2, 0, 3\}.$$

EXAMPLE 4.8 Let $f : [-2, 3] \to R$, $f(x) = x^3 - x$ and $g : [0, 5] \to R$, $g(x) = x^3 - 2x^2$. Find

(i) Supp(f) (ii) Supp (g)

(iii) Dom $\left(\frac{f}{g}\right)$ (iv) Dom $\left(\frac{g}{f}\right)$

(v) $\left(\frac{f}{g}\right)(2)$ (vi) $\left(\frac{g}{f}\right)(2)$

(vii) $\left(\frac{f}{g}\right)(1/3)$ (viii) $\left(\frac{g}{f}\right)(1/3)$

SOLUTION

(i) As $x^3 - x = x(x-1)(x+1)$,

$\text{Supp}(f) = [-2, -1) \cup (-1, 0) \cup (0, 3]$

(ii) As $x^3 - 2x^2 = x^2(x-2)$, $\text{Supp}(g) = [0, 2) \cup (2, 5]$

(iii) Dom $\left(\frac{f}{g}\right)$ = Dom(f) $\cap$ Supp(g)

$= [-2, 3] \cap ((0,2) \cup (2, 5]) = (0, 2) \cup (2, 3]$

(iv) Dom $\left(\frac{g}{f}\right)$ = Dom(g) $\cap$ Supp (f)

$= [0, 5] \cap ([-2, -1) \cup (-1, 0) \cup (0, 3]) = (0, 3]$

(v) $\left(\frac{f}{g}\right)(2)$ is undefined, as $2 \notin (0, 2) \cup (2, 3]$

(vi) $\left(\frac{g}{f}\right)(2) = \frac{g(2)}{f(2)} = \frac{0}{6} = 0$

(vii) $\left(\frac{f}{g}\right)(1/3) = \frac{-\frac{8}{27}}{-\frac{5}{27}} = \frac{8}{5}$

(viii) $\left(\frac{g}{f}\right)(1/3) = \frac{-\frac{5}{27}}{-\frac{8}{27}} = \frac{5}{8}$

EXAMPLE 4.9 Find domain of the function :

$$f(x) = 2\sqrt{(x-1)} + \sqrt{(1-x)} + \sqrt{(x^2+x+1)}$$

SOLUTION The given function can be considered to be addition of three separate functions. We know that scalar multiplication of a function does not change the domain. As such, domain of $2\sqrt{(x-1)}$ is same as that of $\sqrt{(x-1)}$. For $\sqrt{(x-1)}$ and $\sqrt{(1-x)}$ to be defined,

$x - 1 \geq 0 \Rightarrow x \geq 1,$

$1 - x \geq 0 \Rightarrow x \leq 1$

Now, we use sign rule for the third function :

$x^2 + x + 1 \geq 0$

Here, coefficient of x^2 is positive and D is negative. Hence, the function is positive for all real x. Hence, the domain of third function is R. Domain of the given function is intersection of the three domains.

It is clear that only x = 1 is common to the three domains.

Therefore, the domain is {1}.

EXAMPLE 4.10 Find the domain of the function

$$f(x) = \frac{\sec^{-1} x}{\sqrt{x - [x]}},$$

where [x] denotes the greatest integer less than or equal to x.

SOLUTION $f(x) = \dfrac{\sec^{-1} x}{\sqrt{x - [x]}}$.

$\sec^{-1} x$ is defined for all $x \in R - (-1, 1)$ and $\dfrac{1}{\sqrt{x - [x]}}$ is defined for all $x \in R - I$.

So the given function is defined for

$x \in R - \{(-1, 1) \cup \{n, n \in I\}\}$.

EXAMPLE 4.11 Let $f(x) = \begin{cases} |x+1|; & x < 1 \\ 1 - x; & x \geq 1 \end{cases}$ and $g(x) = \begin{cases} x - 2; & x < 0 \\ x + 3; & x \geq 0 \end{cases}$ then find (f + g) (x) and (fg)(x).

SOLUTION We have $f(x) = \begin{cases} -x-1; & x < -1 \\ x+1; & -1 \leq x < 1 \\ 1-x; & x \geq 1 \end{cases}$

and $g(x) = \begin{cases} x - 2; & x < 0 \\ x + 3; & x \geq 0 \end{cases}$

Making the domain sub-intervals same for f and g

$$f(x) = \begin{cases} -x-1; & x < -1 \\ x+1; & -1 \leq x < 0 \\ x+1; & 0 \leq x < 1 \\ 1-x; & x \geq 1 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x-2; & x < -1 \\ x-2; & -1 \leq x < 0 \\ x+3; & 0 \leq x < 1 \\ 1+3; & x \geq 1 \end{cases}$$

Now (f + g) (x)

$$= \begin{cases} -x-1+x-2 = -3; & x < -1 \\ x+1+x-2 = 2x-1; & -1 \leq x < 0 \\ x+1+x+3 = 2x+4; & 0 \leq x < 1 \\ 1-x+x+3 = 4; & x \geq 1 \end{cases}$$

and (fg) (x)

$$= \begin{cases} (-x-1)(x-2) = -x^2+x+2; & x < -1 \\ (x+1)(x-2) = x^2-x-2; & -1 \leq x < 0 \\ (x+1)(x+3) = x^2+4x+3; & 0 \leq x < 1 \\ (1-x)(x+3) = x^2-2x+3; & x \geq 1 \end{cases}$$

EXAMPLE 4.12 Find the domain of the function $f(x) = \dfrac{x}{[x-2][x+1]}$.

SOLUTION The function has rational form. Denominator consists of product of two greatest integer functions.

We can consider, the function as product of three individual functions :

$$f(x) = x \times \frac{1}{[x-2]} \times \frac{1}{[x-1]}$$

The domain of x is R. We, now, analyze the individual greatest integer functions such that it does not become zero. If we recall the graph of greatest integer function, then we can realize that the value of greatest integer [x] is equal to zero for the interval given by $0 \leq x < 1$. Following this clue, we find the intervals in which greatest integer functions are zero.

$[x-2] = 0$, if $0 \leq x - 2 < 1$

$\Rightarrow$ $[x-2] = 0$, if $2 \leq x < 3$

It means that given function is undefined for this interval of x. The domain of the function for this condition is :

$D_1 = R - [2, 3)$

Similarly,

$[x + 1] = 0$, if $0 \le x + 1 < 1$

$\Rightarrow$ $[x + 1] = 0$, if $-1 \le x < 0$

The domain for this condition is :

$D_2 = R - [-1, 0)$

Hence, domain of the given function is intersection of the three domains. Hence, the domain of f is

$x \in (-\infty, -1) \cup [0, 2) \cup [3, \infty)$.

PRACTICE PROBLEMS [C]

1. Examine whether the following pair of functions are identical or not

   (i) $f(x) = \text{sgn}(x)$ and $g(x) = \begin{cases} \dfrac{x}{|x|} & x \ne 0 \\ 0 & x = 0 \end{cases}$

   (ii) $f(x) = \sin^{-1}x + \cos^{-1}x$ and $g(x) = \dfrac{1}{x}$.

2. Find for what values of x, the following functions are identical.

   (a) $f(x) = \ln(x - 1) - \ln(x - 2)$ and $g(x) = \ln\left(\dfrac{x-1}{x-2}\right)$.

   (b) $f(x) = \log_x e$, $g(x) = \dfrac{1}{\log_e x}$.

   (c) $f(x) = \sin(\cos^{-1} x)$; $g(x) = \cos(\sin^{-1} x)$.

   (d) $f(x) = \sec^2 x - \tan^2 x$; $g(x) = \text{cosec}^2 x - \cot^2 x$.

3. Show that the following pairs of functions are not identical: $y = \sin(\sin^{-1}|x|)$, $y = |x|$.

4. Show that the functions $f(x) = \ln e^x$, $g(x) = |x| \text{ sgn } x$, and $h(x) = \cot(\cot^{-1} x)$ have the same graph as that of $f(x) = x$.

5. The graphs of which of the following pairs differ ?

   (i) $y = \dfrac{\sin x}{\sqrt{1+\tan^2 x}} + \dfrac{\sin x}{\sqrt{1+\tan^2 x}}$; $y = \sin 2x$

   (ii) $y = \tan x \cot x$; $y = \sin x \text{ cosec } x$

   (iii) $y = |\cos x| + |\sin x|$; $y = \dfrac{|\sec x| + |\text{cosec} x|}{|\sec x||\text{cosec} x|}$

6. Let functions f and g be defined by

   $f(x) = x^3 - 4x^2 + 5x - 2$, and $g(x) = 1/x$. Find h(x) if

   (i) $h = f(g)$
   (ii) $h = f + g$
   (iii) $h = g(f)$
   (iv) $h = fg$
   (v) $h = 5fg^2$

7. Let $f : [-5, 3] \to R$, $f(x) = x^4 - 16$ and $g : [-4, 2] \to R$, $g(x) = |x| - 4$. Find

   (i) Dom $(f + g)$
   (ii) Dom (fg)

(iii) Dom $\left(\frac{f}{g}\right)$ (iv) Dom $\left(\frac{g}{f}\right)$

(v) $(f+g)(2)$ (vi) $(fg)(2)$

(vii) $\left(\frac{f}{g}\right)(2)$ (viii) $\left(\frac{g}{f}\right)(2)$

(ix) $\left(\frac{f}{g}\right)(1)$ (x) $\left(\frac{g}{f}\right)(1)$

8. Let f, g, h : $\{1, 2, 3, 4\} \to \{1, 2, 10, 1993\}$ be given by
$f(1) = 1, f(2) = 2, f(3) = 10, f(4) = 1993$
$g(1) = g(2) = 2, g(3) = g(4) - 1 = 1,$
$h(1) = h(2) = h(3) = h(4) + 1 = 2.$
Compute the following :
(i) $(f+g+h)(3)$ (ii) $(fg+gh+hf)(4)$
(iii) $f(1+h(3))$

9. Let $f(x) = \begin{cases} x^2-4x+3, & 0 \le x < 3 \\ x-4, & x \ge 3 \end{cases}$ $g(x) = \begin{cases} x-3, & x < 4 \\ x^2-7x+10, & 4 \le x \le 6 \end{cases}$ define f/g and write its domain.

4.4 | DOMAIN OF A FUNCTION

Natural Domain

The natural domain of a function is the largest set of real number inputs that will give a real number output of the function.

We shall adopt the convention that if no explicit description of the domain of a function is given, then its domain is assumed to be the largest set of real numbers that makes sense.

We work with different function types, which are combined in various ways to form a function. The domain of such functions are determined in accordance with rules for function operations.

Steps to find domain :

(i) Find domain of the each individual function, which forms the given function. The individual functions may be of different types.

(ii) The domain of a function is unchanged when it is multiplied with a scalar (i.e. a constant).

(iii) The resulting domain after addition, subtraction and multiplication of two functions is given by the intersection of domains, "$D = D_1 \cap D_2$".

(iv) In the case of division, we need to remove values for which denominator is zero.
Domain = $D_1 \cap D_2 - \{$values of x for which denominator is zero$\}$.

Note

(i) For domain of $\varphi(x) = \{f(x)\}g(x)$, conventionally, the conditions are $f(x) > 0$ and $g(x)$ must be defined.

(ii) For domain of $\varphi(x) = {}^{f(x)}C_g(x)$ or $\varphi(x) = {}^{f(x)}P_{g(x)}$ conventionally, the conditions are $f(x) \ge g(x)$ and $f(x) \in N$ and $g(x) \in W$.

EXAMPLE 4.1 Find the natural domain of $f(x) = \frac{1}{x^2-x-6}$.

SOLUTION For output to be a real number, the denominator must not vanish. We must have $x^2 - x - 6 = (x + 2)(x - 3) \neq 0$, and so $x \neq -2$ nor $x \neq 3$. Thus, the natural domain of the function is $R - \{-2, 3\}$.

EXAMPLE 4.2 Find the natural domain of the function

$$f(x) = \sqrt{x+3}.$$

SOLUTION For output to be a real number, the quantity under the square root must be positive. Hence $x + 3 \geq 0 \Rightarrow x \geq -3$ and the natural domain is the interval $[-3, \infty)$.

EXAMPLE 4.3 Find the natural domain of

$$g(x) = \frac{2}{\sqrt{x+3}}$$

SOLUTION The denominator must not vanish and hence the quantity under the square root must be positive, therefore $x > -3$ and the natural domain is the interval $(-3, \infty)$.

EXAMPLE 4.4 Find the natural domain of the function

$$f(x) = \sqrt{1-x} + \frac{1}{\sqrt{1+x}}.$$

SOLUTION We need simultaneously $1 - x \geq 0$ (which implies that $x \leq 1$) and $1 + x > 0$ (which implies that $x > -1$), so $x\ (-1, 1]$.

Restricted Domain

If a function is defined such that its domain is a subset of the natural domain, then the domain is said to be restricted.

Consider the function $g(x) = \sqrt{1-x} + \dfrac{1}{\sqrt{1+x}}$, $0 \leq x \leq 1$.

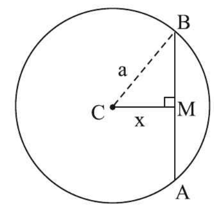
We noticed in the previous example, that the natural domain is $x \in (-1, 1]$. The domain given with g is a subset of $(-1, 1]$, and hence it is a restricted domain.

Domain of definition

The domain of definition is the domain which is specified by the conditions or meaning of the problem.

EXAMPLE 4.5 Consider a circle of radius a, as shown in the figure. Let f(x) be the length of a chord AB of this circle at a distance x from the centre of the circle. Find a formula for f(x) and write its domain of definition.

SOLUTION



f(x) = length of AB

Let M be the midpoint of the chord AB and let C be the centre of the circle. Observe that CM = x and CB = a. By Pythagoras theorem,

$BM = \sqrt{a^2 - x^2}$. Hence $AB = 2\sqrt{a^2 - x^2}$.

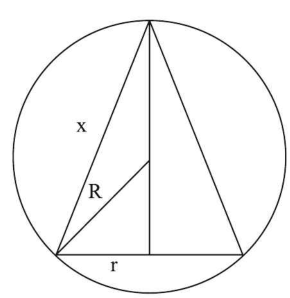
Thus, $f(x) = 2\sqrt{a^2 - x^2}$ describes the function f in algebraic terms.

Note that f(x) produces real values for $-a \leq x \leq a$.

But the distance x cannot be negative.

Hence, the domain of definition of the function f is $[0, a]$.

EXAMPLE 4.6 A right cone is inscribed in a sphere of radius R. Find the functional relationship between the lateral surface areas of the cone and its generatrix x. Indicate the domain of definition of this function.



SOLUTION $S = \pi rx$. Using mensuration we find that

$$r = \frac{x}{2R}\sqrt{4R^2 - x^2}$$

$$S = \frac{\pi x^2}{2R}\sqrt{4R^2 - x^2}$$

$$= \frac{\pi x^2}{2R}(2R).\sqrt{1-\left(\frac{x}{2R}\right)^2}$$

$$S = \pi x^2\sqrt{1-\left(\frac{x}{2R}\right)^2}.$$

The domain of definition is $0 < x < 2R$.

EXAMPLE 4.7 Find domain of the following functions:

(i) $f(x) = \sqrt{x-1} + \sqrt{6-x}$

(ii) $f(x) = \log_{1/2}(x^2 + 4x + 3)$

(iii) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

SOLUTION (i) $\sqrt{x-1}$ is defined if $x - 1 \geq 0$.

$\sqrt{6-x}$ is defined if $6 - x \geq 0$.

$\therefore$ $f(x)$ is defined $\forall x \in \{x : x-1 \geq 0\} \cap \{x : 6-x \geq 0\}$

Domain of f is $x \in [1, 6]$.

(ii) $f(x) = \log_{1/2}(x^2 + 4x + 3)$

f is defined if $x^2 + 4x + 3 > 0$

$\Rightarrow$ if $(x + 1)(x + 3) > 0$

$\Rightarrow$ if $x > -1$ or $x < -3$

Domain of f is $x \in (-\infty, -3) \cup (-1, \infty)$.

(iii) f is defined if $|x| - x > 0$

If $x \geq 0$ then $|x| = x$

If $x < 0$ then $|x| > 0$ and hence $|x| > x$.

$\therefore$ Domain of f is $x \in (-\infty, 0)$.

EXAMPLE 4.8 Find domain of the following functions :

(i) $f(x) = 2^{\sin^{-1}x} + \sqrt{x+2} + \dfrac{1}{\log_{10}(x+1)}$

(ii) $f(x) = \sin^{-1}\sqrt{4 - x^2}$

(iii) $f(x) = \ln(-2 + 3x - x^2)$

SOLUTION (i) For f(x) to be defined $-1 \leq x \leq 1$,

$x + 2 \geq 0 \quad \Rightarrow \quad x \geq -2,$

$x + 1 > 0 \quad \Rightarrow \quad x > -1$ and

$x + 1 \neq 1 \quad \Rightarrow \quad x \neq 0.$

So, domain of f is $x \in (-1, 0) \cup (0, 1]$.

(ii) $f(x) = \sin^{-1}\sqrt{4 - x^2}$

For f(x) to be defined $0 \leq 4 - x^2 \leq 1$

$\Rightarrow$ $x^2 - 4 \leq 0$ and $x^2 - 3 \geq 0$

$\Rightarrow$ $x \in [-2, 2]$ and $x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

$\Rightarrow$ $x \in [-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$

So the domain of f is $x \in [-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$.

(iii) $f(x) = \ln(-2 + 3x - x^2)$

For f(x) to be defined $-2 + 3x - x^2 > 0$

$\Rightarrow$ $x^2 - 3x + 2 < 0$

$\Rightarrow$ $(x - 1)(x - 2) < 0 \Rightarrow x \in (1, 2)$

So the domain of f is $x \in (1, 2)$.

EXAMPLE 4.9 Find the domain of the function

$$y = \frac{1}{\sqrt{\sqrt[x-1]{81} - 3}}$$

SOLUTION For the function to be defined,

$x - 1 \geq 2$, $x - 1 \in N$ and $\sqrt[x-1]{81} > 3$

$\Rightarrow$ $x \geq 3$, $x \in N$, and $81 > 3^{(x-1)}$

$\Rightarrow 3^4 > 3^{(x-1)}$

$\Rightarrow x - 1 < 4 \Rightarrow x < 5$

Hence, the domain is $x \in \{3, 4\}$.

EXAMPLE 4.10 Find the domain of the function

$$f(x) = \frac{1}{\sqrt{[x]^2 - 7[x] + 10}},$$

where $[\ \cdot\ ]$ denotes the greatest integer function.

SOLUTION $[x]^2 - 7[x] + 10 > 0$

$\Rightarrow ([x] - 5)([x] - 2) > 0$

$\Rightarrow [x] < 2$ and $[x] > 5$

$[x] < 2 \Rightarrow x < 2$ and $[x] > 5 \Rightarrow x \geq 6$

$\Rightarrow x \in (-\infty, 2) \cup [6, \infty)$.

EXAMPLE 4.11 Find the domain of $f(x) = \cos^{-1}[2x^2 - 3]$, where $[\cdot]$ denotes greatest integer function.

SOLUTION We should have $-1 \leq [2x^2 - 3] \leq 1$

$\Rightarrow -1 \leq 2x^2 - 3 < 2$

$\Rightarrow 1 \leq x^2 < \dfrac{5}{2} \quad \Rightarrow x \in \left(-\sqrt{\dfrac{5}{2}}, -1\right] \cup \left[1, \sqrt{\dfrac{5}{2}}\right)$.

EXAMPLE 4.12 Find the domain of following functions:

(i) $f(x) = \sqrt{\sin x} - \sqrt{16 - x^2}$

(ii) $f(x) = \dfrac{3}{\sqrt{4 - x^2}} \log(x^3 - x)$

(iii) $f(x) = x^{\cos^{-1} x}$

SOLUTION (i) $\sqrt{\sin x}$ is real if $\sin x \geq 0$

$\Rightarrow x \in [2n\pi, 2n\pi + \pi],\ n \in I$.

$\sqrt{16 - x^2}$ is real if $16 - x^2 \geq 0 \Rightarrow -4 \leq x \leq 4$.

Thus the domain of the given function is

$\{x : x \in [2n\pi, 2n\pi + \pi],\ n \in I\} \cap [-4, 4]$

which is $[-4, -\pi] \cup [0, \pi]$.

(ii) Domain of $\sqrt{4 - x^2}$ is $[-2, 2]$,

but $\sqrt{4 - x^2} = 0$ for $x = \pm 2$.

$\Rightarrow x \in (-2, 2)$

$\log(x^3 - x)$ is defined for $x^3 - x > 0$

i.e. $x(x - 1)(x + 1) > 0$.

$\therefore$ Domain of $\log(x^3 - x)$ is $(-1, 0) \cup (1, \infty)$.

Hence the domain of the given function is

$\{(-1, 0) \cup (1, \infty)\} \cap (-2, 2)$

$= (-1, 0) \cup (1, 2)$.

(iii) $x > 0$ and $-1 \leq x \leq 1$

$\therefore$ Domain is $(0, 1]$

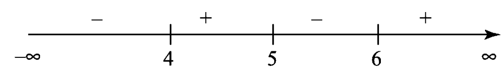
EXAMPLE 4.13 Find the domain of definition of the function $y = \log_{10} \dfrac{x - 5}{x^2 - 10x + 24} - \sqrt[3]{x + 5}$

SOLUTION For y to be defined

$$\frac{x - 5}{x^2 - 10x + 24} > 0 \qquad \ldots(1)$$

$x^2 - 10x + 24 = 0 \quad \Rightarrow x = 4, 6$

Sign scheme for $\dfrac{x - 5}{x^2 - 10x + 24}$ is as follows.



$$\therefore \quad \frac{x - 5}{x^2 - 10x + 24} > 0$$

if $4 < x < 5$ or $6 < x < 5$ $\ldots(2)$

Now $(x + 5)^{\frac{1}{3}}$ is defined for all x $\ldots(3)$

Taking intersection of (2) and (3), we get the domain as $x \in (4, 5) \cup (6, \infty)$.

EXAMPLE 4.14 Find the domain of the function $y = f(x)$ implicitly defined by $x^2 - \cos^{-1}y = \pi$.

SOLUTION $x^2 - \cos^{-1}y = \pi$

$\Rightarrow \cos^{-1}y = x^2 - \pi$, $\ldots(1)$

$\Rightarrow y = \cos(x^2 - \pi)$.

From (1), $0 \leq y \leq \pi$

$\Rightarrow 0 \leq x^2 - \pi \leq \pi$

$\Rightarrow \pi \leq x^2 \leq 2\pi$

$\Rightarrow x \in \left[-\sqrt{2\pi}, -\sqrt{\pi}\right] \cup \left[\sqrt{\pi}, \sqrt{2\pi}\right]$.

EXAMPLE 4.15 Find the domain of

$f(x) = \log_{(4-x)}(x - 1) - \sin^{-1}[2x - 3]$.

SOLUTION For f(x) to be defined, both $\log_{(4-x)}(x - 1)$ and $\sin^{-1}[2x - 3]$ should be defined. Therefore, for $\log_{(4-x)}(x - 1)$ to be defined

$4 - x > 0, \quad 4 - x \neq 1$ and $x - 1 > 0$

$\Rightarrow$ $x > 4$, $\quad x \neq 3 \quad$ and $x > 1$

$\Rightarrow$ $x \in (1, 4) - \{3\}$ $\quad$ ...(1)

For $\sin^{-1}[2x - 3]$ to be defined

$-1 \leq [2x - 3] \leq 1$

$\Rightarrow$ $-1 \leq 2x - 3 < 2$

$\Rightarrow$ $2 \leq 2x < 5 \quad \Rightarrow 1 \leq x < 5/2$ $\quad$ ...(2)

From (1) and (2), we have $1 < x < 5/2$

Hence, the domain of the function f is $(1, 5/2)$.

EXAMPLE 4.16 Find the domain of the function given by $f(x) = \log_e \cos(x - 5) + \sqrt{(9 - x^2)}$.

SOLUTION Here, we observe that argument (input) of logarithmic function is itself a trigonometric function. We know that cosine function is real for all real values of x. The important point to realize here is that we have to evaluate logarithmic function for the values of trigonometric function $\cos(x - 5)$ – not for independent variable x. Now, the argument of logarithmic function is a positive number. It means that :

$\cos(x - 5) > 0$

The basic interval is $(-\pi/2, \pi/2)$. The solution of the cosine inequality is the domain of the logarithmic function :

$$D_1 : 2n\pi - \frac{\pi}{2} < (x - 5) < 2n\pi + \frac{\pi}{2}, \; n \in I$$

$$\Rightarrow \left(2n - \frac{1}{2}\right)\pi + 5 < x < \left(2n + \frac{1}{2}\right)\pi + 5, \; n \in I$$

For the algebraic function, the expression within the square root must be a non-negative number :

$\Rightarrow$ $9 - x^2 \geq 0 \quad \Rightarrow \quad x^2 - 9 \leq 0$

$\Rightarrow$ $(x + 3)(x - 3) \leq 0$

$\Rightarrow$ $D_2 : -3 \leq x \leq 3$

The domain of the given function, f(x), is intersection of two domains i.e. $D = D_1 \cap D_2$

The common interval is between $-5\pi/2$ and $-3\pi/2$ as obtained for $n = -1$.

We should draw a rough number line for few values of n.

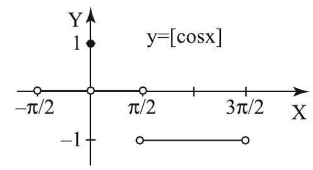
Domain is $x \in \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right)$.

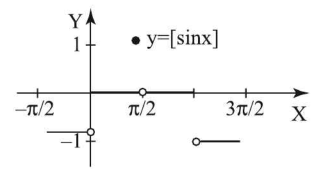
EXAMPLE 4.17 Find the domain of the function $f(x) = \log_e \dfrac{1}{\sqrt{[\cos x] - [\sin x]}}$.

SOLUTION For f(x) to be defined

$[\cos x] - [\sin x] > 0$

$\Rightarrow$ $[\cos x] > [\sin x]$





It is clear that $[\cos x] > [\sin x]$ is true in the fourth quadrant i.e. $x \in \left[-\frac{\pi}{2}, 0\right]$. Hence, the domain of the function is $x \in \bigcup_{n \in I} \left\{\left[2n\pi - \frac{\pi}{2}, 2n\pi\right]\right\}$.

EXAMPLE 4.18 Find the domain of the func-tion

$$f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)} \times \frac{1}{x^2 - 36}.$$

SOLUTION For f(x) to be defined $x \neq -6, 6$ and

$$\log_{0.4}\left(\frac{x-1}{x+5}\right) \geq 0, \; \frac{x-1}{x+5} > 0.$$

Since $\log_a x$ for $0 < a < 1$ is a decreasing function, we have $\frac{x-1}{x+5} \leq 1$ and $\frac{x-1}{x+5} > 0$.

$$\Rightarrow \frac{x - 1 - x - 5}{x + 5} \leq 0 \text{ and } x < -5, x > 1$$

$$\Rightarrow \frac{-6}{x + 5} \leq 0 \text{ and } x < -5, x > 1$$

$\Rightarrow$ $x > -5$ and $x < -5$, $x > 1$

$\Rightarrow$ $x > 1$

$\Rightarrow$ The domain of f(x) is $(1, \infty) - \{6\}$.

EXAMPLE 4.19 Find the domain of definition of the functions :

(a) $f(x) = \sqrt{(x^2 - 3x - 10) \cdot \ell n^2 (x - 3)}$

(b) $f(x) = \sqrt{\dfrac{1 - 5^x}{7^{-x} - 7}}$

SOLUTION (a) $(x - 5)(x + 2) \ln^2 (x - 3) \geq 0$

Equality gives $x = 5$; $x = 4$

(Note for $x = -2$, $\ln(x - 3)$ is meaningless)

Now $(x - 5)(x + 2) > 0$ and $x - 3 > 0$

$\Rightarrow$ $x > 5$

Hence the domain is $\{4\} \cup [5, \infty)$.

(b) For f(x) to be defined $\dfrac{1-5^x}{7^{-x}-7} \geq 0$

$7 - x \neq 7 \quad \Rightarrow \quad x \neq -1$

Now $\dfrac{1 - 5^x}{7^{-x} - 7} = 0 \quad \Rightarrow \quad x = 0$

Now we solve $\dfrac{1 - 5^x}{7^{-x} - 7} > 0$

$1 - 5x > 0$ and $7^{-x} - 7 > 0 \Rightarrow x \in (-\infty, -1)$

or $1 - 5^x < 0$ and $7^{-x} - 7 < 0 \Rightarrow x \in (0, \infty)$

Hence the domain is $(-\infty, -1) \cup [0, \infty)$.

EXAMPLE 4.20 Find the domain of definition of the following functions :

(a) $f(x) = \tan^{-1} \sqrt{x(x+3)} + \sec^{-1} \dfrac{1}{\sqrt{x^2+3x+1}}$

(b) $f(x) = \ln(1 - 2|\cos x|) + e^{\cos^{-1}(2x/\pi)}$

(c) $f(x) =$

$$\sqrt{(\log_{0.2} x)^3 + (\log_{0.2} x^3)(\log_{0.2} 0.0016x) + 36}$$

(d) $f(x) = \dfrac{1}{\sqrt{\{x-1\} - x^2 + 2x}}$

(e) $f(x) = \sqrt{-\log_{\sin x} \tan 2x}$

SOLUTION (a) The inverse tan function is defined for every real number. For the square root operation to be defined, we have

$x(x + 3) \geq 0$ ...(1)

For the inverse sec function to be defined, we have

$$\left| \frac{1}{\sqrt{x^2+3x+1}} \right| \geq 1$$

i.e. $\dfrac{1}{\sqrt{x^2+3x+1}} \geq 1$

i.e. $x^2 + 3x + 1 \leq 1$

i.e. $x(x + 3) \leq 0$ ...(2)

The required domain of definition is the intersection of (1) and (2) which gives $x(x + 3) = 0$, i.e. $x \in \{-3, 0\}$.

(b) For the log operation to be defined, we have

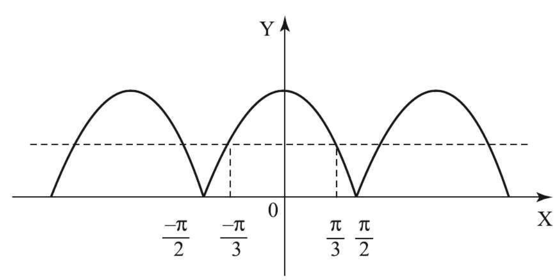
$1 - 2|\cos x| > 0$

i.e. $|\cos x| < 1/2$ ...(1)

For the inverse cos operation to be defined, we have $-1 \leq \dfrac{2x}{\pi} \leq 1$

i.e. $\dfrac{-\pi}{2} \leq x \leq \dfrac{\pi}{2}$ ...(2)

To find the values of x satisfying both (1) and (2), let us plot the curve $y = |\cos x|$ and $y = 1/2$ in the interval $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ as shown below.



The required domain of definition is :

$$x \in \left[\frac{-\pi}{2}, \frac{-\pi}{3}\right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2}\right].$$

(c) f(x) =

$\sqrt{(\log_{0.2} x)^3 + (\log_{0.2} x^3)(\log_{0.2} 0.0016x) + 36}$

$= \sqrt{(\log_{0.2} x)^3 + (3\log_{0.2} x)\{\log_{0.2} x + \log_{0.2}(0.2)^4\} + 36}$

$= \sqrt{\lambda^3 + 3\lambda(\lambda + 4) + 36}$ [putting log0.2 x = λ]

$= \sqrt{\lambda^3 + 3\lambda^2 + 12\lambda + 36}$

$= \sqrt{(\lambda + 3)(\lambda^2 + 12)}$

For the square root operation to be defined, we have $(\lambda + 3)(\lambda^2 + 12) \geq 0$

$\Rightarrow \lambda + 3 \geq 0$ [$\lambda^2 + 12$ is a +ve quantity]

$\Rightarrow \log_{0.2} x \geq -3$

$\Rightarrow 0 < x \leq (0.2)^{-3} \Rightarrow 0 < x \leq 125$

Hence, the domain of the function is $x \in (0, 125]$.

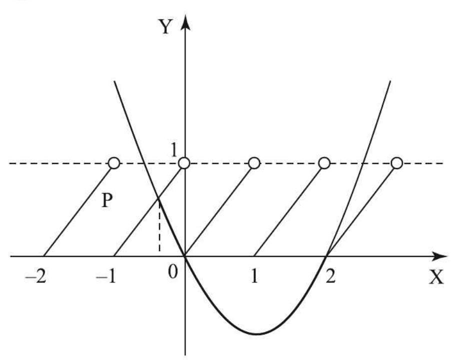
(d) The function is defined for values of x which satisfy the inequality

$\{x + 1\} - x^2 + 2x > 0$

i.e. $x^2 - 2x < \{x + 1\}$

To solve the inequality, let us plot the graph of $y = x^2 - 2x$ and $y = \{x + 1\}$ on the same reference frame. The curve $y = x^2 - 2x$

i.e. $y - 1 = (x - 1)^2$ is parabola having vertex at (1, –1),



The darkened portion of the curve $y = x^2 - 2x$ lies below the curve $y = \{x + 1\}$. The x-coordinate of the intersection point P is given by the equation $x^2 - 2x = x + 1$

i.e. $x^2 - 3x - 1 = 0 \Rightarrow x = \dfrac{3 \pm \sqrt{13}}{2}$.

The x coordinate of P is $\dfrac{3 - \sqrt{13}}{2}$.

Hence, the domain of the function is

$$x \in \left(\frac{3 - \sqrt{13}}{2}, 2\right) - \{0\}$$

Note that x = 0 does not satisfy the inequality (1), since the two curves intersect at x = 0.

(e) For the log operation to be defined, we have

$0 < \sin x < 1$

i.e. x lies in the 1st and the 2nd quadrant but

$x \neq 0, \pi/2, \pi$.

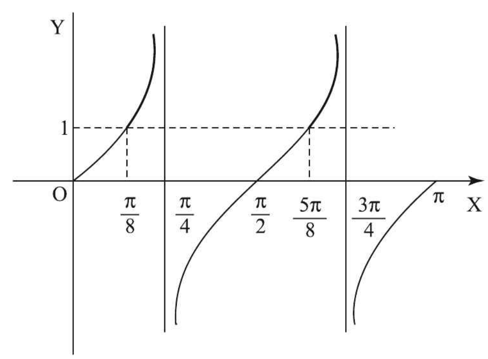
For the square root operation to be defined, we have $\log_{\sin x} \tan 2x \leq 0$

$\Rightarrow \tan 2x \geq 1$

This inequality can be solved by plotting the curves

y = tan 2x (period π/2) and y = 1 on the same frame as shown in the figure.

Let us concentrate only on the interval $0 < x < \pi$.



The darkened portions of the curve y = tan 2x curve lies above the line y = 1. The other values of x lie in the intervals $2x\pi < x < \pi + 2n\pi$.

Hence, the domain of the function is

$$\frac{\pi}{8} + 2n\pi \leq x < \frac{\pi}{4} + 2n\pi,\ n \in I$$

and $\dfrac{5\pi}{8} + 2n\pi \leq x < \dfrac{3\pi}{4} + 2n\pi,\ n \in I$.

EXAMPLE 4.21 Find the domain of the function f(x) =

$$\log_{\left[x+\frac{1}{x}\right]}\left|x^2-x-6\right|+{}^{16-x}C_{2x-1}+{}^{20-3x}P_{2x-5}$$

where [.] denotes the greatest integer function.

SOLUTION $\left[x+\frac{1}{x}\right]>0$ and $\left[x+\frac{1}{x}\right]\neq 1$

$\Rightarrow$ For x > 0 we have $\left[x+\frac{1}{x}\right]\geq 2$ ...(1)

Also $x^2-x-6\neq 0$

$\Rightarrow (x-3)(x+2)\neq 0$

$\Rightarrow x\neq -2$ and $x\neq 3$ ...(2)

Now, $16-x\geq 2x-1$; $16-x\in N$ and $2x-1\in W$

$\Rightarrow 3x\leq 17;\ 16\quad -x\geq 1,\ x\in I;\ 2x-1\geq 0,\ 2x\in I$

$\Rightarrow x\leq \frac{17}{3};\ x\leq 15,\ x\geq \frac{1}{2},\ x\in I$

$\Rightarrow x\in\{1, 2, 3, 4, 5\}$ ...(3)

Further, $20-3x\geq 2x-5,\ 20-3x\geq 1,\ 3x\in I;$
$2x-5\geq 0,\ 2x\in I$

$\Rightarrow x\leq 5;\ x\leq \frac{20}{3},\ x\geq \frac{5}{2},\ x\in I$

$\Rightarrow x\in\{3, 4, 5,\}$ ...(4)

Taking intersection of (1), (2), (3) and (4), we get $x\in\{4, 5\}$. This is the required domain.

EXAMPLE 4.22 Find the domain of the function $f(x)=\log_4\log_3(\log_2(x^2-2x+3)-\log_{\sqrt{2}}(2x-1))$.

SOLUTION $f(x)=\log_4\log_3(\log_2(x^2-2x+3)$
$-\log_{\sqrt{2}}(2x-1))$

We have $x^2-2x+3>0,\ \forall\ x\in R$

and $2x-1>0$

$\Rightarrow x>1/2$

Also $\log_3(\log_2(x^2-2x+3)-2\log_2(2x-1)>0$

$$\Rightarrow \log_2\frac{x^2-2x+3}{(2x-1)^2}>1$$

$$\Rightarrow \frac{x^2-2x+3}{(2x-1)^2}>2$$

$\Rightarrow x^2-2x+3>2[4x^2+1-4x]$

$\Rightarrow 7x^2-6x-1<0$

$\Rightarrow (x-1)(7x+1)$

$\Rightarrow -\frac{1}{7}<x<1.$

The common solution is $x\in\left(\frac{1}{2},1\right)$.

The domain of the function is $x\in(1/2, 1)$.

EXAMPLE 4.23 Find the domain of the function $f(x)=\sqrt{(x^{14}-x^{14}+x^6-x^3+x^2+1)}$.

SOLUTION Clearly, the function is real for values of x for which the expression within the square root is a non negative number. We note that the independent variable is raised to positive integers. If $x\geq 1$, then x^n evaluates to higher value for higher power. If x lies between 0 and 1, then x^n evaluates to lower value for higher power. Further, a negative x yields negative value when raised to odd power and positive value when raised to even power. We shall use these properties to evaluate the expression for three different intervals of x.

$$x^{14}-x^{11}+x^6-x^3+x^2+1\geq 0$$

We consider different intervals of values of x, which cover the complete interval of real numbers.

(i) Let $x\geq 1$

In this case, $x^a>x^b$, if a > b. Evaluating in groups,

$(x^{14}-x^{11})+(x^6-x^3)+(x^2+1)>0.$

(ii) Let $0\leq x<1$

In this case, $x^a<x^b$, if a > b. Rearranging in groups,

$x^{14}-\{(x^{11}-x^6)+(x^3-x^2)\}+1$

Here, $\{(x^{11}-x^6)+(x^3-x^2)\}$ is negative. Hence the total expression is positive,

$\Rightarrow\quad x^{14}-\{(x^{11}-x^6)+(x^3-x^2)\}+1>0$

(iii) Let x < 0

Rearranging in groups,

$x^{14} + (-x^{11}) + x^6 + (-x^3) + x^2 + 1$

Here, all terms are positive. Hence, the total expression is positive.

We see that expression is positive for all values of x. Hence, the domain of the function is :

$x \in (-\infty, \infty)$.

PRACTICE PROBLEMS [D]

1. Find the domain of the following functions :

   (i) $f(x) = \sqrt{\dfrac{(2x+1)}{x^3 - 3x^2 + 2x}}$.

   (ii) $f(x) = \sin|x| + \sin^{-1}(\tan x) + \sin(\sin^{-1} x)$.

   (iii) $f(x) = \sqrt{\log_{0.3}\left(\dfrac{3x - x^2}{x - 1}\right)}$.

2. Find the domain of the following functions:

   (i) $f(x) = \log_{(x^2 - x + 1)}\left(2x^2 - 7x + 9\right)$.

   (ii) $f(x) = \sqrt{\dfrac{\log_{0.3}(x-1)}{x^2 - 2x - 8}}$

   (iii) $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$

   (iv) $f(x) = \sqrt{3 - 2^x - 2^{1-x}} + \sqrt{\sin^{-1} x}$

3. Find the domain of the following functions:

   (i) $y = \dfrac{1}{\sqrt{\log_{\frac{1}{2}}(x^2 - 7x + 13)}}$

   (ii) $y = \sqrt{\log_3(\cos(\sin x))}$

   (iii) $y = \sqrt{\log(\log x) - \log(4 - \log x) - \log 3}$

   (iv) $y = \sqrt{4^x + 8^{\frac{2}{3}(x-2)} - 13 - 2^{2(x-1)}}$

4. Find the domain of following functions :

   (i) $f(x) = \dfrac{1}{\sqrt{x^6 - 13x^4 + 36x^2}}$

   (ii) $f(x) = \dfrac{1}{\sqrt{x^5 - 13x^3 + 36x}}$

   (iii) $f(x) = \dfrac{1}{\sqrt{|x^4 - 13x^2 + 36|}}$

5. Find the domain of the following functions:

   (i) $f(x) = \log_x \dfrac{(|x| - 3)(x - 2)}{(x-1)x}$

   (ii) $f(x) = \log_{10}\left(\sqrt{x - 4} + \sqrt{6 - x}\right)$.

(iii) $f(x) = \sqrt{\sin^{-1}(\log_2 x)} + \sqrt{\cos(\sin x)} + \sin^{-1}\left(\frac{1+x^2}{2x}\right)$

(iv) $f(x) = \frac{3}{[x/2]} - 5^{\cos^{-1} x^2} + \frac{(2x+1)!}{\sqrt{x+1}}$,

where [ ] denotes greatest integer function.

6. Find the domain of the following functions, where $[\cdot]$ is G.I.F. and $\{\cdot\}$ is F.P.F.:

(i) $\sqrt{2\{x\}^2 - 3\{x\} + 1} + \sin^{-1}x$

(ii) $\cos^{-1}\left[\log_2\left(\frac{x^2}{2}\right)\right]$

(iii) $\sin^{-1}\sqrt{\log_{[x]}\frac{|x|}{x}}$

(iv) $\sqrt{\frac{x-1}{x-2\{x\}}}$

7. Find the domain of the following functions :

(i) $\sqrt{\log_{0.3}\left(\frac{3\sin x - \sin^2 x}{\sin x - 1}\right)}$

(ii) $\frac{1}{\sqrt{4\cos x + 1}} + \sqrt{\sin^2 x - \sin x}$

(iii) $\sqrt{\cos(\cos x) + \sin(\cos x) - 1}$

8. Let $f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-1/2}$. Prove that the domain of the function is R.

9. Find the domain of the implicit function given by :

$$3^y + 2^{x^4} = 2^{4x^2-3}.$$

10. Find the domain of the function, $f(x) = (x+0.5)^{\log_{0.5+x}\frac{x^2+2x-3}{4x^2-4x-3}}$

11. Find the domain of the function $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$.

12. Let the function $f : D \to R, f(x) = \log_5\left(\log_{1/3}\log_8(2x+1)\right)$ where D is the natural domain of f. If S represents the sum of the absolute values of all integers from D, then find the value of S.

4.5 | METHODS OF FINDING RANGE

1. Basic methods

We are familiar with the least value, greatest value and range of the most standard functions of all origin. Consider constant, identity, reciprocal, modulus, greatest integer, least integer, fraction part, trigonometric, inverse trigonometric, exponential and logarithmic functions. All these functions have been described in detail and we know their properties with respect to least and greatest values and also the range. Greatest value of sine function, for example, is 1. On the other hand, exponential and logarithmic functions neither have minimum value nor maximum value.

In case, the function can be reduced to the standard forms having least and greatest values, then it is possible to know its range. If “c” and

"d" be the least and greatest values of a continuous function in a finite interval, then range of the function is given by :

Range = [least value, greatest value] = [c, d]

We should note that determining range is a comparatively more difficult proposition than determining domain.

We are already aware of some techniques of finding the minimum and maximum values of functions based on the concepts of (i) quadratic functions, (ii) A.M.–G.M. inequality, and (iii) trigonometric ratios. The following examples use some of these techniques.

EXAMPLE 4.1 Find the range of the following functions:

(i) $f(x) = \dfrac{1}{8-3\sin x}$

(ii) $f(x) = x^2 - 7x + 5$

SOLUTION (i) $f(x) = \dfrac{1}{8-3\sin x}$.

We know that $-1 \le \sin x \le 1$

$\Rightarrow -3 \le 3\sin x \le 3$

$\Rightarrow 5 \le 8 - 3\sin x \le 11$

$\therefore$ Range (f) = $\left[\dfrac{1}{11}, \dfrac{1}{5}\right]$.

(ii) $f(x) = x^2 - 7x + 5$

$\Rightarrow f(x) = \left(x - \dfrac{7}{2}\right)^2 - \dfrac{29}{4}$

$\Rightarrow$ Range (f) = $\left[-\dfrac{29}{4}, \infty\right)$.

EXAMPLE 4.2 Find the range of $y = \sin^2 x + \sin x + 2$.

SOLUTION Let $\sin x = t$

$\Rightarrow y = t^2 - t + 2, -1 \le t \le 1$

The minimum value is obtained at $t = 1/2$ (vertex) and the maximum value is obtained at $t = -1$(one of the endpoints).

Hence, the range is $y \in \left[\dfrac{7}{4}, 4\right]$.

EXAMPLE 4.3 Find the range of
$y = 4^x + 3 \cdot 2^x + 1$

SOLUTION Let $2^x = t$

$\Rightarrow y = t^2 + 3t + 1, 0 < t < \infty$

$\Rightarrow y \Rightarrow (1, \infty)$

Note that the vertex $t = -3/2$ is ineffective since it lies outside $0 < t < \infty$.

EXAMPLE 4.4 Find the range of the function

$$f(x) = \frac{x^2}{x^4+1}.$$

SOLUTION Clearly, $f(x) \ge 0$ for any $x \in R$.

For $x \ne 0$, $f(x) = \dfrac{1}{x^2 + \dfrac{1}{x^2}}$.

Since $x^2 + \dfrac{1}{x^2} \ge 2$, we have $f(x) \le \dfrac{1}{2}$.

At $x = 0$, $f(x) = 0$.

Hence the range of f is [0, 1/2].

EXAMPLE 4.5 Let $f(x) = \sqrt{ax^2 + bx}$. Find the set of real values of 'a' for which there is at least one positive real value of 'b' for which the domain of f and the range of f are the same set.

SOLUTION If $a > 0$, then the domain of f(x) is : $ax^2 + bx \ge 0$

$\Rightarrow x \in \left(-\infty, -\dfrac{b}{a}\right] \cup [0, \infty)$

Since the range does not have negative values, $a > 0$ does not satisfy the conditions of the problem.

Also $a = 0$ does not work.

Hence $a < 0$.

For domain, $x(ax+b) \ge 0 \Rightarrow 0 \le x \le -\dfrac{b}{a}$.

We have $f\left(-\dfrac{b}{2a}\right) = \sqrt{a\dfrac{b^2}{4a^2} - \dfrac{b^2}{2a}} = \sqrt{-\dfrac{b^2}{4a}}$

$\Rightarrow$ Range of f (x) is $\left[0, f\left(-\dfrac{b}{2a}\right)\right] = \left[0, \sqrt{-\dfrac{b^2}{4a}}\right]$

Since domain of f = range of f

$\Rightarrow -\dfrac{b}{a} = \sqrt{-\dfrac{b^2}{4a}}$ $\quad\Rightarrow \dfrac{b^2}{a^2} = -\dfrac{b^2}{4a}$

$\Rightarrow 4a + a^2 = 0$ $\quad\Rightarrow a = 0$ or -4

Hence, $a \Rightarrow \{0, -4\}$.

EXAMPLE 4.6 Find the range of

$$f(x) = \frac{x^2+x+1}{x^2+x-1}.$$

SOLUTION $f(x) = \dfrac{x^2+x+1}{x^2+x-1}$

$\{x^2 + x + 1$ and $x^2 + x - 1$ have no common factor$\}$

$$y = \frac{x^2+x+1}{x^2+x-1}$$

$\Rightarrow yx^2 + yx - y = x^2 + x + 1$

$\Rightarrow (y - 1)x^2 + (y - 1)x - y - 1 = 0$

If $y = 1$, then the above equation reduces to $-2 = 0$. which is not true.

Further if $y \neq 1$, then $(y - 1)x^2 + (y - 1)x - y - 1 = 0$ is a quadratic and has real roots if

$(y - 1)^2 - 4(y - 1)(-y - 1) \geq 0$

i.e. if $y \leq -3/5$ or $y \geq 1$ but $y \neq 1$.

Thus, the range is $(-\infty, -3/5] \cup (1, \infty)$.

EXAMPLE 4.7 Find the range of the expression

$y = \dfrac{x^2-2x-8}{x^2-4x-5}$, for all permissible value of x.

SOLUTION We have $x^2y - 4xy - 5y = x^2 - 2x - 8$

$(y - 1)x^2 + 2x(1 - 2y) + 8 - 5y = 0 \ (y \neq 1)$

Since $x \in R$, $D \geq 0$

$4(1 - 2y)^2 - 4(y - 1)(8 - 5y) \geq 0$

$(4y^2 - 4y + 1) - (13y - 8 - 5y^2) \geq 0$

$9y^2 - 17y + 9 \geq 0.$ ...(1)

Since coefficient of $y^2 > 0$ and $D = 289 - 324 < 0$, (1) is always true

Also if $y = 1$ we have $x = 3/2$.

Hence, the range of y is $(-\infty, \infty)$.

EXAMPLE 4.8 Let $f(x) = \dfrac{3x^2+mx+n}{x^2+1}$. If the range of this function is $[-4, 3)$ then find the value of $(m^2 + n^2)$.

SOLUTION $y = \dfrac{3(x^2+1)+mx+n-3}{1+x^2}$

$\Rightarrow y = 3 + \dfrac{mx+n-3}{1+x^2}$

For y to lie in $[-4, 3)$, $\quad mx + n - 3 < 0 \ \forall \ x \in R$

This is possible only if $m = 0$.

When, $m = 0$, then $y = 3 + \dfrac{n-3}{1+x^2}$

note that $n - 3 < 0 \quad \Rightarrow \quad n < 3$

If $x \to \infty$, $\ y \to 3-$

Now y_{min} occurs at $x = 0$ (as $1 + x^2$ is minimum)

$y_{min} = 3 + n - 3 = n \Rightarrow n = -4$.

Hence the value of $m^2 + n^2$ is 16.

EXAMPLE 4.9 Find the range of the continuous function $f(x) = a\cos x + b\sin x$, where a and b are constants.

SOLUTION Since we know the least and greatest values of sine and cosine functions, we shall attempt to reduce the given function in terms of either sine or cosine function .

Let $a = r\cos\alpha$ and $b = r\sin\alpha$ where,

$$r = \sqrt{(a^2+b^2)}$$

Substituting in the given function, we have :

$f(x) = r\cos\alpha \cos x + r\sin\alpha \sin x = r\cos(x - \alpha)$

$= \sqrt{(a^2+b^2)}\ \cos(x - \alpha)$

We know that the minimum and maximum values of cosine function are -1 and 1 respectively. Hence,

$f(x)_{min} = -\sqrt{(a^2+b^2)}$

$f(x)_{max} = \sqrt{(a^2+b^2)}$ and the range is

$y \in [\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$

EXAMPLE 4.10 Find the range of

$$y = \sqrt{3}\cos x + \sin\left(x + \frac{\pi}{3}\right).$$

SOLUTION

$$y = \sqrt{3}\cos x + \sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3}$$

$$= \frac{3\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x$$. Using the above result, the range is $-\sqrt{7} \le y \le \sqrt{7}$.

EXAMPLE 4.11 Find the range of

$y = \sqrt{3}\cos x + \sin x + 1$, if

(i) $x \in R$, and

(ii) $x \in \left[\frac{\pi}{6}, \frac{4\pi}{3}\right]$

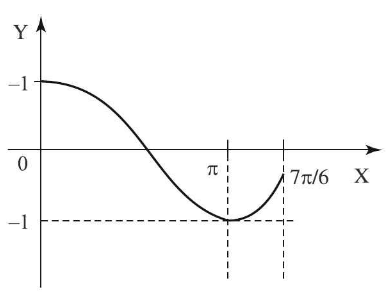
SOLUTION (i) $y = \sqrt{3}\cos x + \sin x + 1$

$$= 2\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right) + 1 = 2\cos\left(x - \frac{\pi}{6}\right) + 1$$

The range is [– 1, 3].

(ii) When $x \in \left[\frac{\pi}{6}, \frac{4\pi}{3}\right]$, we have a restricted domain and hence we draw the graph of y = cos $\left(x - \frac{\pi}{6}\right)$.

Since, $\frac{\pi}{6} \le x \le \frac{4\pi}{3}$ we have $0 \le x - \frac{\pi}{6} \le \frac{7\pi}{6}$.



From the above graph we find that the range of $\cos\left(x - \frac{\pi}{6}\right)$ in the given interval is [– 1, 1].

Hence, the range of the function is [– 1, 3].

EXAMPLE 4.12 Find the range of

(i) $y = \dfrac{1}{2\cos^2 x + 4\sin x \cos x + 4}$

(ii) $y = \dfrac{1}{\cos 2x + 2\sin 2x}$

SOLUTION $y = \dfrac{1}{\cos 2x + 2\sin 2x + 5}$

We have $-\sqrt{5} \le \cos^2 x + 2\sin^2 x \le \sqrt{5}$

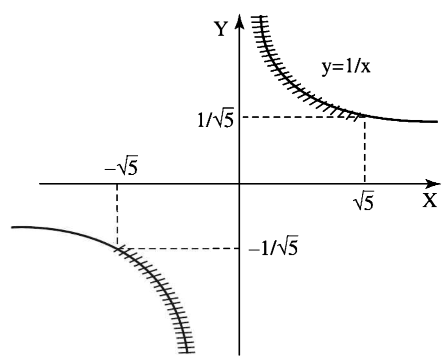
$\Rightarrow 5 - \sqrt{5} \le \cos^2 x + 2\sin^2 x + 5 \le \sqrt{5} + 5$

$\therefore\ y \in \left[\frac{1}{5+\sqrt{5}}, \frac{1}{5-\sqrt{5}}\right]$.

(ii) $y = \dfrac{1}{\cos 2x + 2\sin 2x}$

We have $-\sqrt{5} \le \cos^2 x + 2\sin^2 x \le \sqrt{5}$

$\therefore$ Now consider the graph of y = 1/x when $x \in [-\sqrt{5}, \sqrt{5}]$



Thus the range of the given function is

$$\left(-\infty, -\frac{1}{\sqrt{5}}\right] \cup \left[\frac{1}{\sqrt{5}}, \infty\right).$$

EXAMPLE 4.13 Find the range of

$$f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$$

SOLUTION Here, we notice that the domain consists of a small number of distinct points.

$\left|\dfrac{1+x^2}{2x}\right| \le 1 \quad \Rightarrow 1 + x^2 \le 2|x|$

$|x|^2 - 2|x| + 1 \le 0 \quad \Rightarrow (|x| - 1)^2 \le 0$

Thus, the domain is $x \in \{\pm 1\}$.

The range is obtained by directly evaluating the function's value at these points.

Hence, the range is $y \in \left\{\pm\dfrac{\pi}{2}\right\}$.

EXAMPLE 4.14 Find the range of

(i) $y = 3\tan^2 x + 12\cot^2 x + 5$

(ii) $y = 2\cos x + \sec^2 x,\ 0 < x < \dfrac{\pi}{2}$

SOLUTION (i) Recall the AM–GM inequality $\dfrac{a+b}{2} \ge \sqrt{ab}$, a, b > 0.

Hence

$$\frac{3\tan^2 x + 12\cot^2 x}{2} \ge \sqrt{3\tan^2 x + 12\cot^2 x}$$

$\Rightarrow 3\tan^2 x + 12\cot^2 x \ge 12$

$\Rightarrow 3\tan^2 x + 12\cot^2 x + 5 \ge 17$

When x approaches $\dfrac{\pi}{2}$, y approaches ∞.

Hence, the range is $y \in [17, \infty)$.

(ii) $y = 2\cos x + \sec^2 x,\ 0 < x < \dfrac{\pi}{2}$

$$\Rightarrow \frac{\cos x + \cos x + \sec^2 x}{3} \ge \sqrt[3]{\cos x.\cos x.\sec^2 x}$$

$\Rightarrow 2\cos x + \sec^2 x \ge 3$

The least value of 3 is attained if $\cos x = \sec^2 x$.

$\cos^3 x = 1 \Rightarrow x = 0$, which lies in the given domain.

When x approaches $\dfrac{\pi}{2}$, y approaches ∞.

Hence, the range is $y \in [3, \infty)$.

2. Graphical Method

We can find the range of function using graph. The set of y–coordinates of the graph of a function is the range.

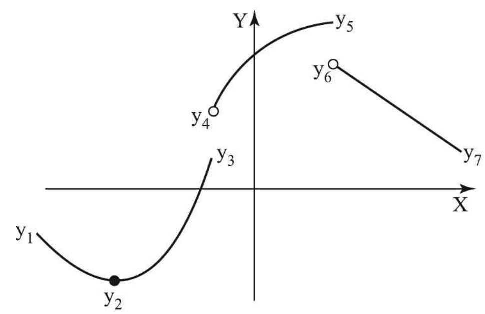
(a) Continuous function

If y = f(x) is continuous in its domain then range of f(x) is $[f(x)_{min.}, f(x)_{max.}]$

(b) Piece-wise continuous function

In case of piece-wise continuous functions, range will be union of $[f(x)_{min.}, f(x)_{max.}]$ over all those intervals where f(x) is continuous, as shown by the following example.

Let the graph of a sample function y = f(x) be as follows:



Then, the range of the above piece-wise continuous function is $[y_2, y_3] \cup (y_4, y_5] \cup (y_6, y_7]$.

EXAMPLE 4.15 Find the range of

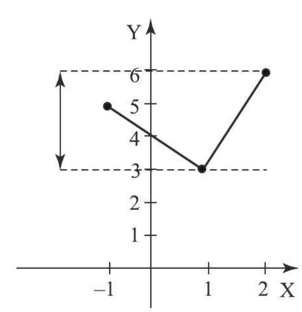
$y = 2|x - 1| + |x + 2|,\ -1 \le x \le 2$.

SOLUTION

$$y = \begin{cases} -2(x-1) + x + 2, & -1 \le x < 1 \\ 2(x-1) + x + 2, & 1 \le x \le 2 \end{cases}$$

$$= \begin{cases} 4 - x, & -1 \le x < 1 \\ 3x, & 1 \le x \le 2 \end{cases}$$

The graph of the function is shown below:



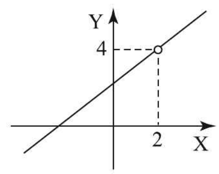
Hence, the range is $y \in [3, 6]$.

EXAMPLE 4.16 Find the range of

$$f(x) = \frac{x^2 - 4}{x - 2}$$

SOLUTION $f(x) = \dfrac{x^2 - 4}{x - 2} = x + 2;\ x \neq 2$

The graph of f(x) is shown below :

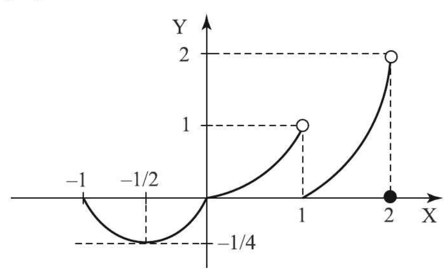


The range of f(x) is R – {4}

EXAMPLE 4.17 Find the range of y = x {x}, $-1 \le x \le 2$.

SOLUTION $y = \begin{cases} x(x+1), & -1 \le x < 0 \\ x^2, & -1 \le x < 0 \\ x(x-1), & 1 \le x < 2 \\ 0, & x = 2 \end{cases}$

The graph of the function is shown below:



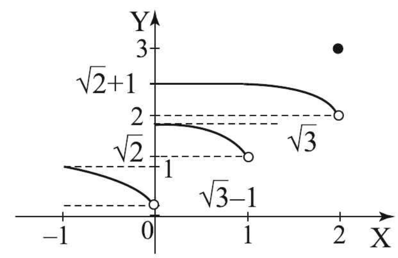
Hence, the range is $y \in \left[-\frac{1}{4}, 2\right)$.

EXAMPLE 4.18 Find the range of

$$y = \sqrt{3 - x} + [x],\ -1 \le x \le 2.$$

SOLUTION $y = \begin{cases} \sqrt{3-x} - 1, & -1 \le x < 0 \\ \sqrt{3-x}, & 0 \le x < 1 \\ \sqrt{3-x} + 1, & 1 \le x < 2 \\ 3, & x = 2 \end{cases}$.

The graph of the function is shown below:



$\therefore\ y \in \left(\sqrt{3} - 1, 1\right] \cup \left[\sqrt{2}, \sqrt{3}\right] \cup \left(2, \sqrt{3} + 1\right] \cup \{3\}$.

3. Representing x in terms of y

The definition of a function is usually represented as y (i.e. f(x) which is the dependent variable) in terms of an expression of x (which is the independent variable).

To find range we rewrite the given definition so as to represent x in terms of an expression of y and thus obtain range (possible values of y).

Let $y = f(x) \Rightarrow x = g(y)$

Write conditions, if any, on transformation from $y = f(x)$ to $x = g(y)$ and also write condition on 'y' so that g(y) produces real values of x lying in given domain.

Sometimes in more complex functions – particularly those, which involves transcendental functions, it is difficult to solve for x explicitly in terms of y. In such cases, we may be stuck with problem of finding range with this method.

EXAMPLE 4.19 Find the range of the function

$$y = \frac{e^x - 1}{e^x + 1}.$$

SOLUTION $y = \dfrac{e^x - 1}{e^x + 1}$

In order to find range, we solve the function for x.

$\Rightarrow ye^x + y = e^x - 1$

$\Rightarrow e^x(y - 1) = -y - 1$

$\Rightarrow ex = \dfrac{-y-1}{y-1} = \dfrac{1+y}{1-y}$

$\Rightarrow x = \ell n\left(\frac{y+1}{1-y}\right)$

For x to be real, $\frac{y+1}{1-y} > 0 \Rightarrow \frac{y+1}{y-1} < 0$

$\Rightarrow -1 < y < 1.$

Hence, the range is $y \in (-1, 1)$.

EXAMPLE 4.20 Find the range of the function

$$y = \cos^{-1}\frac{x^2}{1+x^2}$$

if (i) $x \in R$, and (ii) $x \in (-\infty, 1]$.

SOLUTION

(i) $y = \cos^{-1}\frac{x^2}{1+x^2}$, $x \in R$.

$\Rightarrow \cos y = \frac{x^2}{1+x^2}$,

we must note that $0 \le y \le \pi$ ...(1)

$\Rightarrow \cos y + x^2 \cos y = x^2$

$\Rightarrow x^2 (1 - \cos y) = \cos y$

$\Rightarrow x^2 = \frac{\cos y}{1-\cos y}$

$\Rightarrow x = \pm\sqrt{\frac{\cos y}{1-\cos y}}$

For x to be real, $\frac{\cos y}{1-\cos y} \ge 0$

$\Rightarrow 0 \le \cos y < 1$ ...(2)

Solving (1) and (2), we get $y \in \left(0, \frac{\pi}{2}\right]$.

Hence, the range is $\left(0, \frac{\pi}{2}\right]$.

(ii) $y = \cos^{-1}\frac{x^2}{1+x^2}$, $x \le 1$

$\Rightarrow x = \pm\sqrt{\frac{\cos y}{1-\cos y}}$

$\Rightarrow x_1 = -\sqrt{\frac{\cos y}{1-\cos y}} \le 1$ is true.

$\Rightarrow \frac{\cos y}{1-\cos y} \ge 0 \Rightarrow 0 \le \cos y < 1$

$\Rightarrow y \in \left(0, \frac{\pi}{2}\right]$ ...(3)

Now $x_2 = \sqrt{\frac{\cos y}{1-\cos y}} \le 1$

$\Rightarrow \frac{\cos y}{1-\cos y} \ge 0$, $\cos y \le \frac{1}{2}$,

$\Rightarrow 0 \le \cos y \le \frac{1}{2}$

$\Rightarrow y \Rightarrow \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ ...(4)

Union of (3) and (4) gives $y \in \left[0, \frac{\pi}{2}\right]$. This is the required range.

EXAMPLE 4.21 Find the range of the function

$$f(x) = \frac{\{x\}}{1+\{x\}},$$

where {.} denotes the fractional part function.

SOLUTION Let $y = \frac{\{x\}}{1+\{x\}} \in \{x\} = \frac{y}{1-y}$.

Since $\{x\} \in [0, 1)$ we have $0 < \frac{y}{1-y} < 1$

$\Rightarrow y \in \left[0, \frac{1}{2}\right)$.

EXAMPLE 4.22 Find the range of the following functions:

$$y = \frac{1}{2+\sin 3x + \cos 3x}.$$

SOLUTION We have $y = \frac{1}{2+\sin 3x+\cos 3x}$

i.e. $\sin 3x + \cos 3x = \frac{1}{y} - 2$

$\Rightarrow \sqrt{2}\sin\left(3x + \frac{\pi}{4}\right) = \frac{1}{y} - 2$

$$\Rightarrow \sin\left(3x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\left(\frac{1}{y}-2\right).$$

Since, $\left|\sin\left(3x+\frac{\pi}{4}\right)\right| \le 1$, therefore we have

$$\left|\frac{1}{y}-2\right| \le \sqrt{2} \qquad \Rightarrow -\sqrt{2} \le \frac{1}{y}-2 \le \sqrt{2}$$

$$\Rightarrow 2-\sqrt{2} \le \frac{1}{y} \le 2\sqrt{2} \Rightarrow \frac{1}{2+\sqrt{2}} \le y \le \frac{1}{2-\sqrt{2}}$$

Hence, the range is $y \in \left[\frac{1}{2+\sqrt{2}}, \frac{1}{2-\sqrt{2}}\right]$.

4. Using Monotonicity / Maxima–Minima

For a continuous function in a finite interval, we need to compare the minimum and maximum values and values at endpoints to determine which are the least and greatest values.

In order to find the maximum and minimum of a continuous function f(x) in [a, b]:

Find out all the critical points of f(x) in (a, b) ie. interior points where f'(x) = 0 or it does not exist. Let $c_1, c_2, \ldots, c_n$ be the different critical points.

Find the value of the function at these critical points. Let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at the critical points.

Let $M = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$ and $m = \min\{f(a), f(c), f(c_2), \ldots, f(c_n), f(b)\}$

Then M is the maximum (greatest value) of f(x) in [a, b] and m is the minimum (least value) of f(x) in [a, b]. Thus, the range is [m, M].

Some functions are monotonic throughout the domain. Linear polynomial, for example, is either strictly increasing or decreasing. Similar is the case with exponential and logarithmic functions. If we investigate these functions in a finite interval, then function values at the end points of the closed interval are the least and greatest values in that interval. Further, functions which are not monotonic, can also be monotonic in a suitably selected interval. For example, sine function is a strictly increasing function in the interval [–π/2, π/2].

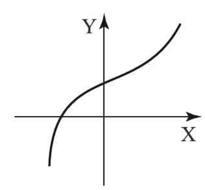
There are certain cases in which functions have exactly a pair of intervals characterized by opposite strict monotonic nature i.e. strictly increasing and strictly decreasing. In such cases, the point at which function changes its monotonic nature should be either minimum and maximum. This in turn means that function has exactly one minimum or maximum. Clearly, this minimum or maximum corresponds to least or greatest function value of the function in its domain.

We shall use these properties to determine range of a function. In order to understand application of these concepts, we need to go through the examples as given here carefully.

EXAMPLE 4.23 Find the range of $f(x) = x^3 + x + 1, x \in R$.

SOLUTION $f(x) = x^3 + x + 1$

$f'(x) = 3x^2 + 1$



$f'(x) > 0 \ \forall x \in R$

$\Rightarrow$ f(x) is strictly increasing.

$f(-\infty) = -\infty, f(\infty) = \infty$.

Hence, the range is $(-\infty, \infty)$.

EXAMPLE 4.24 Find the range of $f(x) = -2x + \cos x$.

SOLUTION $f(x) = -2x + \cos x$.

$f'(x) = -2 - \sin x < 0 \ \forall x \in R$

$\Rightarrow$ f(x) is strictly decreasing.

$f(-\infty) = \infty, \ f(\infty) = -\infty$.

Hence, the range is $(-\infty, \infty)$.

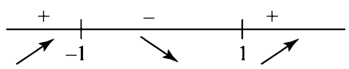
EXAMPLE 4.25 Find the range of $f(x) = x^3 - 3x + 1, -2 \le x \le 2$.

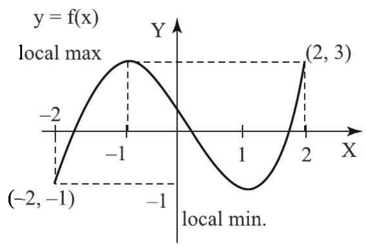
SOLUTION $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$

$f'(x) = 0 \in x = \pm 1$ (critical points)

Sign scheme of $f'(x)$





We evaluate f(x) at endpoints and critical points, from which we choose the least / greatest values :

$f(-2) = -1$ $\quad$ $f(2) = 3$

$f(-1) = 3$ $\quad$ $f(1) = -1$

The least value is – 1 and the greatest value is 3. Hence, the range is $y \in [-1, 3]$.

EXAMPLE 4.26 Find the range of

(i) $f(x) = e^x - e^{-x}$, $[-1, 2]$

(ii) $f(x) = x - 2\sin x$, $\quad x \in [0, \pi]$

SOLUTION (i) $f(x) = e^x - e^{-x}$

$f'(x) = e^x + e^{-x} > 0 \ \forall \quad x \in [-1, 2]$.

$\Rightarrow$ f(x) is strictly increasing

$f(-1) = \frac{1}{e} - e$ $\qquad$ least value

$f(2) = e^2 - \frac{1}{e^2}$ $\qquad$ greatest value

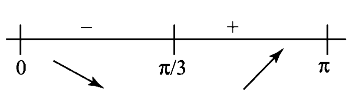
Hence, the range is $y \in \left[\frac{1}{e} - e,\ e^2 - \frac{1}{e^2}\right]$.

(ii) $f(x) = x - 2\sin x$, $x \in [0, \pi]$

$f'(x) = 1 - 2\cos x$

We solve $1 - 2\cos x = 0$ for critical points.

$\in \quad \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3} \in [0, \pi]$



$f(0) = 0$,

$f(\pi) = \pi$ $\qquad$ greatest value

$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$ least value

Hence, the range is $y \in \left[\frac{\pi}{3} - \sqrt{3},\ \pi\right]$.

EXAMPLE 4.27 A function is defined as $f(x) = \frac{e^x}{1+[x]}$. If domain of the function is $[0, \infty)$, find the range of the function.

SOLUTION The function contains greatest integer function in its denominator, whereas numerator of function is an exponential function. Greatest integer function, GIF, is defined in discontinuous intervals of length 1. It returns integral value equal to the starting value of discontinuous interval. The function, however, is continuous in sub-intervals of 1 i.e. in a finite interval. It is clear that function values and function nature will change in accordance with the values of GIF in sub-intervals of the domain. We need to determine nature of function in few of the initial sub-intervals and extrapolate the results to determine the range of function as required.

In order to determine nature of function, we write function and its derivative in the initial sub-intervals of the given domain. Note that domain of function starts with 0.

$0 \le x < 1, \quad f(x) = ex, \quad f'(x) = ex$

$1 \le x < 2, \quad f(x) = \frac{e^x}{2}, \quad f'(x) = \frac{e^x}{2}$

$2 \le x < 3, \quad f(x) = \frac{e^x}{3}, \quad f'(x) = \frac{e^x}{3}$

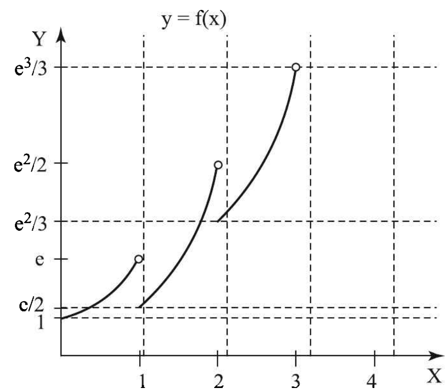
The derivatives in each interval are positive for values of x in the domain. It means that function is strictly increasing in each of the intervals. The least and greatest values, therefore, correspond to the end values in each finite sub-intervals. Using these least and greatest values, we determine range of the initial sub-intervals as given here :

$$0 \le x < 1 \quad \Rightarrow e^0 \le y < e^1 \Rightarrow 1 \le y < e$$

$$1 \le x < 2 \quad \Rightarrow \frac{e}{2} \le y < \frac{e^2}{2}$$

$$2 \le x < 3 \quad \Rightarrow \frac{e^2}{3} \le y < \frac{e^3}{3}$$

The plot of the function is drawn in the figure. Note that the function values in adjacent intervals overlap each other, but exceeds the greatest value in the preceding intervals. Clearly, function value begins from 1 and continues towards infinity. Hence, range of the function is $[1, \infty)$



EXAMPLE 4.28 Find range of the function

$$f(x) = \sqrt{(x-1)} + \sqrt{(5-x)}$$

SOLUTION The given function is continuous. Also, it involves square root of polynomial. It means that function is defined in a subinterval of real number set R. We, therefore, need to find the domain of the function first. Then we investigate the nature of the function in its domain and determine the least and the greatest function values. For individual square roots to be real,

$$x - 1 \ge 0 \quad \Rightarrow \quad x \ge 1$$

$$5 - x \ge 0 \quad \Rightarrow \quad x \le 5$$

Hence, domain of the function is a finite interval [1,5]. Now, we differentiate the function with respect to independent variable to determine the nature of the function.

$$f'(x) = \frac{1}{2\sqrt{(x-1)}} - \frac{1}{2\sqrt{5-x)}}$$

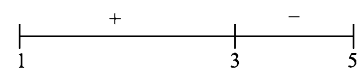
In order to draw sign scheme, we find the roots of the equation by putting f′(x) = 0 and solving for x:

$$2\sqrt{(x-1)} = 2\sqrt{(5-x)}$$

$$\Rightarrow x - 1 = 5 - x \Rightarrow \quad x = 3.$$

x = 3 is called as critical point.

Using test point, x = 4, f′(x) in negative. Hence, corresponding sign scheme of f′(x) is :



Since the function is increasing to the left and decreasing to the right of x = 3, the function value at x = 3 is maximum. Clearly, there is only one maximum. Thus, it is the greatest value. Further, either of two function values at endpoints is the least value of the function in the domain. Now, function values at end points and critical point are :

$$f(1) = 0 + 2 = 2$$

$$f(3) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

$$f(5) = 2 + 0 = 2$$

Clearly, the greatest value is $2\sqrt{2}$ and the least value is 2. Hence, the range of the function is $[2, 2\sqrt{2}]$.

5. Composition

We now take up the method of evaluating range of a composite function. Range of a composite function is evaluated from inside to outside. This means that we need to evaluate innermost function and then the one outside it. We shall highlight these aspects while working with the examples.

EXAMPLE 4.29 Find the range of

$$y = \sin^{-1}\left(\frac{\cos^2 x + 1}{2}\right).$$

SOLUTION Let $t = \dfrac{\cos^2 x + 1}{2}$

The range of t is $\left[\frac{1}{2}, 1\right]$.

Now, $y = \sin^{-1}t, t \in \left[\frac{1}{2}, 1\right]$

Then, $y \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

Hence, the range of the given function is $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

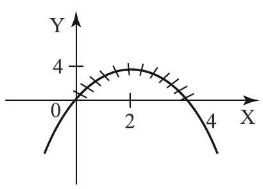
EXAMPLE 4.30 Find the range of $y = \log_2(4x - x^2)$.

SOLUTION $y = \log_2(4x - x^2)$

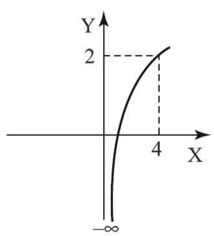
Let $t = 4x - x^2$

The range of t is $(-\infty, 4]$.

$y = \log_2 t, t \in (0, 4]$.



Note that we can feed only positive input to log function.



Hence, the range is $y \in (-\infty, 2)$.

EXAMPLE 4.31 Find the range of

$$y = 2^{\sqrt{x(x-1)}} + \sqrt{x(x-1)}$$

SOLUTION Let $t = x(x - 1) \Rightarrow t \in \left[-\frac{1}{4}, \infty\right)$.

Let $u = \sqrt{t}, t \in [0, \infty)$.

$\Rightarrow u \in [0, \infty)$

$\Rightarrow y = 2u + u, u \in [0, \infty)$

This is an increasing function.

The least value of y is obtained at $u = 0$, and y approaches ∞ as u approaches ∞.

$\Rightarrow y \in [1, \infty)$.

EXAMPLE 4.32 Find the range of the following functions:

(i) $y = \sin^{-1}\left(\frac{x^2+1}{x^2+2}\right)$,

(ii) $y = \ln\left(\frac{x^2+e}{x^2+1}\right)$

(iii) $f(x) = \log_2(\log_{1/2}(x^2 + 4x + 4))$.

SOLUTION (i) We have $\frac{x^2+1}{x^2+2} = 1 - \frac{1}{x^2+2}$

Now, we have $2 \le x^2 + 2 < \infty$

$\Rightarrow \frac{1}{2} \ge \frac{1}{x^2+2} > 0 \Rightarrow \frac{-1}{2} \le \frac{-1}{x^2+2} < 0$

$\Rightarrow 1 - \frac{1}{2} \le 1 - \frac{1}{x^2+2} < 1 \Rightarrow \frac{1}{2} \le \frac{x^2+1}{x^2+2} < 1$

$\Rightarrow \sin^{-1}\frac{1}{2} \le \sin^{-1}\left(\frac{x^2+1}{x^2+2}\right) < \sin^{-1} 1$

$\Rightarrow \frac{\pi}{6} \le y < \frac{\pi}{2}$.

Hence, the range is $y \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$.

(ii) $y = \ell n\left(\frac{x^2+e}{x^2+1}\right)$

$= \ln\left(\frac{x^2+1-1+e}{x^2+1}\right) = \ln\left(1 + \frac{e-1}{x^2+1}\right)$

Clearly range is (0, 1].

(iii) $f(x) = \log_2(\log_{1/2}(x^2 + 4x + 4))$

Since $0 \le (x^2 + 4x + 4) < \infty\ \forall\ x \in$ domain,

$-\infty < \log_{1/2}(x^2 + 4x + 4) < \infty$.

We adjust to $0 < \log_{1/2}(x^2 + 4x + 4) < \infty$.

$\Rightarrow -\infty < \log_2(\log_{1/2}(x^2 + 4x + 4)) < \infty$

Hence, the range of f is $(-\infty, \infty)$.

EXAMPLE 4.33 Find the range of the function defined by $y = \left[\frac{1}{[x-1]}\right]$, where [·] represents the greatest integer function.

SOLUTION Now the domain of the function is R – [1, 2). Here we cannot easily express x in terms of y. But as the function is in the form of $\left[\frac{1}{n}\right]$, where n is any nonzero integer, we have $-1 \le \frac{1}{n} \le 1$.

$\Rightarrow \left[\frac{1}{n}\right] = -1, 0, 1$

Hence, the values of y are –1, 0, 1.
Thus, the range of the function is {–1, 0, 1}.

EXAMPLE 4.34 Find the range of the function f defined by $f(x) = \left[\frac{1}{\sin\{x\}}\right]$ (where [.] and {.} respectively denote the greatest integer and the fractional part functions).

SOLUTION $\{x\} \in [0, 1)$

$\sin\{x\} \in [0, \sin 1)$ but f(x) is defined if $\sin\{x\} \neq 0$

$\Rightarrow \frac{1}{\sin\{x\}} \in \left(\frac{1}{\sin 1}, \infty\right)$

$\Rightarrow \left[\frac{1}{\sin\{x\}}\right] \in \{1, 2, 3, ...\}$

Thus the range is the set of natural numbers.

EXAMPLE 4.35 Find the range of

$f(x) = \left[\ln(\sin^{-1}\sqrt{x^2 + x + 1})\right]$

where [ ] denotes the greatest integer function.

SOLUTION We have $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ which is a positive quantity whose minimum value is 3/4.

Also, for the function

$f(x) = \left[\ln(\sin^{-1}\sqrt{x^2 + x + 1})\right]$

to be defined, we have $x^2 + x + 1 \le 1$

Thus, we have $\frac{3}{4} \le x^2 + x + 1 \le 1$

$\Rightarrow \frac{\sqrt{3}}{2} \le \sqrt{x^2 + x + 1} \le 1$

$\Rightarrow \frac{\pi}{3} \le \sin^{-1}(\sqrt{x^2 + x + 1}) \le \frac{\pi}{2}$

$\Rightarrow \sin^{-1} x$ is an increasing function, the inequality sign remains same]

$\Rightarrow \ln\left(\frac{\pi}{3}\right) \le \ln(\sin^{-1}\sqrt{x^2 + x + 1} \le \ln\left(\frac{\pi}{2}\right)$.

We have $0 < \ln\left(\frac{\pi}{3}\right) < \ln\left(\frac{\pi}{2}\right) < 1$.

$\Rightarrow \left[\ln(\sin^{-1}\sqrt{x^2 + x + 1})\right] = 0$ for all x in the domain. Hence, the range is $f(x) \in \{0\}$.

EXAMPLE 4.36 Find the range of the function $f(x) = \cot^{-1}\log_{0.5}\left(x^4 - 2x^2 + 3\right)$.

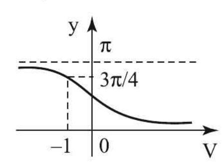
SOLUTION Let $u = x^4 - 2x^2 + 3 = (x^2 - 1)^2 + 2$

$\Rightarrow 2 \le u < \infty$

Let $v = \log_{0.5}(u)$, $2 \le u < \infty$

$\Rightarrow -\infty < v \le -1$

Now $f(x) = \cot^{-1}v$, $-\infty < v \le -1$



From the graph of $y = \cot^{-1}v$, the range is $\left[\frac{3\pi}{4}, \pi\right)$.

EXAMPLE 4.37 Find the domain and range of

$f(x) = \sin\left\{\ell n\left(\frac{\sqrt{4 - x^2}}{1 - x}\right)\right\}$.

SOLUTION $\sqrt{4 - x^2}$ is positive and $x^2 < 4$

$\Rightarrow -2 < x < 2$

$1 - x$ should also be positive. $\therefore x < 1$

Thus the domain of $\ell n\left(\frac{\sqrt{4 - x^2}}{1 - x}\right)$ is $-2 < x < 1$.

sine being defined for all values, the domain of

$\sin\left\{\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right\}$ is the same as the domain of $\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)$.

To study the range consider the function $\frac{\sqrt{4-x^2}}{1-x}$.

As x varies from –2 to 1, $\frac{\sqrt{4-x^2}}{1-x}$ varies in the open interval $(0, \infty)$ and hence $\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)$ varies from $-\infty$ to ∞.

Therefore, the range of $\sin\left(\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right)$ is $[-1, 1]$

EXAMPLE 4.38 Find the domain and range of the following functions :

(i) $f(x) = \sin[\ln(5x^2 - 8x + 4)]$

(ii) $f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

SOLUTION (i) $5(x - 4/5)^2 + 4/5 > 0$ is always true.

$\Rightarrow \ln(5x^2 - 8x + 4)$ is defined for all $x \in (-\infty, \infty)$.

Hence the domain is $D_f = (-\infty, \infty)$.

$$y = f(x) = \sin u, \text{ where } u = \ell n\left(5\left(x - \frac{4}{5}\right)^2 + \frac{4}{5}\right)$$

$u_{min} = \ln 4/5$ at $x = 4/5$.

i.e. $\ln 4/5 \le u < \infty$ ...(1)

For $u \in [\ln 4/5, \infty)$, $\sin u$ assumes all values between – 1 and 1.

$\Rightarrow \quad -1 \le y \le 1$.

Hence, the range is $R_f = [-1, 1]$.

(ii) Let $u = \frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}} = \frac{\sin x - \cos x}{\sqrt{2}} + 3$

$\Rightarrow 2 \le u \le 4$ since $\frac{\sin x - \cos x}{\sqrt{2}}$ lies between –1 and 1

$\log_2 u$ is defined for all $x \in (-\infty, \infty)$. Hence the domain is $x \in (-\infty, \infty)$.

Now, $y = \log_2 u$, $2 \le u \le 4 \Rightarrow \log_2 2 \le y \le \log_2 4$

$\Rightarrow 1 \le y \le 2$.

Hence the range is [1, 2].

EXAMPLE 4.39 $f(x) = \cos^{-1}\left(\log\left[\sqrt{[x^3+1]}\right]\right)$, find the domain and range of f(x) (where [.] denotes the greatest integer function).

SOLUTION For f(x) to be defined,

$$-1 \le \log\left[\sqrt{[x^3+1]}\right] \le 1$$

$$\Rightarrow e^{-1} \le \left[\sqrt{[x^3+1]}\right] \le e$$

$$\Rightarrow 0.37 \le \left[\sqrt{[x^3+1]}\right] \le 2.7$$

$\Rightarrow 1 \le \sqrt{[x^3+1]} < 3 \quad \Rightarrow 1 \le [x^3 + 1] < 9$

$\Rightarrow 1 \le [x^3] + 1 < 9 \quad \Rightarrow 0 \le [x^3] < 8$

$\Rightarrow 0 \le x < 2$

$\therefore$ Domain of f(x) is $x \in [0, 2)$.

When $0 \le x < 2$

$1 \le x^3 + 1 < 9$

$\therefore \ 1 \le [x^3 + 1] \le 8$

$\Rightarrow 1 \le \sqrt{[x^3+1]} \le 2\sqrt{2} \Rightarrow 1 \le \sqrt{[x^3+1]} \le 2.8$

Case 1: $1 \le \sqrt{[x^3+1]} < 2$ then $\left[\sqrt{[x^3+1]}\right] = 1$

Case 2: $2 \le \sqrt{[x^3+1]} \le 2.8$ then $\left[\sqrt{[x^3+1]}\right] = 2$

$\therefore$ f(x) has two outputs $\cos^{-1}\log 1$ and $\cos^{-1}\log 2$

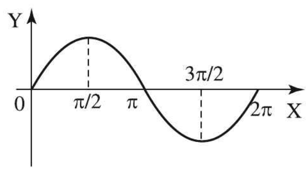
$\Rightarrow$ The range is $\{0, \cos^{-1}(\log 2)\}$

EXAMPLE 4.40 Let

$$f(x) = \max.\{\sin t : 0 \le t \le x\}$$
$$g(x) = \min.\{\sin t : 0 \le t \le x\}$$
$$\text{and } h(x) = [f(x) - g(x)]$$

where [ ] denotes greatest integer function, then find the range of h(x).

SOLUTION



$$f(x) = \begin{cases} \sin x; & 0 \le x < \dfrac{\pi}{2} \\ 1; & \dfrac{\pi}{2} \le x \le 2\pi \end{cases}$$

$$g(x) = \begin{cases} 0; & 0 \le x < \pi \\ \sin x; & \pi < x < \dfrac{3\pi}{2} \\ -1; & \dfrac{3\pi}{2} \le x \le 2\pi \end{cases}$$

$$h(x) = \begin{cases} 0; & 0 \le x < \dfrac{\pi}{2} \\ 1; & \dfrac{\pi}{2} \le x < \dfrac{3\pi}{2} \\ 2; & x \ge \dfrac{3\pi}{2} \end{cases}$$

Hence the range of h(x) is {0, 1, 2}.

PRACTICE PROBLEMS [E]

1. Find the domain and range of the function
$$f(x) = \frac{x+2}{x^2 - 8x - 4}.$$

2. Find the range of the following functions :
   (i) $f(x) = \dfrac{\sin x}{\sqrt{1+\tan^2 x}} - \dfrac{\cos x}{\sqrt{1+\cot^2 x}}$
   (ii) $f(x) = \max\{10 - |x-1|, (x-3)^2\}, x \in [0, 5]$.

3. Find the range of the functions :
   (i) $f(x) = \tan\left(\sqrt{\dfrac{\pi^2}{9} - x}\right)$
   (ii) $f(x) = \cos(\sin^{-1}\sqrt{x^2 + x + 1})$
   (iii) $f(x) = \dfrac{1 + 2x - 2[x]}{1 + x - [x]}$
   (iv) $f(x) = 2^{-2x-1}(1 - \cos 2x)$

4. Find the range of the functions :
   (i) $f(x) = \sqrt{^{x^2+4x}C_{2x^2+3}}$ (ii) $f(x) = 8^x + 4^x + 8^{-x} + 4^{-x} + 5$

5. Find the range of the functions :
   (i) $f(x) = \sin^4 x(1 + \sin^2 x) + \cos^4 x(1 + \cos^2 x)$
   (ii) $\sqrt{\ln(\cos(\sin x))}$

6. Find the range of the function f(x) = min (sin x, cos 2x).
7. If the range of $f(x) = \ell n\left(2\sin x + \tan x - \frac{3x}{\pi} + 1\right)$ for $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ is [a, b], then find the value of [a + b], where [.] represent greatest integer function.
8. If range of the function $f(x) = \frac{2\sin^2 x + 2\sin x + 3}{\sin^2 x + \sin x + 1}$ is [p, q], then find the value of 3p + 6q + 1.
9. Find the range of the functions :
   (i) $f(x) = 2x^3 - 15x^2 + 36x + \sin^{-1} x$
   (ii) $f(x) = |\sin x|^2 + 2|\sin x| - |\cos x|$
10. Find the range of $f(x) = \begin{cases} 2 - x^2 & , x < -1 \\ \sin^{-1} x & , -1 \le x \le 1 \\ \{x^2\} & x > 1 \end{cases}$.
11. Find the range of the function
    $f(x) = (\tan^{-1} x)^2 + \frac{2}{\sqrt{1+x^2}}$
12. Find the range of the function :
    (i) $f(x) = x^2 + \frac{1}{x^2+1}$
    (ii) $f(x) = 3^x (2^x + 1) + 3^{-x}(2^{-x} + 1) + 2$.
13. Find the range of the following functions :
    (i) $f(x) = \sqrt{x-1} + 2\sqrt{3-x}$
    (ii) $f(x) = \log_2 (\sqrt{x} - 2x^2)$
    (iii) $f(x) = \sec^{-1} (x^3 - 3x), x \in [-1, 1]$
    (iv) $f(x) = x^3 + 3x^2 + 20x - 10 \sin x$
14. If $f(x) = ax^2 - bx + c$ is an integer for $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, then prove that the value of f(x) is an integer for any integral x.

4.6 COMPOSITION OF FUNCTIONS

A function is a special mapping between two sets. It emerges that it is possible to combine two functions with the help of composition.

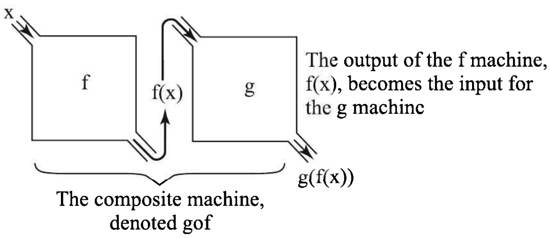
Let f and g be functions. Suppose that x is such that f(x) is in the domain of g. Then the function that assigns to x the value g(f(x)) is called the composition of g and f. It is denoted by gof. It is also called function of a function.

Thus if f(x) = u and g(u) = y, then (gof)(x) = y.

(gof is read as "g of f" or as "g composed with f.").

In practical terms, the definition says: "To compute gof, first apply f and then apply g to the result."

Thinking of functions as input-output machines, we may consider gof as the machine built by hooking the machine for g onto the machine for f, as shown in the figure.



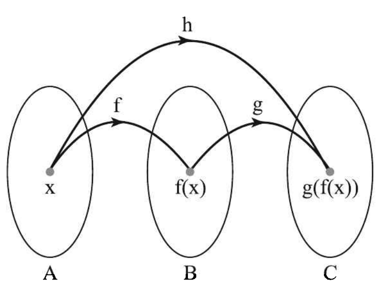
Notice that f(x) also serves as the input variable to the function g. Hence, outputs from f are the inputs to g ; and the expression g(f(x)) is the final output variable from this two-stage process.

Let there be two functions defined as :

$f : A \to B, y = f(x)$

$g : B \to C, y = g(x)$

Observe that set B is common to the two functions. It means that every element x of set A has an image f(x) in set B. Our objective here is to define a new function $h : A \to C$ and its rule. For an element x in A, we have an image f(x) in B, and corresponding to f(x), there is a unique image in set C. Following the symbolic notation, the image of f(x) is denoted by g(f(x)). The figure here depicts the relationship among three the sets via two functions and the composition function h.



It follows, then, that for every element x in A, there exists an element g(f(x)) in set C. This concluding statement is the definition of a new function :

$h : A \to C$, $h(x) = g(f(x))$ for all $x \in A$

By convention, we call this new function h as gof and is read as "g of f" or "g composed with f".

$gof(x) = g(f(x))$ for all $x \in A$

The two symbolical representations are equivalent.

EXAMPLE 4.1 Let two functions be defined as:

$f : R \to R,\ f(x) = x + 1$

$g : R \to R,\ g(x) = x^2$

Determine gof and fog.

SOLUTION According to definition,

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(x+1) \\ &= (x+1)^2 \end{aligned}$$

Again, according to definition,

$$\begin{aligned} fog(x) &= f(g(x)) \\ &= f(x^2) \\ &= (x^2+1) \end{aligned}$$

Note that gof (x) $\neq$ fog (x). It indicates that composition of functions is not commutative, in general.

EXAMPLE 4.2 Let $f(x) = \dfrac{x+2}{3x-1}$, $g(x) = x^2$ and $h(x) = \sqrt{x}$. Find h(f(g(x))) .

SOLUTION

$$h(f(g(x))) = h(f(x^2)) = h\left(\frac{x^2+2}{3x^2-1}\right) = \sqrt{\frac{x^2+2}{3x^2-1}} .$$

EXAMPLE 4.3 A function $f : R \to R$ is such that $f\left(\dfrac{1-x}{1+x}\right) = x$ for all $x \neq -1$. Then prove that

(i) $f((x)) = x$

(ii) $f\left(\dfrac{1}{x}\right) = -f(x),\ x \neq 0$

(iii) $f(-x-2) = -f(x) - 2$

SOLUTION (i) $f\left(\dfrac{1-x}{1+x}\right) = x$

Let $\dfrac{1-x}{1+x} = t \Rightarrow t + tx = 1 - x$

$\Rightarrow \quad x(1+t) = 1 - t \Rightarrow x = \dfrac{1-t}{1+t}$

$$f(t) = \frac{1-t}{1+t}$$

$\therefore \quad f(x) = \dfrac{1-x}{1+x}$ ...(1)

$\therefore \quad f(f(x)) = \dfrac{1-f(x)}{1+f(x)} = \dfrac{1-\dfrac{1-x}{1+x}}{1+\dfrac{1-x}{1+x}}$

$= \dfrac{1+x-1+x}{1+x+1-x} = \dfrac{2x}{2}$

$\therefore \quad f(f(x)) = x.$

(ii) From (1), $f\left(\dfrac{1}{x}\right) = \dfrac{1-(1/x)}{1+(1/x)} = \dfrac{x-1}{x+1}$

$f\left(\dfrac{1}{x}\right) = -\left(\dfrac{1-x}{1+x}\right) = -f(x).$

(iii)

Also $f(-x-2) = \dfrac{1+(x+2)}{1-(x+2)} = -\left(\dfrac{x+3}{x+1}\right)$ ...(2)

and $\quad -f(x) - 2 = -\left(\dfrac{1-x}{1+x} + 2\right)$

$= -\left(\dfrac{1-x+2+2x}{1+x}\right)$

$= -\left(\dfrac{x+3}{x+1}\right)$ ...(3)

From (2) and (3), $f(-x-2) = -f(x) - 2$.

EXAMPLE 4.4 Suppose $f(x) = ax + b$ and $g(x) = bx + a$, where a and b are positive integers. If $f(g(50)) - g(f(50)) = 28$ then find all possible values of the product ab.

SOLUTION

$f[g(x)] = a(bx + a) + b$

$= abx + a^2 + b$ ...(1)

$g[f(x)] = b(ax + b) + a$

$= abx + b^2 + a$ ...(2)

(1) – (2) gives $f[g(50)] - g[f(50)] = a^2 - b^2 + b - a$

$\therefore \quad (a^2 - b^2) + (b - a) = 28$

$(a - b)(a + b - 1) = 28 = 1 \times 28$

or $\quad 2 \times 14$ or 4×7

Let $\quad a - b = 1$ and $a + b - 1 = 28$

and, $\quad 2a - 2 = 28$

$\Rightarrow \quad a = 15 ; b = 14$

$\therefore \quad ab = 210$

If $\quad a - b = 2$ and $a + b - 1 = 14$ (not possible)

If $\quad a - b = 4$ and $a + b - 1 = 7$

$2a - 1 = 11$

$\Rightarrow \quad a = 6$ and $b = 2$

$\therefore \quad ab = 12$

Hence the possible values of the product ab are 210 and 12.

Properties of composition

(i) The composition is generally not commutative except for some special functions.

$gof(x) \neq fog(x)$

For example, if $f(x) = x^2 + 1$ and $g(x) = x - 1$.

$gof(x) = g(x^2 + 1) = (x^2 + 1) - 1 = x^2$

$fog(x) = f(x - 1) = (x - 1)^2 + 1 = x^2 - 2x$

Notice $gof(x) \neq fog(x)$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions then

$(hog)of = ho(goh)$

i.e. composition among three functions is independent of parentheses.

Proof: We have

$ho(gof)(x) = h(gof(x)) = h(g(f(x))).$

and $(hog)of(x) = hog(f(x)) = h(g(f(x))).$

Hence, $ho(gof) = (hog)of$.

Domain and range of composition

Let there be two functions defined as :

$f : D_1 \to R_1, y = f(x)$

$g : D_2 \to R_2, y = g(x).$

and the set $A = \{x \in D_1 : f(x) \in D_2\}$. If A is not an empty set, then the function h defined on A by

$h(x) = g(f(x))$ is called composite function of g and f and is denoted by gof.
Note that the domain of gof is A, which is a subset of D_1, the domain of f.
Thus, the domain of gof (x) i.e. g(f(x))
$= \{x : x \in \text{Dom } f, \text{ and } f(x) \in \text{Dom } g\}$
Corresponding to A, there is a set $B = R_1 \cap D_2$, which acts as the domain of g for the composition. The range of g under the domain B is the range of gof, C. Note that the range of gof is C, which is a subset of the range of g.
We now define the composite function gof as
$gof : A \to C$
$gof\,(x) = g(f(x))$.

EXAMPLE 4.5 Let two functions be defined as:
$f = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$
$g = \{(0, 4), (3, 2), (4, 3), (5, 1)\}$
Define gof.

SOLUTION Here,
Domain of $f = D_1 = \{1, 2, 3, 4\}$
Range of $f = R_1 = \{2, 3, 4, 5\}$
Domain of $g = D_2 = \{0, 3, 4, 5\}$
Range of $g = R_1 = \{1, 2, 3, 4\}$
$B = R_1 \cap D_2 = \{3, 4, 5\}$
The domain of gof is the set A corresponding to which the output of the function f is B.
Hence $A = \{2, 3, 4\}$
We can see that A is a subset of D_1.
The range of gof is the set C which is the output under g, corresponding to elements of set B.
Hence, $C = \{1, 2, 3\}$.
We can see that C is a subset of R_2.
$gof : \{2, 3, 4\} \to \{1, 2, 3\}$
$gof = \{(2, 2), (3, 3), (4, 1)\}$
Composition of two functions results in a new rule. The expression of the new rule may prohibit certain elements of the original domain set. For example, consider the function,

$$f(x) = \frac{1}{(1-x)}.$$

Clearly, the domain of function is $R - \{1\}$. Let us now see the expression of composition of function with itself,

$$f \circ f = f\,(f\,(x)) = \frac{1}{1 - \dfrac{1}{(1-x)}} = \frac{1-x}{1-x-1}$$

$$\Rightarrow\ f \circ f = \frac{x-1}{x}.$$

It seems that this expression is valid for all real values of x when $x \neq 0$. However, fof is also not defined for $x = 1$, since f is undefined at $x = 1$. Thus, the domain of the composition fof(x) is $R - \{0,1\}$.

EXAMPLE 4.6 If $f(x) = x^2 + 1$, $g(x) = \dfrac{1}{x-1}$, then find (fog) (x) and (gof) (x) and write their domains.

SOLUTION Given, $f(x) = x^2 + 1$ ...(1)

$$g(x) = \frac{1}{x-1} \qquad \text{...(2)}$$

Now $(fog)\,(x) = f(g(x)) = f\left(\dfrac{1}{x-1}\right)$

$$= \left(\frac{1}{x-1}\right)^2 + 1 = \frac{1}{(x-1)^2} + 1.$$

Domain of fog(x) is $x \in R - \{1\}$.
$(gof)\,(x) = g(f(x)) = g(x^2 + 1)$

$$= \frac{1}{x^2 + 1 - 1} = \frac{1}{x^2}$$

Domain of gof(x) is $x \in R - \{0\}$.

EXAMPLE 4.7 Consider the function
$f\,(x) = x + 1;$
$0 \leq x \leq 2$. Determine fof (x).

SOLUTION The composition of the function with itself is :
$fof\,(x) = f\,(f\,(x)) = f\,(x + 1)\ ;\ 0 \leq x \leq 2$
$\Rightarrow\ fof\,(x) = (x + 1) + 1;\ 0 \leq x + 1 \leq 2 \text{ and } 0 \leq x \leq 2$

$\Rightarrow$ fof (x) = x + 2; $-1 \le x \le 1$ and $0 \le x \le 2$

$\therefore$ fof (x) = x + 2; $0 \le x \le 1$.

EXAMPLE 4.8 Define fog and gof for the following functions :

(i) $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

SOLUTION (i) Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[1, \infty)$.

Since range of f is a subset of domain of g,

$\therefore$ Domain of gof is $[-3, \infty)$ {equal to the domain of f }

gof (x) = g(f(x)) = g ($\sqrt{x+3}$) = 1 + (x + 3) = x + 4. Range of gof is $[1, \infty)$.

Further since range of g is a subset of domain of f,

$\Rightarrow$ Domain of fog is R {equal to the domain of g}

fog (x) = f(g(x)) = f(1+ x^2) = $\sqrt{x^2+4}$.

Range of fog is $[2, \infty)$.

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[-1, \infty)$.

Since range of f is a subset of the domain of g,

$\therefore$ Domain of gof is $[0, \infty)$ and g(f(x))= g($\sqrt{x}$) = x − 1. Range of gof is $[-1, \infty)$.

Further since range of g is not a subset of the domain of f i.e. $[-1, \infty) \not\subset [0, \infty)$, fog is not defined on whole of the domain of g.

Domain of fog is {x : x $\in$ domain of g and g(x) $\in$ $[0, \infty)$, the domain of f}.

Thus the domain of fog is D = {x : $0 \le g(x) < \infty$}

i.e. D = { x : $0 \le x^2 - 1$} = { x : $x \le -1$ or $x \ge 1$ }

= $(-\infty, -1] \cup [1, \infty)$.

fog (x) = f(g(x)) = f(x^2−1) = $\sqrt{x^2-1}$. Its range is $[0, \infty)$.

EXAMPLE 4.9 Let $f : (0, \infty) \to \left[\frac{1}{2}, \infty\right)$,

$$f(x) = x^2 + \frac{1}{2}$$

$$g : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ g(x) = \sin^{-1} x.$$

Define gof and write it domain and range.

SOLUTION $\text{gof}(x) = g(f(x)) = g\left(x^2 + \frac{1}{2}\right)$

$$= \sin^{-1}\left(x^2 + \frac{1}{2}\right).$$

We must have $-1 \le x^2 + \frac{1}{2} \le 1$, $x \in (0, \infty)$

$\Rightarrow -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$, $x \in (0, \infty)$

Hence, the domain of gof is $\left(0, \frac{1}{\sqrt{2}}\right]$.

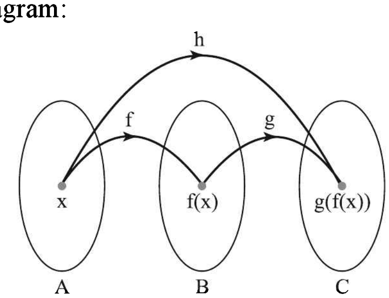
With these values of x, $f(x) \in \left(\frac{1}{2}, 1\right]$.

To find the range of gof, we need to find the range of $g(x) = \sin^{-1}x$, when $x \in \left(\frac{1}{2}, 1\right]$.

$\Rightarrow g(x) \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right]$

Thus, the range of gof is $\left(\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Let us see the details of the process in the following diagram:



EXAMPLE 4.10 Two functions are defined as:

$g(x) = \frac{1-x}{1+x}$; $0 \le x \le 1$

$h(x) = 4x(1 - x)$; $0 \le x \le 1$

Determine the composition goh(x) and hog(x).

SOLUTION We need to interpret domain of the composition as we compose the required function.

Let $y = h(x) = 4x(1-x) = 4x - 4x^2;\ 0 \le x \le 1$

Then,

$$goh(x) = g(y);\ 0 \le y \le 1 \text{ and } 0 \le x \le 1$$

$$\Rightarrow goh(x) = g(4x - 4x^2)\ ;\ 0 \le 4x - 4x^2 \le 1 \text{ and } 0 \le x \le 1$$

Let us first interpret rule of the function :

$$\Rightarrow goh(x) = \frac{1-y}{1+y} + \frac{1-(4x-4x^2)}{1+(4x-4x^2)}$$

$$= \frac{1-4x+4x^2}{1+4x-4x^2}.$$

Now, we interpret the interval $0 \le (4x - 4x^2) \le 1$ in parts. The left part is :

$$(4x - 4x^2) \ge 0 \Rightarrow x(1-x) \ge 0$$

$$\Rightarrow 0 \le x \le 1$$

The right part of the interval is :

$$(4x - 4x^2) \le 1 \Rightarrow (4x^2 - 4x) \ge -1$$

$$\Rightarrow (4x^2 - 4x + 1) \ge 0$$

$$\Rightarrow -\infty < x < \infty$$

The intersection of the two parts is $0 \le x \le 1$. Thus, the interval of composition is intersection of intervals $0 \le x \le 1$ and $0 \le x \le 1$. Therefore,

$$goh(x) = \frac{1-4x+4x^2}{1+4x-4x^2}\ ;\ 0 \le x \le 1.$$

For determining hog(x), Let

$$y = g(x) = \frac{1-x}{1+x};\ 0 \le x \le 1$$

Then, $hog(x) = g(y);\ 0 \le y \le 1$ and $0 \le x \le 1$

$$\Rightarrow hog(x) = g\left(\frac{1-x}{1+x}\right);\ 0 \le \left(\frac{1-x}{1+x}\right) \le 1 \text{ and } 0 \le x \le 1$$

Let us first interpret rule of the function :

$$hog(x) = 4y(1-y) = 4\,\frac{1-x}{1+x}\left(1 - \frac{1-x}{1+x}\right)$$

$$hog(x) = 4\,\frac{1-x}{1+x}\left(\frac{1-x-1+x}{1+x}\right) = \frac{8x(1-x)}{(1+x)}.$$

Now, we interpret the interval $0 \le \left(\frac{1-x}{1+x}\right) \le 1$ in parts. The left part is :

$$\frac{1-x}{1+x} \ge 0$$

$$\Rightarrow 0 \le x \le 1.$$

The right part of the interval is :

$$\frac{1-x}{1+x} \le 1 \Rightarrow \frac{x}{1+x} \ge 0 \Rightarrow x < -1,\ x \ge 0.$$

The intersection of two parts is $0 \le x \le 1$. Thus, interval of composition is intersection of intervals $0 \le x \le 1$ and $0 \le x \le 1$ which is $0 \le x \le 1$. Therefore,

$$hog(x) = \frac{8x(1-x)}{(1+x)^2}\ ;\ 0 \le x \le 1.$$

Iteration / Repeated Composition

Given a function f(x), its iterate at x is f(f(x)), that is, we use its value as the new input.

The iterates at x

$$x,\ f(x),\ f(f(x)),\ f(f(f(x)))....$$

are called 0-th iterate, 1st iterate, 2nd iterate, 3rd iterate, etc. We denote the n-th iterate by f[n]. In some cases it is easy to find the n-th iterate of a function, For example

If $p(x) = x^t$ then $p^{[n]}(x) = x^{tn}$,

If $q(x) = mx$ then $q^{[n]}(x) = m^n x$,

If $r(x) = mx + k$ then $r^{[n]}(x) = m^n x + k\left(\frac{m^n - 1}{m - 1}\right)$.

The above examples are more an exception than the rule. Even if it is possible to find a closed formula for the n-th iterate, in some cases it is very complicated.

EXAMPLE 4.11 Let $f(x) = \frac{1}{1-x}$. Find the n-th iterate of f at x, and determine the set of values of x for which it is defined.

SOLUTION We have

$f^{[2]}(x) = (f \circ f)(x) = f(f(x)) = \dfrac{1}{1-\dfrac{1}{1-x}} = \dfrac{x-1}{x}$.

$f^{[3]}(x) = (f \circ f \circ f)(x) = f(f^{[2]}(x)) = f\left(\dfrac{x-1}{x}\right)$

$= \dfrac{1}{1-\dfrac{x-1}{x}} = x$

Notice now that

$f^{[4]}(x) = (f \circ f^{[3]})(x) = f(x) = f^{[1]}(x)$. We see that f is cyclic with period 3, that is,

$$f^{[1]}(x) = f^{[4]}(x) = f^{[7]}(x) = ... = \frac{1}{1-x}$$

$$f^{[2]}(x) = f^{[5]}(x) = f^{[8]}(x) = ... = \frac{x-1}{x}$$

$$f^{[3]}(x) = f^{[6]}(x) = f^{[9]}(x) = ... = x.$$

The formulae above hold for $x \notin \{0, 1\}$.

$$fn(x) = \begin{cases} \dfrac{1}{1-x}, & n = 3m+1 \\ \dfrac{x-1}{x}, & n = 3m+2 \\ x, & n = 3m,\ m \in I \end{cases}.$$

EXAMPLE 4.12 A function f(x) is given as : $f(x) = \{2 - x^n\}^{1/n}$, where $x > 0$ and n is a positive integer. Prove that $f\{f(x)\} + f\{f\left(\dfrac{1}{x}\right)\} \geq 2$.

SOLUTION The domain of the given function is a positive number as $x > 0$. In order to prove the inequality, we need to determine each composition on the left hand side of the given inequality.

If $f(x) = \{a - x^n\}^{1/n}$, then $f\{f(x)\} = x$.

Hence, if $f(x) = \{2 - x^n\}^{1/n}$, then $f\{f(x)\} = x$. Similarly, we determine $f\{f(1/x)\}$. Here,

$$f\{f(x)\} = f(y) = \left[a - \left\{\left(a - \frac{1}{x^n}\right)^{1/n}\right\}^n\right]^{1/n}$$

$$= \left(a - a - \frac{1}{x^n}\right)^{1/n}$$

$$\Rightarrow f\left\{f\left(\frac{1}{x}\right)\right\} = \left(\frac{1}{x^n}\right)^{1/n} = \frac{1}{x}.$$

Substituting these values in the LHS of the inequality, we have :

$$\text{LHS} = f\{f(x)\} + f\left\{f\left(\frac{1}{x}\right)\right\} = x + \frac{1}{x}$$

$$= \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2.$$

But, the square term is a non-negative number. Hence, LHS ≥ 2

$\Rightarrow f\{f(x)\} + f\{f(1/x)\} \geq 2$.

EXAMPLE 4.13 If $f(x) = px + q$ and $f(f(f(x))) = 8x + 21$, where p and q are real numbers, then find the value of p + q.

SOLUTION Using $f(x) = px + q$,

$$f(f(x)) = f(px + q) = p(px + q) + q$$
$$= p^2x + pq + q$$

and $f(f(f(x))) = f(p^2x + pq + q)$

$$= p(p^2x + pq + q) + q$$
$$= p^3x + p^2q + pq + q$$

Equating this to 8x + 21 , we get p = 2 and q = 3.

Thus $p + q = 2 + 3 = 5$.

EXAMPLE 4.14 Let $f(x) = x^2 + kx$ where k is a real number. Find the set of values of k for which the equation f (x) = 0 and f(f(x)) = 0 have exactly the same real solution set.

SOLUTION

$$f(f(x)) = f(x^2 + kx) = (x^2 + kx)(k + x^2 + kx)$$
$$f(f(x)) = f(x)(k + f(x)) = 0$$

For f (x) and f(f(x)) to have the same solution set, k + f (x) = 0 should have no solution

$$x^2 + kx + k = 0 \quad \ldots(1)$$

$D < 0$ (for no solution)

$$k^2 - 4k < 0$$

$$k(k-4) < 0$$

$$k \in (0, 4)$$

Also for k = 0 both f (x) = 0 and f(f(x)) = 0 has same solution set.

$\therefore$ $k \in [0, 4)$.

EXAMPLE 4.15 Let $f_1(x) = x, f_2(x) = 1 - x, f_3(x) = \frac{1}{x}, f_4(x) = \frac{1}{1-x}, f_5(x) = \frac{x}{x-1}, f_6(x) = \frac{x-1}{x}$.

Suppose that $f_6(f_m(x)) = f_4(x)$ and $f_n(f_4(x)) = f_3(x)$ then find m and n.

SOLUTION Given $f_6(x) = \frac{x-1}{x}$ ...(1)

$$f_6(f_m(x)) = f_4(x) = \frac{1}{1-x} \quad \text{(given)}$$

$$\Rightarrow f_6(f_m(x)) = \frac{f_m(x)-1}{f_m(x)} = \frac{1}{1-x}$$

(using given relation $f_6(x) = \frac{x-1}{x}$)

Put $f_m(x) = k$, $\frac{k-1}{k} = \frac{1}{1-x}$

$$\Rightarrow k - kx - 1 + x = k \quad \Rightarrow \quad k = \frac{x-1}{x}$$

$$\Rightarrow f_m(x) = \frac{x-1}{x} = f_6(x) \quad \Rightarrow \quad m = 6.$$

Again, $f_n(f_4(x)) = f_3(x) = \frac{1}{x}$

$$f_n\left(\frac{1}{1-x}\right) = \frac{1}{x}.$$

Let $\frac{1}{1-x} = t \Rightarrow t - tx = 1 \Rightarrow x = \frac{t-1}{t}$

$$\Rightarrow f_n(t) = \frac{t}{t-1} \Rightarrow f_n(x) = \frac{x}{x-1} = f_5(x)$$

Hence, n = 5.

EXAMPLE 4.16 Let $f(x) = \frac{x-1}{x+1}$, $f^2(x) = f(f(x))$, $f^3(x) = f(f^2(x))$,..........$f^{k+1}(x) = f(f^k(x))$ for $k \in N$.

Find $f^{1998}(x)$.

SOLUTION $f(x) = \frac{x-1}{x+1}$,

$$f^2(x) = f(f(x)) = \frac{f-1}{f+1} = \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = \frac{-1}{x}$$

$$f^3(x) = f(f^2(x)) = \frac{f^2(x)-1}{f^2(x)+1} = \frac{\frac{-1}{x} - 1}{\frac{-1}{x} + 1} = \frac{x+1}{x-1}$$

$$f^4 = f(f^3(x)) = \frac{f^3(x)-1}{f^3(x)+1} = \frac{\frac{x+1}{x-1} - 1}{\frac{x+1}{x-1} + 1} = x$$

$$f^5(x) = f(f^4(x)) = \frac{x+1}{x-1} = f(x).$$

Thus, we can see that $f^k(x)$ repeats itself at intervals of k = 4.

Hence, we have $f^{1998}(x) = f^2(x) = \frac{x+1}{x-1}$.

$[\because 1998 = 499 \times 4 + 2]$

Composition of piecewise defined functions

EXAMPLE 4.17 A function is defined as

$$f(x) = \begin{cases} -1 & ; -2 \le x < 0 \\ x-1 & ; 0 \le x \le 2 \end{cases}$$

Find composition f(|x|) and its domain.

SOLUTION The function is defined by different rules in two intervals.

The composition consists of two functions f(x) and |x|. We know that modulus is defined for all values of x. However, domain of f(x) is [–2, 2].

$$f(|x|) = \begin{cases} -1 & ; -2 \le |x| < 0 \text{ and } R \\ |x|-1 & ; 0 \le |x| \le 2 \text{ and } R \end{cases}$$

The interval $-2 \le |x| \le 0$ can be interpreted in parts. The left part $|x| \ge -2$, is always true. The right part $|x| < 0$ is meaningless, which yields no solution for x. Therefore, f(|x|) is domainless.

On the other hand, the interval $0 \le |x| \le 2$ has two parts. The left part $|x| \ge 0$ is true for all values of x. The right part is $|x| \le 2$. This expands to $-2 \le x \le 2$.

The intersection of $-2 \le x \le 2$ and R is $-2 \le x \le 2$.

Hence, the composition is

$$f(|x|) = |x| - 1; -2 \le x \le 2.$$

EXAMPLE 4.18 Two functions are given as :

$$f(x) = -1 + |x - 1|; -1 \le x \le 3$$

$$g(x) = 2 - |x + 1|; \quad -2 \le x \le 2.$$

Find fog(x).

SOLUTION Let $y = g(x) = (2 - |x + 1|); -2 \le x \le 2$

Now, let us first determine the composition,

$$fog(x) = f(y) = f(2 - |x + 1|)| = -1 + |2 - |x + 1 - 1|$$

This is the rule of the composition function. In order to find the domain of the composition, we write domains of two functions as an intersection :

$$fog(x) = -1 + |2 - |x + 1| - 1|;$$

$$\text{when } -1 \le y \le 3 \text{ and } -2 \le x \le 2$$

We interpret the interval $-1 \le y \le 3$ as :

$$-1 \le (2 - |x + 1|) \le 3$$

$$\Rightarrow -3 \le (-|x + 1|) \le 1$$

Multiplying each term with –1 and reversing inequality, we have :

$$3 \ge (|x + 1|) \ge -1$$

But, $|x + 1| \ge -1$ is true for all values of x. hence, the above inequality is equal to the interval given by the first part $|x + 1| \le 3$ as :

$$-3 \le x + 1 \le 3$$

$$\Rightarrow -4 \le x \le 2$$

Hence, interval of the composition is intersection of intervals $-4 \le x \le 2$ and $-2 \le x \le 2$

∴ The domain is $-2 \le x \le 2$.

The composition, therefore, is

$$fog(x) = -1 + |1 - |x + 1||; -2 \le x \le 2.$$

EXAMPLE 4.19 A function is defined as:

$$f(x) = \begin{cases} 1+x & ; x \ge 0 \\ 1-x & ; x < 0 \end{cases}$$

Determine the composition f{f(x)}.

SOLUTION For f(1 + x) in the interval $x \ge 0$

$$f(1+x) = \begin{cases} 1+1+x & ; 1+x \ge 0 \text{ and } x \ge 0 \\ 1-1-x & ; 1+x < 0 \text{ and } x \ge 0 \end{cases}$$

$$= \begin{cases} 2+x & ; x \ge -1 \text{ and } x \ge 0 \\ -x & ; x < -1 \text{ and } x \ge 0 \end{cases}$$

The intersection of upper intervals $x \ge -1$ and $x \ge 0$ is equal to $x \ge 0$. There is no common interval for the intersection of lower intervals. Hence,

$$f(1 + x) = 2 + x; x \ge 0.$$

For f(1 – x) in the interval x < 0,

$$f(1-x) = \begin{cases} 1+1-x & ; 1-x \ge 0 \text{ and } x \le 0 \\ 1-1+x & ; 1-x < 0 \text{ and } x < 0 \end{cases}$$

$$= \begin{cases} 2-x & ; x \le 1 \text{ and } x < 0 \\ x & ; x > 1 \text{ and } x < 0 \end{cases}.$$

The intersection of upper intervals $x \le 1$ and $x < 0$ is equal to $x < 0$. There is no common interval for the intersection of the lower intervals. Hence,

$$f(1 - x) = 2 - x; x < 0$$

Therefore, the composition is :

$$f\{f(x)\} = \begin{cases} 2+x & ; x > 0 \\ 2-x & ; x < 0 \end{cases}.$$

EXAMPLE 4.20 If $f(x)=\begin{cases} x, & \text{when } x \text{ is rational} \\ 1-x, & \text{when } x \text{ is irrational}\end{cases}$, then find fof (x).

SOLUTION

$$\text{fof}(x)=\begin{cases} f(x), & \text{when } f(x) \text{ is rational} \\ 1-f(x), & \text{when } f(x) \text{ is irrational}\end{cases}$$

$$=\begin{cases} x, & \text{when } x \text{ is rational} \\ 1-(1-x), & \text{when } x \text{ is irrational}\end{cases} = x,\ x\in R$$

fof (x) = x for all real x.

EXAMPLE 4.21 Let $f(x)=\begin{cases} x+1, & x<0 \\ x^2, & x\ge 0\end{cases}$

and $g(x)=\begin{cases} 2x, & -1\le x<1 \\ 3-x, & 1x\ge 1\end{cases}$.

Define gof and gog.

SOLUTION $\begin{cases} 2f(x), & -1\le f(x)<1 \\ 3-f(x), & f(x)\ge 1\end{cases}$

$$=\begin{cases} 2(x+1), & -1\le x+1<1, & x<0 \\ 3-(x+1), & x+1\ge 1, & x<0 \\ 2x^2, & -1\le x^2<1, & x\ge 0 \\ 3-x^2, & x^2\ge 1, & x\ge 0\end{cases}$$

$$\text{gof}(x)=\begin{cases} 2x+2, & -2\le x<0 \\ 2x^2, & 0\le x<1 \\ 3-x^2, & x\ge 1\end{cases}$$

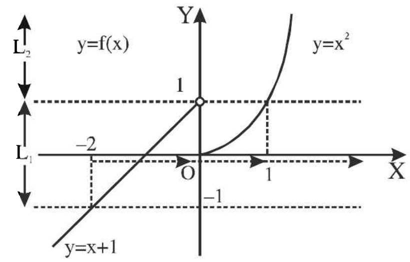
Now, $g(g(x))=\begin{cases} 2g(x), & -1\le g(x)<1 \\ 3-g(x), & g(x)\ge 1\end{cases}$

$$=\begin{cases} 2(2x), & -1\le 2x<1, & -1\le x<1 \\ 3-2x, & 2x\ge 1, & -1\le x<1 \\ 2(3-x), & -1\le 3-x<1, & x\ge 1 \\ 3-(3-x), & 3-x\ge 1, & x\ge 1\end{cases}$$

$$=\begin{cases} 4x, & -\frac{1}{2}\le x<\frac{1}{2} \\ 3-2x, & \frac{1}{2}\le x<1 \\ 2(3-x), & 2<x\le 4 \\ x, & 1\le x\le 2\end{cases}$$

Graphical method

$$\text{gof}(x)=\begin{cases} 2f(x), & -1\le f(x)<1 & L_1 \\ 3-f(x), & f(x)\ge 1 & L_2\end{cases}$$



From the figure, the inequalities $-1\le f(x)<1$ and $f(x)\ge 1$ can be solved. Thus,

$$\text{gof}(x)=\begin{cases} 2(x+1), & -2\le x<0 \\ 2x^2, & 0\le x<1 \\ 3-x^2, & x\ge 1\end{cases}.$$

EXAMPLE 4.22 If $f(x)=||x-3|-2|,\ 0\le x\le 4$ and $g(x)=4-|2-x|,\ -1\le x\le 3$, then find fog(x).

SOLUTION $f(x)=||x-3|-2|,\ 0\le x\le 4$

$$=\begin{cases} |x-1| & 0\le x<3 \\ |x-5| & 3\le x\le 4\end{cases}=\begin{cases} 1-x & 0\le x<1 \\ x-1 & 1\le x<3 \\ 5-x & 3\le x\le 4\end{cases}$$

$g(x)=4-|2-x|,\ -1\le x\le 3$

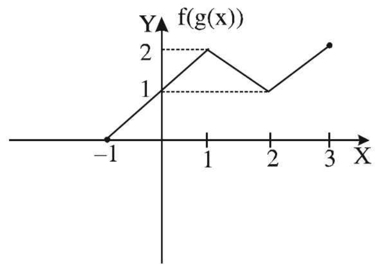
$$=\begin{cases} 4-(2-x) & -1\le x<2 \\ 4-(x-2) & 2\le x\le 3\end{cases}$$

$$=\begin{cases} 2+x & -1\le x<2 \\ 6-x & 2\le x\le 3\end{cases}$$

$$\therefore\ \text{fog}(x)=\begin{cases} 1-g(x) & 0\le g(x)<1 \\ g(x)-1 & 1\le g(x)<3 \\ 5-g(x) & 3\le g(x)\le 4\end{cases}$$

$$=\begin{cases} 1-(2+x) & 0\le 2+x<1 & \text{and} & -1\le x<2 \\ 2+x-1 & 1\le 2+x<3 & \text{and} & -1\le x<2 \\ 5-(2+x) & 3\le 2+x\le 4 & \text{and} & -1\le x<2 \\ 1-6+x & 0\le 6-x<1 & \text{and} & 2\le x\le 3 \\ 6-x-1 & 1\le 6-x\le 3 & \text{and} & 2\le x\le 3 \\ 5-6+x & 3\le 6-x\le 4 & \text{and} & 2\le x\le 3\end{cases}$$

$$= \begin{cases} -1-x & -2 \le x < 1 \quad \text{and} \quad -1 \le x < 2 \\ 1+x & -1 \le x < 1 \quad \text{and} \quad -1 \le x < x \\ 3-x & 1 \le x \le 2 \quad \text{and} \quad -1 \le x < 2 \\ x-5 & -6 \le -x < -5 \quad \text{and} \quad 2 \le x \le 3 \\ 5-x & -5 \le -x < -3 \quad \text{and} \quad 2 \le x \le 3 \\ x-1 & -3 \le -x \le -2 \quad \text{and} \quad 2 \le x \le 3 \end{cases}$$

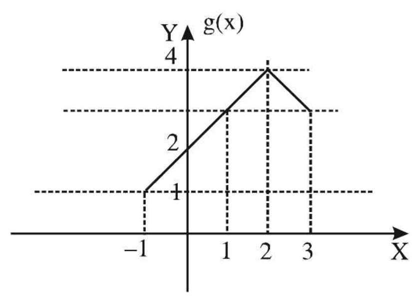


$$= \begin{cases} -1-x & -2 \le x < -1 \quad \text{and} \quad -1 \le x < 2 \\ 1+x & -1 \le x < 1 \quad \text{and} \quad -1 \le x < 2 \\ 3-x & 1 \le x \le 2 \quad \text{and} \quad -1 \le x < 2 \\ x-5 & 5 < x \le 6 \quad \text{and} \quad 2 \le x \le 3 \\ 5-x & 3 < x \le 5 \quad \text{and} \quad 2 \le x \le 3 \\ x-1 & 2 \le x \le 3 \quad \text{and} \quad 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} 1+x & -1 \le x < 1 \\ 3-x & 1 \le x < 2 \\ x-1 & 2 \le x \le 3 \end{cases}$$

Graphical method

$$g(x) = \begin{cases} 2+x & -1 \le x < 2 \\ 6-x & 2 \le x \le 3 \end{cases}$$



$$\therefore \ fog(x) = \begin{cases} 1-g(x) & 0 \le g(x) < 1 \\ g(x)-1 & 1 \le g(x) < 3 \\ 5-g(x) & 3 \le g(x) \le 4 \end{cases}$$

$$= \begin{cases} 1-g(x) & \text{for no value} \\ g(x)-1 & -1 \le x < 1 \\ 5-g(x) & 1 \le x \le 3 \end{cases}$$

$$= \begin{cases} 2+x-1 & -1 \le x < 1 \\ 5-(2+x) & 1 \le x < 2 \\ 5-(6-x) & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} x+1 & -1 \le x < 1 \\ 3-x & 1 \le x < 2 \\ x-1 & 2 \le x \le 3 \end{cases}$$

EXAMPLE 4.23

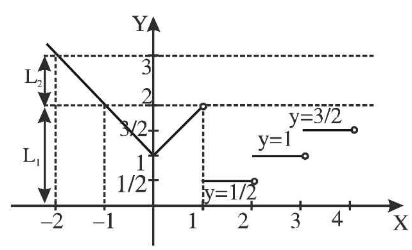
Let $f(x) = \begin{cases} 1+|x|, & x < 1 \\ \dfrac{[x]}{2}, & 1 \le x < 4 \end{cases}$

and $g(x) = \begin{cases} x+1, & x < 2 \\ x^2-1, & 2 \le x < 3 \end{cases}$

Define gof.

SOLUTION

$$g(f(x)) = \begin{cases} f(x)+1, & f(x) < 2 & L_1 \\ f^2(x)-1, & 2 \le f(x) < 3 & L_2 \end{cases}$$



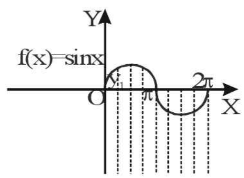
$$\text{Hence, } g(f(x)) = \begin{cases} (|x|+1)^2-1, & -2 < x \le -1 \\ (|x+1|)+1, & -1 < x < 1 \\ \dfrac{[x]}{2}+1, & 1 \le x < 4 \end{cases}$$

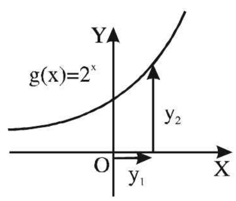
Graph of composite functions

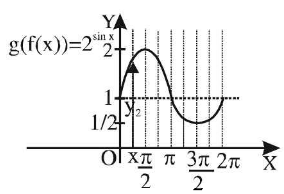
EXAMPLE 4.24 Draw the graph of $y = 2^{\sin x}$

SOLUTION

Let $f(x) = \sin x$, $g(x) = 2^x$ $y = 2^{\sin x} = g(f(x))$





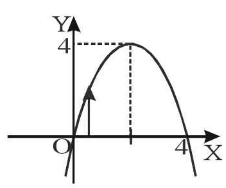


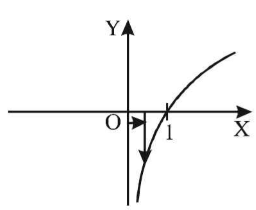
EXAMPLE 4.25 Draw the graph of

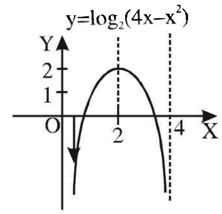
$y = \log_2 (4x - x^2)$

SOLUTION Let $f(x) = 4x - x^2$, $g(x) = \log_2 x$

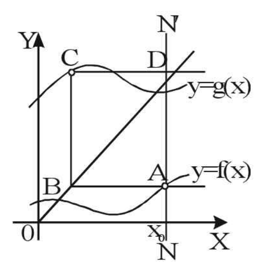
$y = \log_2 (4x - x^2) = g(f(x))$



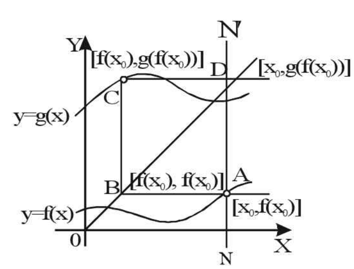




EXAMPLE 4.26 Given the graphs of the two functions, $y = f(x)$ and $y = g(x)$. In the adjacent figure from point A on the graph of the function $y = f(x)$ corresponding to the given value of the independent variable (say x_0), a straight line is drawn parallel to the X-axis to intersect the bisector of the first and the third quadrants at point B. From the point B a straight line parallel to the Y-axis is drawn to intersect the graph of the function $y = g(x)$ at C. Again a straight line is drawn from the point C parallel to the X-axis, to intersect the line NN′ at D . If the straight line NN′ is parallel to Y-axis, then show that the co-ordinates of the point D are $(x_0, g(f(x_0)))$.



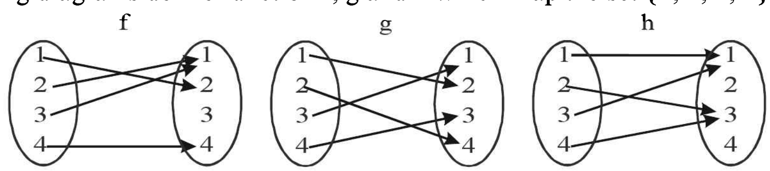
SOLUTION The points have been labelled in the diagram.



CONCEPT PROBLEMS [B]

1. Let f : R → R and g : R → R be defined by $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$. Find the formula defining the composite functions :
   (i) f o g (ii) g o f
   (iii) g o g (iv) f o f

2. The following diagrams define function f, g and h which map the set {1, 2, 3, 4} into itself.



   (i) Find the ranges of f, g and h.
   (ii) Find the composition function
   (a) f o g (b) h o f (c) g o g

3. Consider the functions $a(x) = 1 - x^2$, $b(x) = 1 + 2x$, $c(x) = 4$. Find
   (i) (a + b + c) (2) (ii) (abc) (2)
   (iii) (aoboc) (2)

4. If $f(x) = \dfrac{x}{\sqrt{1+x^2}}$ then show that $(f \text{ o } f \text{ o } f)\, x = \dfrac{x}{\sqrt{(1+3x^2)}}$.

5. Let $f(x) = x^2 - 2x$ and $g(x) = 2x + 1$. Find all x for which (fog) (x) = (gof) (x).

6. Consider $f : \left[-\sqrt{3}, \sqrt{3}\right] \to R$, $f(x) = \sqrt{3 - x^2}$ and $g : [-2, \infty) \to R$, $g(x) = -\sqrt{x+2}$. Find
   (i) Range (f) (ii) Range (g) (iii) Dom (f o g)
   (iv) f o g (v) Dom (g o f) (vi) g o f.

7. Let $f(x) = 2x^2 - 1$ and $g(x) = 4x^3 - 3x$. Show that (f o g)x = g o f)(x).

8. Given $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$ and $g(x) = \dfrac{3x + x^3}{1+3x^2}$ then show that fog (x) equals 3f(x).

9. Let f, g and h be functions from R to R. Show that
   (i) (f + g)oh = foh + goh (ii) (f · g)oh = (foh) · (goh)

10. Let f(x) = 2x + 3. How many functions are there of the form g(x) = ax + b, a and b constants such that f o g = g o f ?

11. Let f(x) = 2x + 3. How many functions are there of the form $g(x) = ax^2 + bx + c$, a, b and c constants. a not 0. such that f o g = g o f ?

12. Let f : R → R be the signum function defined as $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ and g : R → R be the greatest integer function given by g(x) = [x], where [x] is greatest integer less than or equal to x. Then, does fog and gof coincide in (0,1]?

13. Find (g o f o g o f o g o f o g o f) if $f(x) = \sqrt{x^2 + 1}$ and $g(x) = \sqrt{x^2 - 1}$.

14. Let $f^{[1]} = f$ be given by $f(x) = \dfrac{1}{1-x}$ and let $f^{[n+1]} = fof^{[n]}$ for $n \geq 1$. find $f^{[69]}(x)$.

15. Let $f^{[1]}(x) = f(x) = 2x$, $f^{[n+1]} = fof^{[n]}$, $n \geq 1$. Find a closed formula for $f^{[n]}$

16. If $f(x) = \begin{cases} x^2 & x \leq 0 \\ x & x > 0 \end{cases}$ and $g(x) = -|x|$, $x \in R$, then find fog and gof.

17. If $f(x) = \begin{cases} \sqrt{(1-x)}; & x \leq 1 \\ 3-(x-1)^2; & x > 1 \end{cases}$ and $g(x) = \begin{cases} 10-x^2; & x \geq 3 \\ 5-x; & x < 3 \end{cases}$ then find f o g(x).

PRACTICE PROBLEMS [F]

18. Let $f(x) = \ell n \sec^{-1}(2x - x^2)$ and $g(x) = \sqrt{x-3}$ then prove that the function gof(x) is not defined for any real x.

19. Let $f : R^+ \to R$ be defined as $f(x) = (x^2 - x + 2)$ and $g : [1, 2] \to [1, 2]$ be defined as $g(x) = \{x\} + 1$, where $\{x\}$ denotes the fractional part of x.

    If the domain and range of f(g(x)) are [a, b] and [c, d) respectively ($a < b$, $c < d$), then find the value of $\dfrac{b}{a} + \dfrac{d}{c}$.

20. Find the number of distinct real solutions of the equation $P(P(P(x)) = 0$, where $P(x) = x^2 - 1$.

21. Define a numerical function f as follows : If x is an integer, let $f(x) = -x$; if x is not an integer, let $f(x) = x$.
    (a) Show that (f o f)(x) = x for all numbers x.
    (b) How many functions g are there from R to R such that (g o g)(x) = x for all numbers x?

22. Let $f(x) = 2x^2 - 1$,
    (a) Show that f o f commutes with f.
    (b) Find all polynomials of degree 4 that commute with f.

23. If $f(x) = \begin{cases} 1+x; & 0 \leq x \leq 2 \\ 3-x; & 2 < x \leq 3 \end{cases}$, then find fof (x).

24. Let $f(x) = \begin{cases} x+1, & x \leq 1 \\ 2x+1, & 1 < x \leq 2 \end{cases}$ and $g(x) = \begin{cases} \{x\}, & -1 \leq x < 2 \\ [x+2], & 2 \leq x \leq 3 \end{cases}$. Find fog.

25. If $f(x) = \begin{cases} x+1 & , x \leq 1 \\ 2x+1 & , 1 < x \leq 2 \end{cases}$ and $g(x) = \begin{cases} x^2, & -1 \leq x < 2 \\ x+2, & 2 \leq x \leq 3 \end{cases}$. Find fog and gof.

26. If $f(x) = -1 + |x - 2|$, $0 \leq x \leq 4$ and $g(x) = 2 - |x|$, $-1 \leq x \leq 3$. Then find fog (x) and gof (x). Draw rough sketch of the graphs of fog (x) and gof (x).

27. If $a_0 = x$, $a_{n+1} = f(a_n)$, n = 0, 1, 2, ... , find a_n when (i) $f(x) = \sqrt{|x|}$, (ii) $f(x) = \dfrac{1}{1-x}$.

28. If $f(x) = \begin{cases} 1+x & , -2 \le x < 1 \\ 4-2x & , 1 \le x \le 3 \end{cases}$ and $g(x) = \begin{cases} -1+2x & , x < 0 \\ x-1 & , x \ge 0 \end{cases}$. Then find f(g(x)).

4.7 | ONE-ONE AND MANY-ONE FUNCTIONS

We now wish to classify the functions according to some specific properties. In, general, the functions are classified as :

(i) One-one or many-one
(ii) Onto or into
(iii) Even, odd or neither even nor odd
(iv) Periodic or non-periodic
(v) Invertible or non-invertible.

The relation between two sets under a rule has two perspectives. We can look at the relation in the direction from domain set A to co-domain set B. This is the function view. But, we can also look this relation in opposite direction from B to A.

When we see function from domain A to co-domain B, then we find the following possibilities :

(i) Every element of the domain is related to different element of the co-domain (one-one function or injection)
(ii) More than one elements of the domain is related to an element of the co-domain (many-one function)

When we see relation from co-domain B to domain A, then we find the following possibilities :

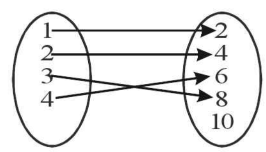
(i) There are elements in co-domain, which are not related to any of the elements in domain (into function).
(ii) There are no elements in co-domain, which are not related to elements in domain (onto function or surjection).
(iii) There are elements in co-domain, which are related to exactly one element in domain. Then the function is called as one-one function.
(iv) There are elements in co-domain, which are related to more than one element in domain. Then the function is called as many-one function.

One-one function (Injection)

A function f : A → B is said to be one-one (injective) if no element of set B is an image of more than one element of set A,

i.e. different elements of domain set A have different images in co-domain set B.

In plain words, every x in A associates with a distinct y in B. For example, consider the mapping



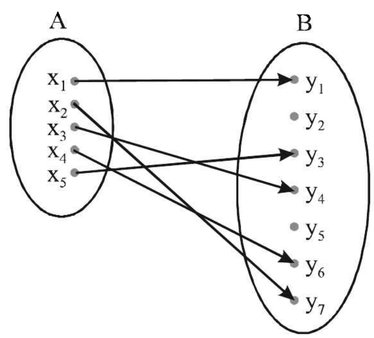
It is an one-one function.

In order to represent the condition of injectivity symbolically, we can think of two different elements x_1 and x_2 in set A. Then, two images $f(x_1)$ and $f(x_2)$ in B, corresponding to these elements in A are not equal. We capture this intent in constructing condition for an injection as :

$f : A \to B$ is an injection, $x_1 \ne x_2$, $f(x_1) \ne f(x_2)$ for all $x_1, x_2 \in A$.

We can also interpret injection by asserting that if two images are equal, then it means that they are images of the same pre-image. The map diagram, corresponding to an injection, is shown in the figure. Note that elements in A are mapped to

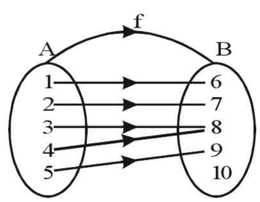
different elements in B.



Many-one function

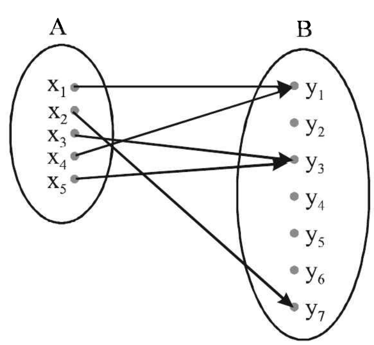
A function f : A → B is said to be many-one if there exists atleast one element in set B which is image of more than one element of set A.

For example, consider the mapping



It is a many-one function since f(3) = f(4) = 8.

The test of condition for many-one function is easy : if a function is not one-one, then it is many-one. Alternatively, we can check literally going by the definition whether there exist such many-one correspondences.



Many-one function

Modulus function is one such many-one function. The function yields the same value for positive and negative arguments of same magnitude.

$f(x) = |x|$

$\Rightarrow f(-1) = |-1| = 1$

$\Rightarrow f(1) = |1| = 1$

We should understand that a function of the type "one-many" is barred from the very definition of function. The element of domain can be related to exactly one element in the co-domain.

EXAMPLE 4.1 Let f be a one-one function with domain {x, y, z} and range {1, 2, 3}. It is given that only one of three conditions given below are true and remaining two are false :

$f(x) = 1$; $f(y) \neq 1$; $f(z) \neq 2$

Determine a for which $f(\alpha) = 1$.

SOLUTION We shall check each such possibility to see whether function is one-one.

First combination :

If $f(x) = 1$ is true, then $f(x) = 1$.
If $f(y) \neq 1$ is false, then $f(y) = 1$.
If $f(z) \neq 2$ is false, then $f(z) = 2$.

The function is clearly not one-one.

Second combination :

If $f(x) = 1$ is false, then $f(x) = 2$ or 3.
If $f(y) \neq 1$ is true, then $f(y) = 2$ or 3.
If $f(z) \neq 2$ is false, then $f(z) = 2$.

As $f(z) = 2$, $f(x) = f(y) = 3$. Again, the function is not one-one.

Third combination :

If $f(x) = 1$ is false, then $f(x) = 2$ or 3.
If $f(y) \neq 1$ is false, then $f(y) = 1$.
If $f(z) \neq 2$ is true, then $f(z) = 1$ or 3.

As $f(y) = 1$, $f(z) = 3$ and $f(x) = 2$. In this case, we see that image and pre-image are related distinctly. The function is one-one. Also, $f(y) = 1$.

Hence, $\alpha = y$.

Tests for one-one and many-one functions

1. Solving $f(x_1) = f(x_2)$

Let $f(x_1) = f(x_2)$. Now, solve for x_2 in terms of x_1.

If the solution is $x_2 = x_1$ only then f is one-one.
If other solutions also exist then f is many-one.

EXAMPLE 4.2 Consider a function defined as

$f : I \to I$ by $f(x) = x^2 + 1$ for all $x \in I$.

Determine whether the function is an injection?

SOLUTION We consider two arbitrary elements of the domain set such that:

$f(x_1) = f(x_2)$

We have deliberately considered a contradictory supposition with respect to the requirement of injectivity. If this supposition yields $x_1 = x_2$, then the given function is an injection; otherwise not. Here,

$\Rightarrow x_1^2 + 1 = x_2^2 + 1$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

This is not a unique solution. Here, x_1 is not uniquely equal to x_2. We conclude that the given function is not an injection. As a matter of fact, we can infer a check on our conclusion as,

$f(1) = f(-1) = 2$.

Thus, we see that two pre-images relate to one image, which is contradictory to the requirement of an injection.

EXAMPLE 4.3 Find whether f(x) = is one-one.

SOLUTION Let $f(x_1) = f(x_2)$

$e^{x_1^3} = e^{x_2^3}$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

The only real solution is $x_1 = x_2$. Hence, f is one-one.

EXAMPLE 4.4 Find whether $f(x) = \dfrac{x}{1+x^2}$ is one-one or many-one.

SOLUTION Let $f(x_1) = f(x_2)$

$$\frac{x_1}{1+x^2} = \frac{x_2}{1+x_2^2}$$

$\Rightarrow x_1 + x_1x_2^2 - x_2 - x_1^2 x_2 = 0$

$\Rightarrow x_1 - x_2 + x_1 x_2(x_2 - x_1) = 0$

$\Rightarrow (x_1 - x_2)(1 - x_1, x_2) = 0$

$x_1 = x_2$ and $x_2 = 1/x_1$.

There exist real solutions other than $x_1 = x_2$. Hence, f is many-one.

EXAMPLE 4.5 Check whether the function $f : R \to R$ given by $f(x) = \dfrac{3x-4}{x^2+1}$ is one-one.

SOLUTION Let $x_1, x_2 \in R$ and let $f(x_1) = f(x_2)$

$\Rightarrow (3x_1 - 4)(x_2^2 + 1) = (3x_2 - 4)(x_1^2 + 1)$

$\Rightarrow 3x_1x_2^2 + 3x_1 - 4x_2^2 - 4 = 3x_1^2x_2 + 3x_2 - 4x_1^2 - 4$

$\Rightarrow 3x_1x_2(x_2 - x_1) + 3(x_1 - x_2) - 4(x_2^2 - x_1^2) = 0$

$\Rightarrow 3x_1x_2 - 3 - 4(x_1 + x_2) = 0$ or $x_1 = x_2$.

The first equation suggests that if $x_1 = 0$ then

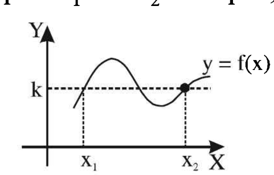
$x_2 = -\dfrac{3}{4}$ where $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

$\therefore$ f is not one-one.

2. Horizontal Line Test

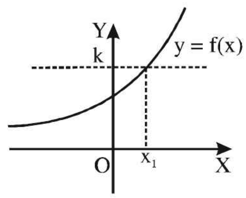
Alternatively, draw plot of the given function. Draw a line parallel to x-axis such that it intersects as many points on the plot as possible. If it intersects the graph only at one point, then the function is one-one.

The graph of a one-one function has the property that every horizontal line meets it in atmost one point. To see why, consider the line y = k in the figure. If it meets the graph of a function f in atleast two distinct points, say (x_1, k) and (x_2, k), then $f(x_1) = k$ and $f(x_2) = k$. This means that f is not a one-one function, since the outputs corresponding to the inputs x_1 and x_2 are equal, namely, k.



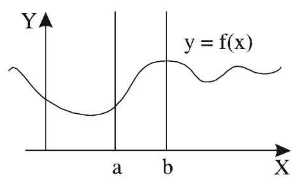
Graph of a many-one function

On the other hand, if each horizontal line meets the graph of a function f in atmost one point, then f is one-one.



Graph of a one-one function

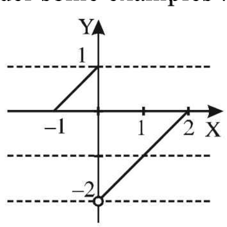
It is generally possible to restrict the domain of a function to some interval so that the function, considered only on that interval, is one-one (See the figure below.)



Restricting f to [a, b] produces a one-one function

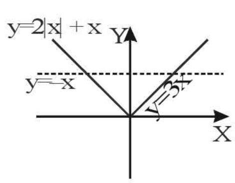
Consider some examples :

(i)



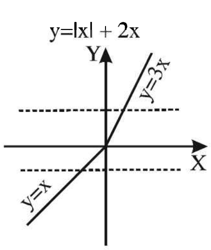
The above graph represents a many-one function.

(ii)



The above graph represents a many-one function.

(iii)

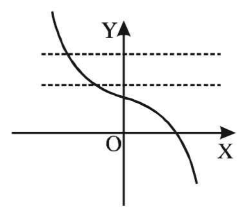


The above graph represents a one-one function.

EXAMPLE 4.6 A function $f : R \to R$ is given by $f(x) = 1 - x^3$. Is the function one-one?

SOLUTION Draw a line parallel to x-axis to intersect the plot of the function as many times as possible.

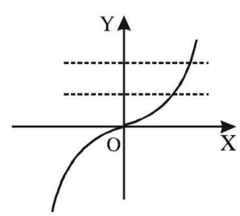
We find that all lines drawn parallel to x-axis intersect the plot only once. Hence, the function is one-one.



EXAMPLE 4.7 A function $f : R \to R$ is given by $f(x) = x|x|$. Is the function one-one?

SOLUTION $f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$

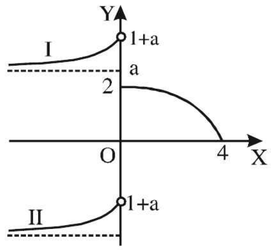
Draw the plot of the function and see intersection of a line parallel to x-axis.



We observe from its plot that there is no line parallel to x-axis, which intersects the functions more than once. Hence, the function is one-one.

EXAMPLE 4.8 Let $y = \begin{cases} 2^x + a, & x < 0 \\ \sqrt{4 - x}, & 0 \le x < 4 \end{cases}$. Find a for which the function is one-one.

SOLUTION No horizontal line should intersect the graph more than once.



Case I : $a \geq 2$

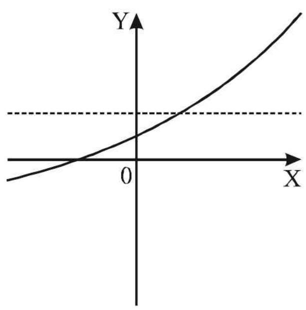
Case II : $1 + a \leq 0 \Rightarrow a \leq -1$.

Hence, $a \in (-\infty, -1] \cup [2, \infty)$

3. Increasing and Decreasing Functions

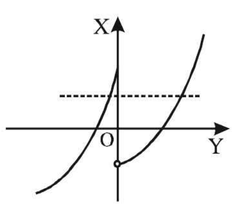
The concept of monotonous functions (increasing or decreasing) can be used to determine whether a function is one-one or not.

Consider the graph of a strictly increasing function shown below. The function is one-one because a straight line parallel to the x-axis intersects the graph only once. The same is the case when a function is strictly decreasing.



one-one function

If the function is either (i) strictly increasing or (ii) strictly decreasing then it is one-one. Note the use of word "strictly". If the graph, as shown in the figure below drops, then we can always find a line parallel to x-axis, which intersects it at two points and the function becomes many-one.



We can associate these characteristics with differential calculus. We can say that :

Either $\frac{dy}{dx} > 0$ for all x

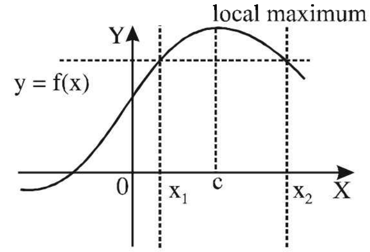
or $\frac{dy}{dx} < 0$ for all x.

As a matter of fact the derivative can be equal to zero for certain distinct values of x – not for an interval of x. Thus, we can write the condition for a strictly increasing function : $\frac{dy}{dx} \geq 0$ for all x, equality holding for certain distinct values of x.

Thus, if a function is continuous and $\frac{dy}{dx} \geq 0$ for all x, equality holding for certain distinct values of x, then the function is one-one.

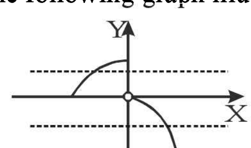
Similarly, if a function is continuous and $\frac{dy}{dx} \leq 0$ for all x, equality holding for certain distinct values of x, then the function is one-one.

A continuous non-monotonous function is many-one. A continuous function f(x) which has atleast one local maximum or local minimum at an interior point, is many-one. Also, every even function and every periodic function is many - one.



many-one function

Note that a non-monotonous function may be one-one as the following graph illustrates:



increasing / decreasing
still one - one

The function is increasing / decreasing, but is still one-one.

EXAMPLE 4.9 Find whether $f(x) = x + e^x$ is one-one.

SOLUTION $f(x) = x + e^x$

$f'(x) = 1 + e^x > 0 \ \forall\ x$

Thus, f(x) is strictly increasing and hence, one- one.

EXAMPLE 4.10 Find whether

$f(x) = x^3 - 2x^2 - x + 1$ is one-one.

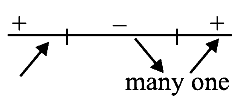
SOLUTION $f(x) = x^3 - 2x^2 - x + 1$

$f'(x) = 3x^2 - 4x - 1$

$f'(x) = 0$ has two distinct real roots.

$f'(x)$ changes sign about these roots. Hence f(x) increases and decreases in different intervals. Since the function is continuous, there would exist a horizontal line which intersects the graph at more than one point. Hence the function is many-one.

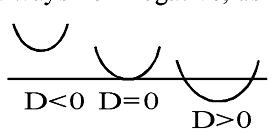
Sign scheme of $f'(x)$



EXAMPLE 4.11 Find all values of p for which the function $f(x) = x^3 - 2x^2 - px + 1$ is one-one.

SOLUTION $f'(x) = 3x^2 - 4x - p$

Since the coefficient of x^2 is positive, we can make f¢(x) always non-negative, using $D \le 0$.



$D \le 0 \quad \Rightarrow \quad 16 + 4 \times 3 \times p \le 0 \quad \Rightarrow \quad p \le -\dfrac{4}{3}$.

With these values of p, the function becomes strictly increasing and hence one-one.

EXAMPLE 4.12 If the function $f : [0, \infty) \to R$ is defined by $f(x) = \dfrac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$, then find whether f(x) is injective or not.

SOLUTION

$$f'(x) = \frac{e^{2x^2} 4x(e^{2x^2} + 1) - e^{2x^2} 4x(e^{2x^2} - 1)}{(e^{2x^2} - 1)^2}$$

$$= \frac{4xe^{2x^2}}{(e^{2x^2} + 1)^2} \ge 0 \quad \forall \quad x \in [0, \infty).$$

Thus, f(x) is strictly increasing and hence injective.

EXAMPLE 4.13 Find whether $f(x) = x - \sin x$ is one-one.

SOLUTION The domain of f(x) is R.

$f'(x) = 1 - \cos x$.

$\therefore$ $f'(x) \ge 0 \ \forall\ x \in$ complete domain and equality holds at distinct points only.

$\therefore$ f(x) is strictly increasing on R.

Hence f(x) is one-one.

EXAMPLE 4.14 Find the values of p if

$f(x) = \cos x - 2px$, defined from $R \to R$, is one-one.

SOLUTION $f(x) = \cos x - 2px$ is one-one, it must be either strictly increasing or strictly decreasing.

f(x) will be strictly decreasing if $f'(x) \le 0$

$\Rightarrow$ $f'(x) = -\sin x - 2p \le 0$ for all x

$\Rightarrow$ $p \ge -\dfrac{1}{2}\sin x$ for all x

$\Rightarrow$ $p \ge \dfrac{1}{2}$ ...(1) $[\because -1 \le \sin x \le 1]$

f(x) will be strictly increasing if $f'(x) \le 0$.

$\Rightarrow$ $f'(x) = -\sin x - 2p \le 0$ for all x

$\Rightarrow$ $p \le -\dfrac{1}{2}\sin x$ for all x

$\Rightarrow$ $p \le -\dfrac{1}{2}$ ...(2) $[\because -1 \le \sin x \le 1]$

From (1) and (2), for f(x) to be one-one

$p \le -$ or $p \ge$.

EXAMPLE 4.15 Let $f : R \to R$ be defined as

$f(x) = \begin{cases} x^2 + 2mx - 1, & x \le 0 \\ mx - 1, & x > 0 \end{cases}$. If f(x) is one-one then find all possible values of m.

SOLUTION Clearly f is continuous at x = 0 and f (0) = –1. For f to be one-one, $f'(x) > 0$ or $f'(x) < 0$ for all x.

Assume that f is strictly increasing.

For $x > 0$, $f'(x) = m > 0$.

For $x < 0$, $f'(x) = 2(x + m)$ cannot be positive for all $x < 0$, when $m > 0$.

Now assume that f is strictly decreasing.

For $x > 0$, $f'(x) = m < 0$.

For $x < 0$, $f'(x) = 2(x + m)$ is always negative for all $x < 0$, when $m < 0$.

$\therefore$ f is strictly decreasing and hence one-one $\forall\, m < 0$.

4. Trial and Error

If possible, find an element in the co-domain which is image of more than one element of the domain, then the function is many-one.

For example, $f(x) = \dfrac{x^2 - x - 2}{x^3 - x + 1}$ is many one because, f(x) = 0 at two distinct values of x (given by the roots of $x^2 - x - 2 = 0$):

$x = -1, 2$

The sole information that f(–1) = f(2) = 0, suggests that the function is many-one.

Sometimes we can try the equation f(x) = f(0), as the following example illustrates:

Let $f(x) = \dfrac{x^2 - x + 2}{x^2 + x - 1}$

We try f(x) = f(0)

$\Rightarrow\ = \dfrac{x^2 - x + 2}{x^2 + x - 1} = \dfrac{2}{-1}$

$\Rightarrow -x^2 + x - 2 = 2x^2 + 2x - 2$

$\Rightarrow 3x^2 + x = 0 \quad \Rightarrow \quad x = 0. -\dfrac{1}{3}$

$\Rightarrow f(0) = f = \left(-\dfrac{1}{3}\right) -2.$

Hence, the function is many-one.

Note that solving f(x) = f(0) may not be always fruitful. Let $f(x) = \dfrac{x^2 - x + 2}{x^2 + 3x + 2}$

f(x) = f(0)

$\Rightarrow \dfrac{x^2 - x + 2}{x^2 + 3x + 2} = \dfrac{2}{2} \quad \Rightarrow \quad x = 0.$

We cannot conclude anything from here.

In fact, we should try something different.

Let $\dfrac{x^2 - x + 2}{x^2 + 3x + 2} = \dfrac{1}{2} \quad \Rightarrow \quad x^2 - 5x + 3 = 0$

Here, D > 0 ensures two distinct roots.

Hence, the function is many-one.

EXAMPLE 4.16 Find whether

$f(x) = \dfrac{x^2 + x - 2}{x^2 - 4x + 3}$ is one-one.

SOLUTION $f(x) = \dfrac{x^2 + x - 2}{x^2 - 4x + 3}$

$= \dfrac{(x-1)(x+2)}{(x-1)(x-3)} = \dfrac{x+2}{x-3},\ x \neq 1.$

This is a linear fractional function, which we know that it is one-one.

EXAMPLE 4.17 A function $f : R \to R$ is given by:

$f(x) = \dfrac{ax^2 + 6x - 8}{a + 6x - 8x^2}$

Is the function one-one for a = 3 ?

SOLUTION The given function is a rational function. Each of numerator and denominator functions is quadratic equation. For a = 3,

$f(x) = \dfrac{3x^2 + 6x - 8}{3 + 6x - 8x^2}.$

In this case, solving f(x) = 0 for x reveals the nature of function.

$\dfrac{3x^2 + 6x - 8}{3 + 6x - 8x^2} = 0$

$\Rightarrow 3x^2 + 5x - 8 = 0$

$\Rightarrow x = \dfrac{-6 \pm \sqrt{36 - 4.3(-8)}}{6} = -1 \pm \dfrac{\sqrt{33}}{3}$

As x is not unique, the given function is not a one-one function. We should emphasize here that solution of function when equated to zero is not a full proof method. In this particular case, it turns out that function value becomes zero for two values

of x. In general, we should resort to techniques outlined earlier to determine function type.

EXAMPLE 4.18 A function $f : R \to R$ is given by:

$$f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}.$$

Is the function one-one ?

SOLUTION We evaluate the function for x = 0. If f(x) = f(0) equation yields multiple values of x, then function in not one-one. Here,

$$f(0) = \frac{30}{18} = \frac{5}{3}$$

Now, f(x) = f(0)

$$\Rightarrow \frac{x^2 + 4x + 30}{x^2 - 8x + 18} = \frac{5}{3}$$

$$\Rightarrow 3x^2 + 12x + 90 = 5x^2 - 40x + 90$$

$$\Rightarrow 2x^2 - 52x = 0 \quad \Rightarrow \quad x = 0, 26$$

We see that f(0) = f(26). It means pre-images are not related to distinct images. Thus, we conclude that the function is not one-one, but many-one.

Alternative 1:

The rational function is a continuous function. We can determine its increasing or decreasing nature by examining the derivative of the function.

$$f'(x) = \frac{(x^2 - 8x + 18)(2x + 4) - (x^2 + 4x + 30)(2x - 8)}{(x^2 - 8x + 18)^2}$$

$$\Rightarrow f'(x) = \frac{-12(x^2 + 2x - 26)}{(x^2 - 8x + 18)^2}$$

The denominator is a square of a quadratic expression, which evaluates to a positive number. On the other hand, the discriminant of the quadratic expression in the numerator is:

$$D = (2)^2 - 4(-26) = 4 + 104 = 108$$

It means that derivative has different signs in the domain. Therefore, the continuous function is a combination of increasing and decreasing nature in different intervals of the domain. Thus, the function is not monotonic in the domain. Hence, we conclude that the function is not an injection.

Alternative 2 :

A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Assume $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1^2 + 4x_1 + 30}{x_1^2 - 8x_1 + 18} = \frac{x_2^2 + 4x_2 + 30}{x_2^2 - 8x_2 + 18}$$

$$\Rightarrow 12x_1^2x_2 - 12x_1x_2^2 + 12x_1^2 - 12x_2^2 - 312x_1 + 312x_2 = 0$$

$$\Rightarrow (x_1 - x_2)\{12x_1x_2 + 12(x_1 + x_2) - 312\} = 0$$

$$\Rightarrow x_1 = x_2 \text{ or } x_1 = \frac{26 - x_2}{1 + x_2}.$$

Since $f(x_1) = f(x_2)$ does not imply $x_1 = x_2$ alone, f(x) is not a one-one function.

PRACTICE PROBLEMS [G]

1. Which of the functions defined below are one-one function(s) ?
   (i) $f(x) = (x + 1), (x \geq -1)$ (ii) $g(x) = x + (1/x)\ (x > 0)$
   (iii) $h(x) = x^2 + 4x - 5, (x > 0)$ (iv) $k(x) = e^{-x}, (x \geq 0)$
2. Determine whether the following functions are one-one:
   (a) $y = x^6 + 3x^2 + 2$; domain $[0, \infty)$
   (b) $y = x/(x + 1)$; domain $R - \{-1\}$
   (c) $x^4 + x + 1$; domain R
3. This table records some of the values of a function f:

x	1	2	3	4	5
fx	3	1	4	3	2

(a) Is f one-one ?
(b) Is f one-one when the domain is restricted to the set {2, 3, 4, 5}?

4. If f(x) is continuous and strictly increasing function, then is {f(x)} always many-one, where {.} represents the fractional part function.

5. Let T = {1, 2, 3, 4, 5}. Obtain a one-one function $f : T \to T$ such that $t + f(t)$ is a perfect square for every t in T.

6. Let $f : R \to R$ where $f(x) = \dfrac{x^2 + 4x + 7}{x^2 + x + 1}$. Is f one-one ? Justify.

7. A function is defined as

   $f : D \to R$, $f(x) = \cot^{-1}(\operatorname{sgn} x) + \sin^{-1}(x - \{x\})$ where $\{x\}$ denotes the fractional part function. Find the largest domain and range of the function. State with reasons whether the function is injective or not. Also draw the graph of the function.

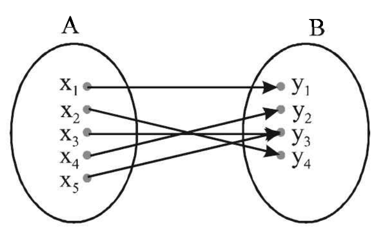
8. Let $f : R \to R$ defined by $f(x) = \dfrac{x^2 + 3x + c}{x^2 + x + 1}$. Show that f is a many one function for all c.

4.8 | ONTO AND INTO FUNCTIONS

Onto function (surjection)

The definition of function puts the restriction on domain that every element in it is related. If we extend this restriction to co-domain also, then we get a function called onto or surjection.

A function $f : A \to B$ is an onto function or surjection, if every element of the co-domain B is the image of some element in the domain A.



Onto function (surjection)

One of the implications of surjection is that all elements of co-domain are related. It reduces the co-domain to range set. In other words, co-domain is also the range of the function.

Range of f = Co-domain of f

This equality of sets form one of the condition for testing a function to be surjection. Alternatively, we can check surjectivity by evaluating the rule of the function for the argument x. If the expression of x is valid for elements in co-domain, then the function is a surjection.

For example, the function $f : R \to R$ given by $f(x) = x^3 + 1$ is onto.

We solve the rule for argument x :

$y = x^3 + 1$

$\Rightarrow x = (y - 1)^{1/3}$

We see that expression on the right hand side is a valid real expression for all values of y in R i.e co-domain. Hence, the given function is an onto function or surjection.

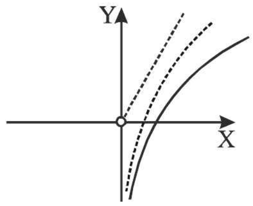
EXAMPLE 4.1 Consider a function defined as:

$f : R^+ \to R$, $f(x) = 2x + \ln x$.

Determine whether the function is onto.

SOLUTION f is onto because the range is R. f is a sum of two continuous increasing functions. Hence, it is also continuous and increasing.

Also $f(0^+) = -\infty$; $f(\infty) = \infty$.



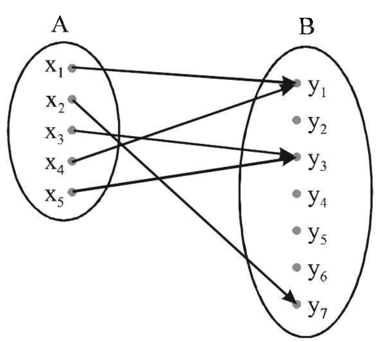
Into function

We have discussed in the beginning of this chapter that there is a possibility that some of the elements of co-domain are not related. In that case, function is said to be an into function.

A function $f : A \to B$ is an into function, if there exists atleast one element in co-domain B, which has no pre-image in the domain A.

One of the implications is that all elements of co-domain are not related to elements in domain set. In other words, range of the function is a proper subset of the co-domain.

Range of $f \subset$ Co-domain of f



Into function

We can check whether a given function is an into function or not by checking whether the function is onto or not. If the function is not an onto function, then it an into function.

EXAMPLE 4.2 Consider a function defined as:

$f : R \to R$ by $f(x) = x^4 + x^2$ for all $x \in R$

Determine whether the function is an into function?

SOLUTION The rule of the function is :

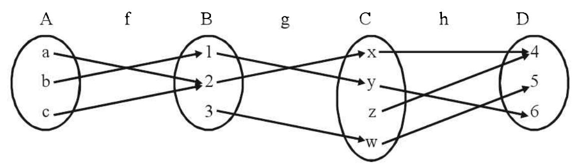
$$y = x^4 + x^2$$

The square of a real number is a positive number for all real number. Hence,

$$y = x^4 + x^2 \geq 0$$

It means that images are only the right half of the real number i.e. from zero to infinity. But, the co-domain of the function is R. It means that left half of the co-domain i.e. from negative infinity to less than zero has no pre-image in A. Therefore, the given function is an into function.

EXAMPLE 4.3 Let the functions $f : A \to B$, $g: B \to C$ and $h: C \to D$ be defined by the diagram.



(i) Determine if each function is onto.

(ii) Find the composition function $h \circ g \circ f : A \to D$ and find whether it is onto.

SOLUTION

(i) The function $f : A \to B$ is not onto since $3 \in B$ is not the image of any element in A.

The function $g : B \to C$ is not onto since $z \in C$ is not the image of any element in B.

The function $h : C \to D$ is onto since each element D is the image of some element of C.

(ii) Now $a \to 2 \to x \to 4$, $b \to 1 \to y \to 6$, $c \to 2 \to x \to 4$.

Hence, $h \circ g \circ f = \{(a, 4), (b, 6), (c, 4)\}$.

We can see that $h \circ g \circ f$ is onto.

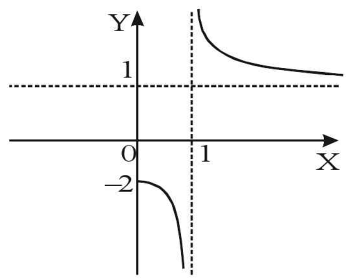
EXAMPLE 4.4 Let $f : [0, \infty) - \{1\} \to B$,

$$f(x) = \frac{x+2}{x-1}.$$

Find B for which the function is onto.

SOLUTION $f(x) = \dfrac{x+2}{x-1} = \dfrac{(x-1)+3}{(x-1)}$

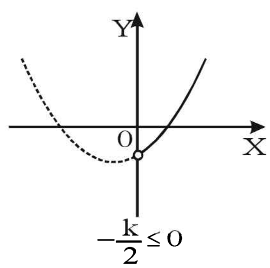
$$f(x) = 1 + \frac{3}{(x-1)}$$



From the graph, the range is $(-\infty, -2] \cup (1, \infty)$. If we use this as the co-domain B, then the function is onto.

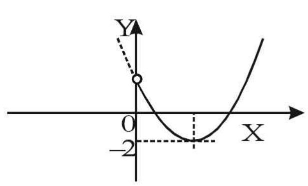
EXAMPLE 4.5 Let $f : R^+ \to [-2, \infty)$, $f(x) = x^2 + kx + k$. Find k such that f is onto.

SOLUTION We draw two possible graphs based on the position of the vertex with respect to the y-axis.



Case 1 : $-\frac{k}{2} \le 0$

Since the function's value at x = 0 is not inclusive, the least value of –2 must be attained at the vertex. This is possible only when the vertex is on the right of y-axis.



Case 2 : $-\frac{k}{2} > 0 \Rightarrow k < 0$

$$f\left(-\frac{k}{2}\right) = -2 \text{ for } k < 0 \Rightarrow \frac{k^2}{4} - \frac{k^2}{2} + k = -2$$

$$\Rightarrow \frac{k^2 - 2k^2 + 4k}{4} = -2$$

$$\Rightarrow -k^2 + 4k = -8 \qquad \Rightarrow k^2 - 4k - 8 = 0$$

$$\Rightarrow k = \frac{4 \pm \sqrt{48}}{2} = 2 \pm 2$$

We choose the negative value of k.
Hence, f is onto for k = 2 – 2.

EXAMPLE 4.6 Find the linear function(s) which map the interval [0, 2] onto [1, 4].

SOLUTION Let $f(x) = ax + b$
If f is increasing then $f(0) = 1$ and $f(2) = 4$

$$\Rightarrow b = 1 \quad \text{and} \quad a = \frac{3}{2}$$

If f is decreasing then $f(0) = 4$ and $f(2) = 1$

$$\Rightarrow b = 4 \quad \text{and} \quad a = -\frac{3}{2}$$

Hence, $f(x) = \frac{3x}{2} + 1$ or $f(x) = 4 - \frac{3x}{2}$.

EXAMPLE 4.7 Identify whether the function $f : R \to R$, $f(x) = -x^3 + 3x^2 - 2x + 4$ is onto or into.

SOLUTION Since, f(x) is a cubic function defined on R, the range of the function is R.
As range = co-domain, the given function is onto.

EXAMPLE 4.8 A function $f(x) = \sqrt{1-2x} + x$ is defined from $A \to B$ and is onto. If the set A is its natural domain then find the set B.

SOLUTION B means range of the function.
Let $y = \sqrt{1-2x} + x$
$\Rightarrow (y - x)^2 = 1 - 2x$
$\Rightarrow y^2 - 2xy + x^2 = 1 - 2x$
$\Rightarrow x^2 + 2x(1 - y) + y^2 - 1 = 0$
as, $x \in R$, $D \ge 0$
$\Rightarrow 4(1 - y)^2 \ge 4(y^2 - 1)$

$\Rightarrow\ 1 + y^2 - 2y \geq y^2 - 1$

$\Rightarrow\ -2y \geq -2$

$\Rightarrow\ y \leq 1 \quad \Rightarrow \quad y \in (-\infty, 1]$.

Hence, the set B is $(-\infty, 1]$.

EXAMPLE 4.9 A function $f : R \to R$ is given by :

$$f(x) = \frac{ax^2 + 6a - 8}{a + 6x - 8x^2}.$$

Find the interval of values of a for which the function is onto.

SOLUTION It is given that the co-domain of the function is R. Since the function is onto the range of the function is equal to co-domain of the function i.e. R.

Let $y = \dfrac{ax^2 + 6a - 8}{a + 6x - 8x^2}$

$ax^2 + 6x - 8 - ay - 6xy + 8x^2y = 0$

Rearranging as quadratic equation in variable x, we have:

$\Rightarrow\ (a + 8y)x^2 + 6(1 - y)x -(8 + ay) = 0$

Since x is real, discriminant of the quadratic equation should be positive,

$\Rightarrow\ 36(1 - y)^2 - 4\,(a + 8y)\{-(8 + ay)\} > 0$

$\Rightarrow\ 9(1 - y)^2 + (a + 8y)\,(8 + ay) > 0$

Expanding and rearranging and as equation in variable y, we have :

$9(1 + y^2 - 2y) + (8a + a^2y + 64y + 8ay^2) > 0$

$\Rightarrow\ (9 + 8a)y^2 + (a^2 + 46)\,y + (9 + 8a) > 0$

Since, the range of the function is R, it means that above inequality holds for all real values of y.

Hence, $9 + 8a > 0 \quad \Rightarrow \quad a > 9/8$ (1)

and $\quad (a^2 + 46)^2 - \{2(9 + 8a)\}^2 < 0$

$\Rightarrow\ (a^2 + 46 + 18 + 16a)\,(a^2 + 46 - 18 - 16a) < 0$

$\Rightarrow\ (a^2 + 16a + 4)\,(a^2 - 16a + 28) < 0$

$\Rightarrow\ (a + 8)^2\,(a - 2a - 14a + 28) < 0$

$(a + 8)^2\,(a - 2)\,(a - 14) < 0$

$\Rightarrow\ (a - 2)\,(a - 14) < 0$

$\Rightarrow\ 2 < a < 14$ (2)

The required interval is the intersection of the two intervals, which is (2, 14).

EXAMPLE 4.10 Let a function f defined from $R \to R$ as $f(x) = \begin{cases} x + m & \text{for } x \leq 1 \\ 2mx - 1 & \text{for } x > 1 \end{cases}$. If the function is surjective then find all values of m.

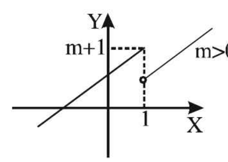
SOLUTION For f to be surjective, the range must be the set of all real numbers.

For $x \leq 1$, $f'(x) = 1 \quad \Rightarrow f$ increasing.

Hence, the maximum value of $f(x)$ for $x \leq 1$ occurs at $x = 1$.

$\therefore$ The maximum value for $x \leq 1$ is $m + 1$.

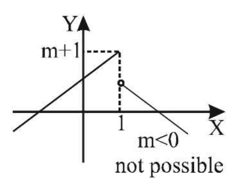
i.e. $f(x) \in (-\infty, m + 1]$ for $x \leq 1$.



For range to be R, the minimum value of f (x) for $x > 1$ must be less than or equal to $m + 1$. Now for $x > 1$, $f'(x) = 2m$

Obviously $m > 0$ (1)

(if $m < 0$ then the range cannot be R)



For $x > 1$, the function approaches the least level of $(2m - 1)$ at $x = 1$.

$\therefore\ 2m - 1 \leq m + 1$

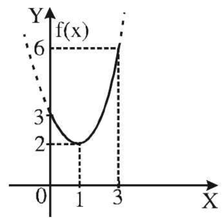
$m \leq 2$ (2)

From (1) and (2), $m \in (0, 2]$.

EXAMPLE 4.11 Let $f : [0, 3] \to A$ where $f(x) = x^2 - 2x + 3$. Find whether f(x) is injective or not. Also find the set A, if f(x) is surjective.

SOLUTION $f'(x) = 2(x - 1),\ 0 \leq x \leq 3$

$$\therefore\ f'(x) = \begin{cases} -ve & ; \quad 0 \leq x < 1 \\ +ve & ; \quad 1 < x < 3 \end{cases}$$



$\therefore$ f(x) is not monotonic. Hence it is not injective.
For f(x) to be surjective, B should be equal to its range. From the graph, the range is [2, 6].
$\therefore$ B = [2, 6].

Trial and Error

If possible, find an element in the co-domain which is not an image of any element of the domain, then function is into.

For example, $f : R \to [0, \infty)$, $f(x) = \dfrac{e^x}{2 + \cos x}$ is into because $f(x) \neq 0$ for any x, though 0 is an element of the co-domain.

EXAMPLE 4.12 Let f: N → I be a function defined as f(x) = x – 1000. Show that f is an into function.

SOLUTION f(x) = x – 1000.
We see that the minimum value of f is – 999 when x = 1. Thus, the function cannot attain integral values – 1000. Hence f is into.

EXAMPLE 4.13 Let $f : R \to [-1, 1]$,

$$f(x) = \frac{\sin x}{x^2 + 1}. \text{ Find whether f is onto.}$$

SOLUTION We note that 1 is an element of the co-domain.
However, the function never attains the value of 1. We prove this as follows :

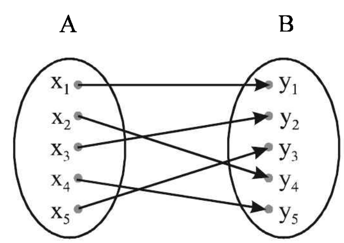
Let $f(x) = \dfrac{\sin x}{x^2 + 1} = 1 \quad \Rightarrow \sin x = x^2 + 1$

L.H.S ≤ 1 and R.H.S. ≥ 1
The only possibility is that

$\sin x = 1$ and $x^2 + 1 = 1$
The second equality holds at x = 0, but the first does not hold then.
Thus, $f(x) \neq 1$ and hence, it is into.

Bijection (One-one onto function)

If a function is both one-one and onto, then the function is a bijection.
The bijection presents the most stringent condition for a function. Every element of both domain and co-domain is related to distinct element. This requirement is fulfilled when a function is both an injection and surjection.



One-one onto function (Bijection)

The injection means that every element of domain is related to a distinct element in co-domain. On the other hand, surjection means that every element of co-domain is related. When conditions of injection and surjection are taken together, then it is also ensured that elements of co-domains are also related to distinct elements only.

EXAMPLE 4.14 Consider a function defined as:

$$f : A \to B \text{ by } f(x) = \frac{x-2}{x-3}.$$

Determine domain A and co-domain B of the function so that it is a bijection.

SOLUTION For determining domain of the function, we inspect the given rule,

$$f(x) = \frac{x-2}{x-3}$$

We observe that the given rational function is defined for all values of x except for x = 3. Hence, its domain is R – {3}.

In order that the given function is a bijection, it should be both an injection and a surjection. For injectivity, we put :

$f(x) = f(y)$

$\Rightarrow \dfrac{x-2}{x-3} = \dfrac{y-2}{y-3}$

$\Rightarrow (x-2)(y-3) = (x-3)(y-2)$

$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6$

$\Rightarrow -3x - 2y = -2x - 3y$

$\Rightarrow x = y$

Hence, function is an injection for the domain as determined above. Now, for surjection we solve the rule of the function for the argument x,

$\Rightarrow y = \dfrac{x-2}{x-3} \quad \Rightarrow \quad xy - 3y = x - 2$

$\Rightarrow x(y-1) = 3y - 2$

$\Rightarrow x = \dfrac{3y-2}{y-1}$

This equation is valid for all real values of y except 1. Hence, the function is surjection for all real values of y except for 1. Hence, for the function to be a surjection the co-domain is $R - \{1\}$.

Thus, we conclude that the given function is bijection for

Domain = $R - \{3\}$

Co-domain = $R - \{1\}$.

EXAMPLE 4.15 Prove that $f : R - \{1\} \to R$, $f(x) = \dfrac{x^{1/3}}{x^{1/3}-1}$ is injective but not surjective.

SOLUTION Let $f(a) = f(b)$

$\Rightarrow \dfrac{a^{1/3}}{a^{1/3}-1} = \dfrac{b^{1/3}}{b^{1/3}-1}$

$\Rightarrow a^{1/3}b^{1/3} - a^{1/3} = a^{1/3}b^{1/3} - b^{1/3}$

$\Rightarrow -a^{1/3} = -b^{1/3}$

$\Rightarrow a = b$,

This implies f is injective. To prove that f is not surjective assume that $f(x) = b$, $b \in R$. Then

$f(x) = b$

$\Rightarrow \dfrac{x^{1/3}}{x^{1/3}-1} = b \quad \Rightarrow \quad x = \dfrac{b^3}{(b-1)^3}$.

The expression for x is not a real number when b = 1 and so there is no real x such that $f(x) = 1$.

EXAMPLE 4.16 Show that $f : N \to N$, given by

$$f(x) = \begin{cases} x+1, \text{ if } x \text{ is odd,} \\ x-1, \text{ if } x \text{ is even} \end{cases}$$

is both one-one and onto.

SOLUTION Suppose $f(x_1) = f(x_2)$. Note that if x_1 is odd and x_2 is even, then we will have $x_1 + 1 = x_2 - 1$, i.e., $x_2 - x_1 = 2$ which is impossible.

Similarly, the possibility of x_1 being even and x_2 being odd can also be ruled out, using the similar argument. Therefore, both x_1 and x_2 must be either odd or even. Suppose both x_1 and x_2 are odd. Then $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$. Similarly, if both x_1 and x_2 are even, then also $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$.

Thus, f is one-one. Also, any odd number $2r + 1$ in the co-domain N is the image of $2r + 2$ in the domain N and any even number $2r$ in the co-domain N is the image of $2r - 1$ in the domain N. Thus, f is onto.

EXAMPLE 4.17 Let $f : R \to R$ be a function defined by $f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$. Then find whether f is one-one and onto.

SOLUTION f is not one-one as $f(0) = 0$ and $f(-1) = 0$.

f is also not onto as there is no x R such that $f(x) = 1$. If there is such an $x \in R$, then

$e^{|x|} - e^{-x} = e^x + e^{-x}$. Clearly $x \neq 0$.

For $x > 0$, this equation gives $e^{-x} = 0$ which is not possible and for $x < 0$, $\dfrac{e^{2x}+1}{e^x} = 0$, which is also not possible.

Hence, f is neither one-one nor onto.

EXAMPLE 4.18 Let f: $R^+ \to R$, $f(x) = e^{|\ln x|} - x^2$. Find whether f is bijective.

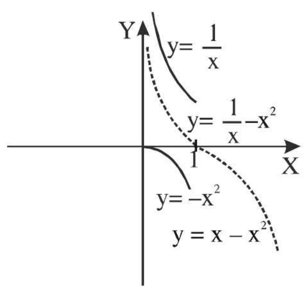
SOLUTION $f(x) = e^{|\ln x|} - x^2$

$$= \begin{cases} e^{-\ln x} - x^2, 0 < x < 1 \\ e^{\ln x} - x^2, x \geq 1 \end{cases}$$

$$= \begin{cases} \frac{1}{x} - x^2, 0 < x < 1 \\ x - x^2, x \geq 1 \end{cases}$$

$= x - x^2$

We obtain the graph of f in (0, 1) by graphical addition.

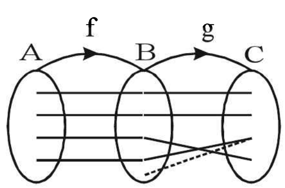


From the graph we can see that f is one-one and onto. Hence, f is bijective.

Note :

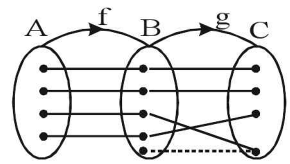
(i) If f : A → B and g : B → C are one-one functions then gof : A → C is also one-one.

(ii) If f : A → B, g : B → C, gof : A → C and gof is one-one, then f must be one-one.



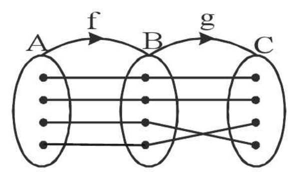
(iii) If f : A → B and g : B → C are onto then gof : A → C is also onto.

(iv) If f : A → B, g : B → C, gof : A → C and gof is onto then g must be onto.



(v) If f : A → B, g : B → C are bijective then gof : A → C is also bijective.

(vi) If f : A → B, g : B → C, gof : A → C and gof is bijective, then f is one-one and g is onto.



EXAMPLE 4.19 Let f : A → B and g : B → C be two functions and gof : A → C is defined. Then explain using mappings whether the following statements are true or false?

(i) If gof is onto then f must be onto.

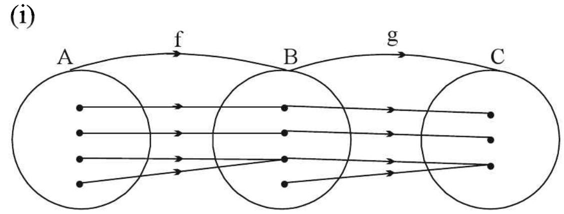
(ii) If f is into and g is onto then gof must be onto function.

(iii) If gof is one-one then g is not necessarily one-one.

(iv) If f is injective and g is surjective then gof must be bijective mapping.

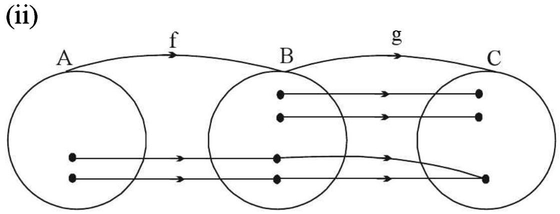
SOLUTION We have f : A → B, g : B → C and gof : A → C

(i)



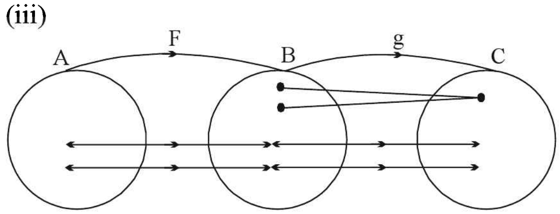
False

(ii)



False

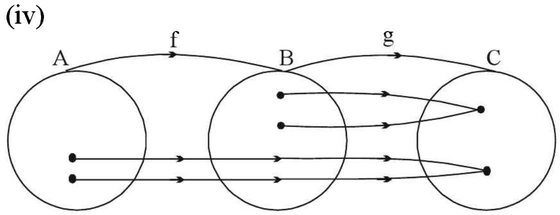
(iii)



True

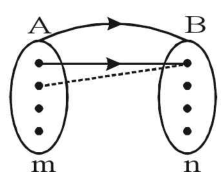
Clearly gof is one-one but g is many one function.

(iv)



False

Number of Functions



No of ordered pairs in A × B = mn.

(i) No. of relations from A to B = 2^{mn}

Since relation is a subset of the cartesian product A × B, the number of subsets formed from a set having mn elements is 2^{mn}.

(ii) No. of functions = n . n ....n (m times) = n^m

(iii) No. of one-one functions
= $n(n-1)(n-2) \ldots (n-m+1)$, $n \geq m$
= ${}^nP_m$

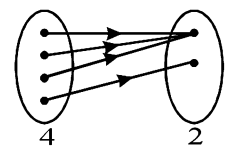
Proof : Let A = $\{x_1, x_2, \ldots, x_m\}$ and B = $\{y_1, y_2, \ldots, y_n\}$.

If f were injective then $f(x_1), f(x_2), \ldots, f(x_m)$ are all distinct and among the y_k. Hence $m \leq n$. In this case, there are n choices for $f(x_1)$, n – 1 choices for $f(x_2)$, ..., n – m + 1 choices for $f(x_m)$. Thus there are

$n(n-1)(n-2)\ldots(n-m+1)$ injections from A to B.

(iv) No. of bijective functions = n!, if m = n

(v) No. of onto functions.



No. of onto functions
= total no. of functions – no. of into functions

For example : No. of onto functions from a set containing 4 elements to a set containing 2 elements
= $2^4 - {}^2C_1 = 14$

No. of onto functions from a set containing 4 elements to a set containing 3 elements
= total no. of functions – no. of into functions leaving 2 elements – number of into functions leaving one element

$= 3^4 - {}^3C_1 . 1$
$- {}^3C_2$ (number of onto functions from 4 → 2)
$= 3^4 - {}^3C_1 - {}^3C_2 (2^4 - 2)$
$= 3^4 - {}^3C_2\, 2^4 + {}^3C_1$.

We also have a formula to generate the number of onto functions from a set containing m elements to a set containing n elements :

No. of onto functions

$$= \sum_{r=1}^{n} (-1)^{n-r} . {}^nC_r . r^m, \quad 1 \leq n \leq m.$$

EXAMPLE 4.20 Let A = {1, 2, 3} and B = {4, 5, 6, 7}. How many functions are there from A to B ? How many functions are there from B to A ? How many injections are there from A to B ? How many surjections are there from B to A.

SOLUTION There are 4.4.4 = 64 functions from A to B, since there are 4 possibilities for the image of 1, 4 for the image of 2 and 4 for the image of 3. Similarly, there are 3.3.3.3 = 81 functions from B to A.

By the above result, there are 4.3. 2 = 24 injections from A to B.

The 34 functions from B to A come in three cases: (i) those that are surjective, (ii) those that map to exactly two elmeents of A, and (iii) those that map to exactly one element of A.

Take a particular element of A. say $1 \in A$. There are 2^4 functions from B to {2, 3}. Notice that some of these may map to the whole set {2, 3} or they may skip an element. Coupling this with the $1 \in A$, this means that there are 2^4 functions from B to A that skip the 1 and may or may not skip the 2 or the 3. Since there is nothing holy about choosing $1 \in A$, we conclude that there are 3.2^4 from B to A that skip either one or two elements of A.

Now take two particular elements of A, say $\{1, 2\} \subseteq A$. There are 1^4 functions from B to {3}. Since there are three 2-element subsets in A –namely {1, 2}, {1, 3} and {2, 3}– this means that there are 3.14 functions from B to A that map precisely into one element of A.

To find the number of surjections from B to A we weed out the functions that skip elements. In considering the difference $3^4 - 3.\ 2^4$, we have taken out all the functions that miss one or two elements of A, but in so doing, we have taken out twice those that miss one element. Hence we put those back in and we obtain.

$$3^4 - 3.\ 2^4 + 3\ .\ 1^4 = 36 \text{ surjections from B to A.}$$

EXAMPLE 4.21 If X = {1, 2, 3, 4, 5} and Y = {a, b, c, d, e, f}and $f : X \to Y$, then find the total number of

(a) Functions (b) One to one function

(c) Many-one function (d) Constant function

(e) Onto function (f) Into function

SOLUTION

(a) Total number of functions = $6^5 = 7776$

(b) Total number of one to one function = ${}^6C_5.5!$ $= 6! = 720$

(c) Total number of many-one function = $6^5 - 6!$ $= 7056$

(d) Total number of constant function = 6

(e) Total number of onto function = 0 (as r > n)

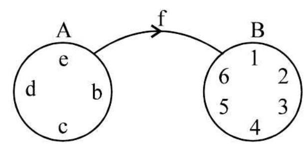
(f) Total number of into function = $6^5 = 7776$.

EXAMPLE 4.22 Suppose set A consists of 4 distinct elements and set B consists of 6 distinct elements. Let (i) the number of injective mapping defined from $A \to B$ be m,

(ii) the number of mapping from $A \to B$ which are not surjective be n. Find m + n.

SOLUTION

(i)



Number of injective mappings from $A \to B$:

4 elements from B can be mapped in ${}^6C_4$ ways and mapped in 4! ways with the elements of A.

Hence number of injective mapping m = ${}^6C_4 \cdot 4!$
$= 15 \cdot 24 = 360$

(ii) All mappings from $A \to B$ are not onto since $4 < 6$.

Hence number of into functions n
= total number of mapping = 6^4
= 1296.

Hence m + n = 360 + 1296 = 1656.

CONCEPT PROBLEMS [C]

1. Is the function $f : N \to N$ defined by $f(x) = 2x + 3$ surjective.
2. If $f : R \to B$, defined by $f(x) = \sin x - \cos x + 1$, is onto, then find the set B.
3. Let $A = \{x : -1 \le x \le 1\} = B$. For each of the following functions from A to B, find whether it is surjective, injective or bijective.

   (i) $f(x) = \frac{x}{2}$ (ii) $g(x) = |x|$ (iii) $h(x) = x|x|$

   (iv) $k(x) = x^2$ (v) $j(x) = \sin \pi x$
4. Give example of a function $f : N \to N$ and $g : N \to W$ such that $g \circ f : N \to N$ is onto but f is not onto.
5. A mapping $f : A \to [-1, 1]$ defined by $f(x) = \sin 2x$, is one-one and onto. Which of the following intervals can A be ?

   (i) $\left[-\frac{\pi}{2}, 0\right]$ (ii) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (iii) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$
6. Find whether the function $f : R \to R$, defined by $f(x) = 2x + 3$ is one-one and onto.
7. Let $f : [0, \infty) \to [0, 2]$ be defined by $f(x) = \frac{2x}{1+x}$, then f find whether f is one-one and onto.
8. Show that if for a map $f : A \to B$ there exists a map $g : B \to A$ such that $(fog)(y) = y$ for every $y \in B$, then f is onto.
9. Show that if for $f : A \to B$ we have $g : B \to A$ such that $(gof)(x) = x$ for all $x \in A$, then f is one-one.
10. $f : R \to R$ defined by $f(x) = \begin{cases} x & \text{if } x \ge 0, \\ x-1 & \text{if } x < 0. \end{cases}$ Show that f is one-one but not onto as $-1/2$ is never attained by the function.
11. Let $f : R - \{3\} \to R - \{1\}$ be defined by $f(x) = \frac{x-2}{x-3}$. Is f bijective ? Give reasons.
12. Determine the kind of mapping if $f : [-1, 2] \to [1, 5]$, where $f(x) = x^2 - 2x + 2$.
13. Consider the function $f(x) = x(x-1)(x+1)$ from R to R. Then show that f is neither one-one nor onto.
14. Find the value of α so that the function $y = -\frac{x^2+\alpha}{x+1}$ from R to R, is onto.
15. For each of the following functions find whether it is one-one or many-one and into or onto

   (i) $f(x) = 2\tan x$; $(\pi/2, 3\pi/2) \to R$ (ii) $f(x) = \frac{1}{1+x^2}$; $(-\infty, 0) \to R$

   (iii) $f(x) = x^2 + \ell n\, x$; $(0, \infty) \to R$
16. Prove that $f : R \to R$ defined by $f(x) = \frac{x}{1+|x|}$ is one-one but not onto.

PRACTICE PROBLEMS [H]

17. For the function defined below, determine which are one-one, which are onto which are both and which are neither.

    (i) $f : (-1, 1) \to R, f(x) = \dfrac{x}{1+x^2}$.

18. Find the set of values of a for which the function $f(x) = ax + 2\sin x + 5$ is one-one and onto.

19. If $a \in [-3, 3]$ then prove that the function $f(x) = x^3 + ax^2 + 4x + \sin x$ is bijective.

20. Which of the following function is surjective but not injective ?

    (i) $f : R \to R \quad f(x) = x^4 + 2x^3 - x^2 + 1$

    (ii) $f : R \to R \quad f(x) = x^3 + x + 1$

    (iii) $f : R \to R^+ \quad f(x) = \sqrt{1+x^2}$

    (iv) $f : R \to R \quad f(x) = x^3 + 2x^2 - x + 1$

21. Prove that the function $f(x) = x - 2\sin\frac{x}{2}$, $f : R \to R$ is bijective.

22. Show that the mapping $f : R \to R$ given by $f(x) = x^3 + ax^2 + bx + c$ is a bijection if and only if $a^2 \le 3b$.

23. Find the number of functions that can be defined from the domain $D = \{1, 2, 3\}$ onto the range $R = \{4, 5\}$

24. Show that the function $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in R$ is one and onto function.

25. Given $X = \{1, 2, 3, 4\}$, find all one-one, into mappings, $f : X \to X$ such that $f(1) = 1$, $f(2) \ne 2$ and $f(4) \ne 4$.

26. Let $X = \{x_1, x_2, x_3, ... x_n\}$ (i.e. n elements) and $Y = \{y_1, y_2, y_3 ...... y_r\}$ (i.e. r elements). Let a function be defined from X to Y. Then show that

    (i) the total number of constant functions = r

    (ii) the total number of many-one functions = $\begin{cases} r^n - {}^rC_n.n! & r \ge n \\ r^n & r < n \end{cases}$

    (iii) the total number of into functions

    $$= \begin{cases} {}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \dots & r \le n \\ r^n & r > n \end{cases}$$

    (iv) The total number of onto functions

    $$= \begin{cases} r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \dots & r < n \\ r! & r = n \\ 0 & r > n \end{cases}$$

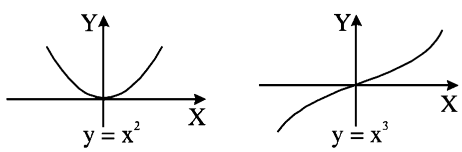
4.9 | EVEN AND ODD FUNCTIONS

Even and odd functions are related to symmetry of functions. The symmetry of a function is visualized by the plot of a function, which may show symmetry with respect to either an axis (y-axis) or a point (origin).

An even function is symmetric about y-axis. If we consider the y-axis as a mirror, then the plot in the first quadrant has its mirror image in second quadrant. Similarly, the plot in fourth quadrant has its mirror image in third quadrant.

An odd function is symmetric about origin of the coordinate system. The plot in first quadrant has its mirror image in third quadrant. Similarly, the plot in second quadrant has its mirror image in fourth quadrant.

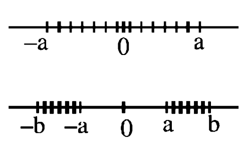
For example, the graph of $y = x^2$ is symmetric about y-axis, while graph of $y = x^3$ is symmetric about origin.



Since functions need not always be symmetric, they may neither be even nor be odd. The parity of a function i.e. whether it is even or odd is determined with certain algebraic steps. Further, symmetry of functions may change under mathematical operations.

Note

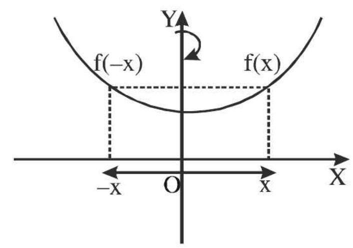
Before we test a function for being even or odd we should check that the domain is symmetric about x = 0.



For example, the can be $(-\infty, \infty)$, $(-a, a)$, $[-b, -a] \cup [a, b]$, etc.

Even functions

The values of even function at x and – x are same i.e. if (x, y) lies on the curve, then (– x, y) also lies on the curve.



$$f(-x) = f(x)\ \forall\ x \in \text{domain}$$

A function f(x) is said to be **even** if for every x, there exists – x in the domain of the function such that $f(-x) = f(x)$.

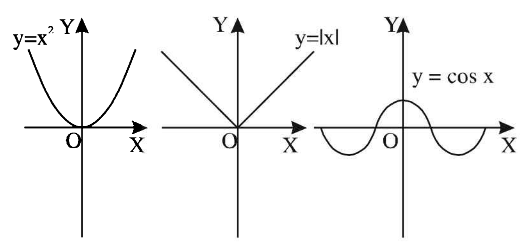
Some examples of even functions are x^2, $|x|$ and cos x. In each case, we see that :

$f(-x) = (-x)2 = x^2 = f(x)$

$f(-x) = |-x| = |x| = f(x)$

$f(-x) = \cos(-x) = \cos x = f(x)$.

The right side is mirror image of left hand side and the left side is mirror image of right hand side of the curve.



It is important to see that if we rotate the curve by 180º about y-axis, then the appearance of the rotated curve is same as the original curve. We can state this alternatively as : if we rotate left hand side of the curve by 180º about y-axis, then we get the right hand curve and vice versa.

EXAMPLE 4.1 Prove that the function $f(x) = x$ is even.

SOLUTION For function to be even, we need to prove that f(–x) = f(x).

Here, $f(-x) = -x\dfrac{a^{-x}-1}{a^{-x}+1} = -x\dfrac{\frac{1}{a^x}-1}{\frac{1}{a^x}+1}$

$$= -x\dfrac{\frac{1-a^x}{a^x}}{\frac{1+a^x}{a^x}} = -x\dfrac{1-a^x}{1+a^x}$$

$$\Rightarrow f(-x) = x\dfrac{a^x-1}{a^x+1} = f(x).$$

EXAMPLE 4.2 Prove that the function

$f(x) = \tan^{-1}(\sin(\cos^{-1}x))$ is even.

SOLUTION In order to determine the nature of the function, we check for f(–x).

$f(-x) = \tan^{-1}(\sin(\cos^{-1}(-x)))$

$= \tan^{-1}(\sin(\pi - \cos^{-1}x))$

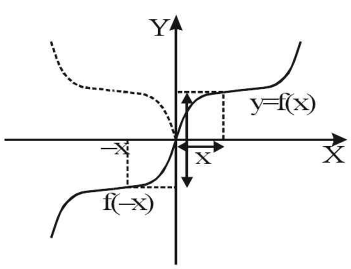
$= \tan^{-1}\sin(\cos^{-1}x)$

$= f(x)$

Hence, the function is even.

Odd functions

The values of odd function at x and – x are equal in magnitude but opposite in sign i.e. if (x, y) lies on the curve, then (– x, – y) also lies on the curve.



$f(-x) = f(x)\ \forall\ x \in$ domain

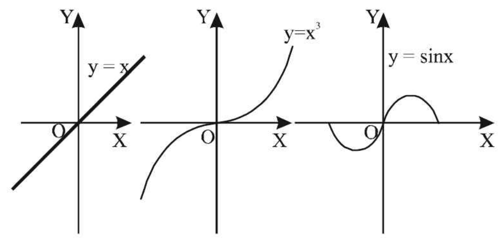
A function f(x) is said to be odd if for every x, there exists –x in the domain of the function such that f(–x) = –f(x).

Some examples of odd functions are x, x^3 and sinx. In each case, we see that :

$f(-x) = -x = -f(x)$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

$f(-x) = \sin(-x) = -\sin x = -f(x)$.

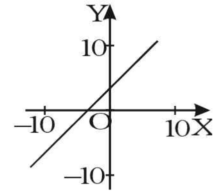


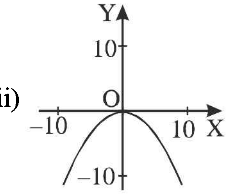
The upper curve of these functions is exactly same as the lower curve across x-axis.

It is important to see that if we rotate the curve by 180° about origin, then the appearance of the rotated curve is same as the original curve. In other words, if we rotate right hand side of curve by 180° about origin, then we get left side of the curve. Further, it is interesting to note that we obtain left hand part of the plot of odd function in two steps : (i) drawing reflection (mirror image) of right hand plot about y-axis and (ii) drawing reflection (mirror image) of "reflection drawn in step (i)" about x-axis.

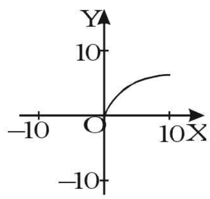
Most functions are neither even nor odd. For instance, $x^3 + x^4$ is neither even nor odd since $(-x)^3 + (-x)^4 = -x^3 + x^4$, which is neither $x^3 + x^4$ nor $-(x^3 + x^4)$.

EXAMPLE 4.3 Which one of the following depicts the graph of an odd function?

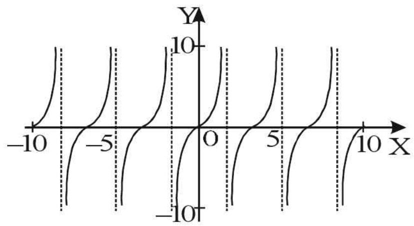
(i)


(ii)


(iii)



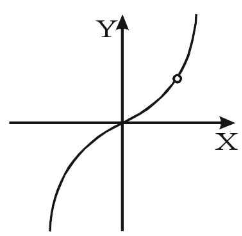
(iv)



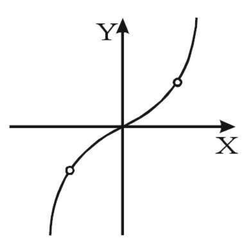
SOLUTION Only in (iv), the graph is symmetric w.r.t. origin.

We must check that the domain is symmetric about x = 0.

For example, the function $y = \dfrac{x^3(x-1)}{(x-1)}$ is neither even nor odd since the domain is unsymmetric about x = 0. See the figure below.



However, $y = \dfrac{x^3(x^2-1)}{(x^2-1)}$ is odd because the domain $R - \{\pm 1\}$ is symmetric about x = 0 and $f(-x) = -f(x)\ \forall\ x \in$ domain.



EXAMPLE 4.4 Determine whether the function

$f(x) = \ln\{x + \sqrt{(x^2+1)}\}$ is an odd function.

SOLUTION Here,

$f(-x) = \ln\{-x + \sqrt{(x^2+1)}\}$

The expression on the right hand side cannot be explicitly interpreted whether it is equal to f(x) or not. Therefore, we rationalize the expression of the logarithmic function,

$$f(-x) = \ln\left[\frac{\{-x+\sqrt{(-x)^2+1}\times\{x+\sqrt{(x^2+1)}\}}{\{x+\sqrt{(x^2+1)}\}}\right]$$

$$= \ln\left[\frac{-x^2+x^2+1}{\left\{x+\sqrt{x^2+1}\right\}}\right] = \ln\left[\frac{1}{\left\{x+\sqrt{x^2+1}\right\}}\right]$$

$\Rightarrow f(-x) = -\ln\{x + \sqrt{x^2+1}\} = -f(x)$

Hence, the given function is an odd function.

EXAMPLE 4.5 Which of the following functions are even / odd ?

(i) f(x) = sin x + cos x

(ii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

SOLUTION

(i) $f(x) = \sin x + \cos x \Rightarrow f(-x) = -\sin x + \cos x$

The resulting function is neither equal to f(x) nor equal to –f(x). Hence, the given function is neither an even nor an odd function.

(ii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

$f(-x) = \sqrt{1-x+x^2} - \sqrt{1+x+x^2} = -f(x)$

$\therefore$ f is an odd function.

EXAMPLE 4.6 Let $f : [-10, 10] \to R$, where $f(x) = \sin x + [x^2/a]$ and [.] denotes the greatest integer function be an odd function. Then find the set of values of parameter a.

SOLUTION sin x is an odd function while $\left[\dfrac{x^2}{a}\right]$ is an even function, in general. To make f an odd function, we need $\left[\dfrac{x^2}{a}\right] = 0$ for all $x \in [-10, 10]$

$\Rightarrow 0 \le \left[\dfrac{x^2}{a}\right] < 1$ for all $x \in [-10, 10]$,

$\Rightarrow a > 100$.

EXAMPLE 4.7 Prove that

$$f(x) = \frac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[\frac{x+2\pi}{\pi}\right] - 3}$$ is an odd function where [·] denotes the greatest integer function.

SOLUTION $$f(x) = \frac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[\frac{x}{\pi}+2\right] - 3}$$

$$= \frac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[\frac{x}{\pi}\right] + 1}$$

Now $$f(-x) = \frac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[\frac{-x}{\pi}\right] + 1}$$

Case 1 : Let $x = n\pi$, $n \in I$, $\left[\frac{-x}{\pi}\right] = -\left[\frac{x}{\pi}\right]$ and the numerator of f (– x) is zero.

$\Rightarrow$ f(– x) = 0 = – f(x).

Case 2 : $x \neq n\pi$, $n \in I$

$$\Rightarrow \left[\frac{-x}{\pi}\right] = -\left[\frac{x}{\pi}\right] - 1$$

$$\therefore\ f(-x) = \frac{2\left(e^x - e^{-x}\right)(\sin x + \tan x)}{-2\left[\frac{x}{\pi}\right] - 1} = -f(x)$$

Hence, f(x) is an odd function.

EXAMPLE 4.8 Let $F(x) = \left[\frac{f(x)+f(-x)}{g(x)-g(-x)}\right]^n$. Show that F(x) is even when n is even and is odd when n is odd.

SOLUTION Let h (x) = f(x) + f(–x) and

k(x) = g(x) – g(–x).

Then h(–x) = f(–x) + f(x) = h(x)

and k(–x) = g(–x) –g(x) = –k(x)

h(x) is even while k (x) is odd.

Thus, $F(x) = \left[\frac{h(x)}{k(x)}\right]^n$ and

$$F(-x) = \left[\frac{h(-x)}{k(-x)}\right]^n = (-1)^n\left[\frac{h(x)}{k(x)}\right]^n = (-1)^n F(x).$$

Hence, f(x) is an even function when n is even, and odd function, when n is odd.

EXAMPLE 4.9 Find whether

$$f(x) = \begin{cases} x^2, & x \in Q \\ -2x^2, & x \notin Q \end{cases}$$ is even or odd.

SOLUTION $$f(x) = \begin{cases} x^2, & x \in Q \\ -2x^2, & x \notin Q \end{cases}$$

$$f(-x) = \begin{cases} (-x)^2, & -x \in Q, & x \in Q \\ -2(-x)^2, & -x \notin Q, & x \notin Q \end{cases}$$

$$= \begin{cases} x^2, & x \in Q \\ -2x^2, & x \notin Q \end{cases} = f(x).$$

Hence, the function is even.

EXAMPLE 4.10 Find whether f(x)

$$= \begin{cases} x\cos x, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1-\sin^4 x, & x < -\frac{\pi}{2},\ x > \frac{\pi}{2} \end{cases}$$ is even or odd.

SOLUTION

$$f(-x) = \begin{cases} -x\cos(-x), & -\frac{\pi}{2} \le -x \le \frac{\pi}{2} \\ 1-\sin^4(-x), & -x < -\frac{\pi}{2},\ -x < -\frac{\pi}{2} \end{cases}$$

$$= \begin{cases} -x\cos x, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1-\sin^4 x, & x < -\frac{\pi}{2},\ x > \frac{\pi}{2} \end{cases}$$

This is neither equal to f(x) nor f(–x).

Hence, the function is neither even nor odd.

EXAMPLE 4.11 Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$ and

$$g(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Then find whether (f – g)(x) is even or odd.

SOLUTION

$$f(x) - g(x) = F(x) = \begin{cases} -x & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}.$$

$$F(-x) = \begin{cases} -(-x) & \text{if } -x \text{ is rational, } x \text{ is rational} \\ -x & \text{if } -x \text{ is irrational, } x \text{ is irrational} \end{cases}$$

$$F(-x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

$= -F(x)$

Hence, F is odd.

EXAMPLE 4.12 Let a function f satisfy $f(x + y) + f(x - y) = 2 f(x) . f(y)$ for all $x, y \in R$, where $f(0) \neq 0$. Prove that f is even.

SOLUTION Putting x = 0 in the given equation, we get $f(y) + f(-y) = 2f(0)f(y)$ (1)

To obtain the value of f(0), we put y = 0 in (1).

$\Rightarrow 2f(0) = 2f^2(0)$

$f(0) = 0, 1,$ given $f(0) \neq 0$.

$\Rightarrow f(0) = 1$

From (1), we have $f(y) + f(-y) = 2f(y)$

$\Rightarrow f(y) = f(-y)$

Hence, the function is even.

Algebra of even / odd functions

It is easy to find the nature of function resulting from mathematical operations, provided we know the nature of operand functions. We first find whether each of the functions involved are even, odd or neither even nor odd.

We shall work with a division operation here to illustrate the point. Let f(x) and g(x) be even and odd functions respectively. Let h(x) = f(x)/g(x). We now substitute x by –x,

$$h(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -h(x)$$

Thus, the division, here, results in an odd function.

The nature of resulting function subsequent to various mathematical operations is tabulated here for reference:

f(x)	= odd,	odd,	even
g(x)	= odd,	even,	even
f(x)±g(x)	= odd,	neither,	even
f(x) g(x)	= even,	odd,	even
f(x)/g(x)	= even,	odd,	even
fog(x)	= odd,	even,	even

Also, the square of an even or odd function is always an even function.

We should emphasize here that we need not memorize this table. We can always carry out particular operation and determine whether a particular operation results in even, odd or neither of two function types.

For example, let us determine the nature of fog function when f is an even and g is an odd function. By definition,

$$fog(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = fog(x)$$

Therefore, resulting fog function is even function.

There is an useful parallel here to remember the results of multiplication and division operations. If we consider even as "plus (+)" and odd as "minus (–)", then the resulting function is same as that resulting from multiplication or division of plus and minus numbers. Product of even (plus) and odd (minus) is minus(odd). Product of odd (minus) and odd (minus) is plus (even). Similarly, division of odd (minus) by even (plus) is minus (odd) and so on.

For example, $f(x) = \dfrac{\text{sgn}\, x + x^2 \sin x}{1 + |x| - \cos^3 x}$

$$= \frac{o + e \times o}{e + e - e} = \frac{o + o}{e} = \frac{o}{e} = o$$

f is an odd function.

EXAMPLE 4.13 Which of the following functions is odd ?

(a) $\text{sgn}\, x + x^{2000}$ (b) $|x| - \tan x$

(c) $x^3 \cot x$ (d) $\text{cosec}\, x^{55}$

SOLUTION (a) $f(-x) = \text{sgn}(-x) + (-x)^{2000}$

$= -\text{sgn } x + x^{2000} \neq \pm f(x)$

$\therefore$ f is neither even nor odd.

(b) $f(-x) = |-x| - \tan(-x) = |x| + \tan x$

$\therefore$ f is neither even nor odd.

(c) $f(-x) = (-x)^3 \cot(-x) = -x^3 (-\cot x) = x^3 \cot x$
$= f(x)$

$\therefore$ f is an even function

(d) $f(-x) = \text{cosec } (-x)^{55} = \text{cosec } (-x^{55})$
$= -\text{cosec } x^{55} = -f(x)$

$\therefore$ f is an odd function.

Alternative :

(a) $f(x) = \text{sgn } (x) + x^{2000} = o + e =$ neither even nor odd.

(b) $f(x) = e - o =$ neither even nor odd.

(c) $f(x) = o \times o = e$ even

(d) $f(x) = o\, o\, o = o$ odd

EXAMPLE 4.14 Find whether $f(x) = (\tan x^5)$ is even or odd.

SOLUTION $f(x) = (\tan (x^5))$

$= O \, o \, O \times 2^{O \times OoO}$

$= o \times 2^{O \times O} = o \times 2^e = o \times e = o.$

The function is odd.

Properties of even / odd functions

(i) A function which is both even and odd is the zero function i.e. $f(x) = 0 \ \forall \ x \in R$.

(ii) A non-zero constant function defined on R is an even function.

(iii) A function whose domain is symmetric about origin can be written uniquely as a sum of an even and an odd function.

Every real function whose domain is symmetric about x = 0, can be considered to be formed from addition of an even and an odd function. This formation is unique for every such function.

$$f(x) = f(x) + \{f(-x) - f(-x)\}$$

Rearranging,

$$f(x) = \frac{1}{2}\{f(x) + f(-x)\} + \frac{1}{2}\{f(x) - f(-x)\}$$
$$= g(x) + h(x)$$

Now, we seek to determine the nature of functions g(x) and h(x). For g(x), we have :

$$\Rightarrow g(-x) = \frac{1}{2}[f(-x) + f\{-(-x)\}]$$
$$= \frac{1}{2}\{f(-x) + f(x)\} = g(x)$$

Thus, g(x) is an even function.

Similarly, $h(-x) = \frac{1}{2}[f(-x) - f\{-(-x)\}]$

$$= \frac{1}{2}\{f(-x) - f(x)\} = -h(x)$$

Clearly, h(x) is an odd function. We, therefore, conclude that a function can be expressed as addition of even and odd functions.

$$\text{Hence } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$

For example, the function

$$f(x) = x^3 + \cos x + x \tan x - \sin x$$

can be expressed as

$$f(x) = (\cos x + x \tan x) + (x^3 - \sin x)$$

where $\cos x + x \tan x$ is even and $x^3 - \sin x$ is odd.

For $f(x) = 2^x + \cos x$ we write

$$= \underbrace{\left(\frac{2^x + 2^{-x}}{2} + \cos x\right)}_{\text{even}} + \underbrace{\left(\frac{2^x - 2^{-x}}{2}\right)}_{\text{odd}}$$

Now consider $f(x) = \dfrac{2}{1 - \sin x}$.

Suppose that we express f(x) as

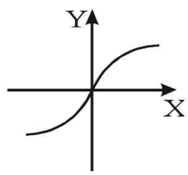
$$\underbrace{\left(\frac{1}{1-\sin x} + \frac{1}{1+\sin x}\right)}_{\text{even}} + \underbrace{\left(\frac{1}{1-\sin x} + \frac{1}{1+\sin x}\right)}_{\text{odd}}$$

the domain of the new representation $\{\sin x \neq \pm 1\}$ is different from that of f(x) $\{\sin x \neq 1\}$, and hence it is a function different from f(x).

This difficulty appears because the domain of f in not symmetric about x = 0. Hence, we cannot express this function as a sum of an even and an odd function.

(iv) An even function is many-one.

(v) If x = 0 lies in the domain of an odd function then f(0) = 0 i.e. the graph of an odd function must pass through origin.



We have f(x) = –f(–x) for all x.
Put x = 0, f(0) = –f(0) $\Rightarrow$ f(0) = 0.

Even and odd extensions of function

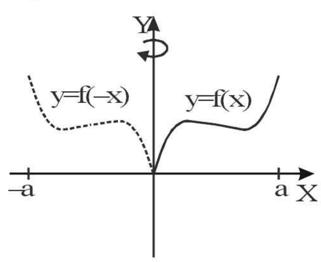
A function has three components – definition (rule), domain and range. What could be the meaning of extension of a function? The idea is to complete a function defined in one half of its representation (say $x \geq 0$) with other half such that resulting function is either even or odd function.

Even extension

Let f(x) be defined in [0, a]. Then, the even extension of f is defined as :

$$g(x) = \begin{cases} f(x); & 0 \leq x \leq a \\ f(-x); & -a \leq x < 0 \end{cases}$$

The graphical interpretation of such extension is that graph of function f(x) is extended in the other half which is mirror image of f(x) in y-axis i.e. image about y-axis.



For example, the even extension of
f(x) = 2x – sin x, $x \geq 0$ is given as

$$fe(x) = \begin{cases} 2^x - \sin x & , \quad x \geq 0 \\ 2^{-x} - \sin(-x) & , \quad x < 0 \end{cases}$$

$$= \begin{cases} 2^x - \sin x & , \quad x \geq 0 \\ 2^{-x} + \sin x & , \quad x < 0 \end{cases}.$$

Let f(x) be defined in [a, b].

Then, its even extension is defined as :

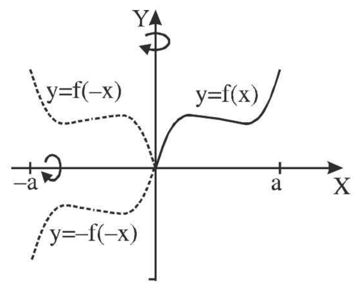
$$g(x) = \begin{cases} f(x), & a \leq x \leq b \\ f(-x), & -b \leq x \leq -a \end{cases}.$$

Odd extension

Let f(x) be defined in [0, a]. Then, its odd extension is defined as :

$$g(x) = \begin{cases} f(x); & 0 \leq x \leq a \\ -f(-x); & -a \leq x < 0 \end{cases}$$

The graphical interpretation of such extension is that the graph of function f(x) is extended in the other half which is mirror image of f(–x) in the x-axis.



Let f(x) be defined in [a, b].
Then, its odd extension is defined as :

$$g(x) = \begin{cases} f(x), & a \leq x \leq b \\ -f(-x), & -b \leq x \leq -a \end{cases}.$$

EXAMPLE 4.15 If $f(x) = \begin{cases} x^3 + x^2, & 0 \leq x \leq 2 \\ x + 3, & 2 < x \leq 4 \end{cases}$. Then find the even and odd extension of f(x).

SOLUTION $f(x) = \begin{cases} x^3 + x^2, & 0 \leq x \leq 2 \\ x + 3, & 2 < x \leq 4 \end{cases}$

The even extension of f(x) is obtained by f(–x). So

$$f(-x) = \begin{cases} (-x)^3 + (-x)^2, & 0 < (-x) \leq 2 \\ (-x) + 3, & 2 < (-x) \leq 4 \end{cases}$$

Even extension of f(x) is

$$g(x) = \begin{cases} -x + 3, & -4 \leq x < -2 \\ -x^3 + x^2, & 2 \leq x \leq 0 \\ x^3 + x^2, & 0 \leq x \leq 2 \\ x + 3, & 2 < x \leq 4 \end{cases}$$

The odd extension of f(x) is obtained by – f(–x). So

$$-f(-x) = \begin{cases} -\{(-x)^3 + (-x)^2\}, & 0 \le (-x) \le 2 \\ -\{(-x)+3\}, & 2 < (-x) \le 4 \end{cases}$$

Odd extension of f(x) is

$$g(x) = \begin{cases} x-3, & -4 \le x < -2 \\ x^3 - x^2, & 2 \le x \le 0 \\ x^3 + x^2, & 0 \le x \le 2 \\ x+3, & 2 < x \le 4 \end{cases}$$

EXAMPLE 4.16 Let $f(x) = e^x + \sin x$ be defined on the interval [–4, 0]. Find the even and odd extension of f(x) in the interval [–4, 4].

SOLUTION Let g_e be the even extension of f(x), then

$$g_e(x) = \begin{cases} f(x) & ; \; x \in [-4,0] \\ f(-x) & ; \; x \in (0,4] \end{cases}$$

$$= \begin{cases} e^x + \sin x & ; \; x \in [-4,0] \\ e^{-x} - \sin x & ; \; x \in (0,4] \end{cases}.$$

We cannot have an odd extension of f(x) because f(0) = 1 ≠ 0. An odd function must have f(0) = 0.

EXAMPLE 4.17 Let f : [–2, 2] → R be a function where $f(x) = \begin{cases} x\tan x, & 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2}[x], & \frac{\pi}{2} < x \le 2 \end{cases}$.

Define f for x ∈ [–2, 0) so that
(i) f is an odd function
(ii) f is an even function,
where [.] denotes the greatest integer function.

SOLUTION Since $f(x) = \begin{cases} x\tan x, & 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2}[x], & \frac{\pi}{2} < x \le 2 \end{cases}$

$$\therefore \; f(-x) = \begin{cases} (-x)\tan(-x), & 0 < -x \le \frac{\pi}{2} \\ \frac{\pi}{2}[-x], & \frac{\pi}{2} < -x \le 2 \end{cases}$$

$$\text{or} \;\; f(-x) = \begin{cases} x\tan x, & -\frac{\pi}{2} \le x < 0 \\ \frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}$$

(i) If f is an odd function then the extended definition in [–2, 0] is

$$f(x) = \begin{cases} -x\tan x, & -\frac{\pi}{2} \le x < 0 \\ -\frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}$$

Note that f(0) = 0 for an odd function.
(ii) If f is an even function then the extended definition in [–2, 0] is

$$f(x) = \begin{cases} x\tan x, & -\frac{\pi}{2} \le x < 0 \\ \frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}.$$

EXAMPLE 4.18 Define the odd entension of $f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ x-3, & 1 < x \le 4 \end{cases}$ in [–4, 4].

SOLUTION The odd extension of

$f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ x-3, & 1 < x \le 4 \end{cases}$ in [–4, 4] is

$$f0(x) = \begin{cases} -(-x-3) & -4 \le x < -1 \\ x+1 & -1 \le x < 0 \\ 0 & x = 0 \\ -(-x+1) & 0 < x \le 1 \\ x-3 & 1 < x \le 4 \end{cases}$$

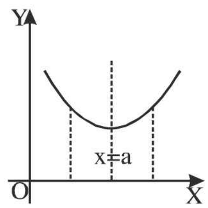
$$= \begin{cases} x+3 & -4 \le x < -1 \\ x+1 & -1 \le x < 0 \\ 0 & x = 0 \\ x-1 & 0 < x \le 1 \\ x-3 & 1 < x \le 4 \end{cases}$$

Note that in an odd function f(0) = 0.

Symmetry about the line x = a

Let us consider the graph of a function which is

symmetric about the line x = a.



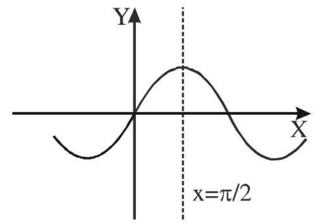
We observe that the function's values at points equidistant from 'a' must have equal value.

$f(a - x) = f(a + x)\ \forall\ x$

If we replace x by x – a we get $f(2a - x) = f(x)\ \forall\ x$.

For example, $\sin(\pi - x) = \sin x\ \forall\ x$.

$\Rightarrow$ The graph of the function f(x) = sinx is symmetric about the line $x = \frac{\pi}{2}$.



Now consider, $f(x) = (x - 2)^2 + (4 - x)^2$

$f(6 - x) = (6 - x - 2)^2 + (4 - (6 - x))^2$

$f(6 - x) = (4 - x)^2 + (x - 2)^2 = f(x)$.

Hence, f is symmetric about the line x = 3.

PRACTICE PROBLEMS [I]

1. Which of these functions are even ? odd ? neither?

   (i) $\sin x^2 + \sqrt[3]{1 + x^2}$ (ii) $\sqrt{1 + x^3} - 2^x \tan 3x$

   (iii) $xe^{-x^2 \tan^2 x} + e^x - e^{-x}$ (iv) $x^2 \cos x - |\sin x| + \frac{(1 + 2^x)^9}{2^x}$

2. If $f(x) = \frac{x}{e^x - 1} + \frac{x}{2}$, prove that f(x) is an even function.
3. Represent the function $y = \sin(x + 1) \sin^3(2x - 3)$ as a sum of an even and an odd function.
4. Represent the function $y = \cos\left(x + \frac{\pi}{8}\right) + \sin\left(2x - \frac{\pi}{12}\right)$ as a sum of an even and an odd function.
5. Let $f(x) = x \sin x + \tan x + 1$ for $x \in (0, \pi)$ then find its odd extension in $\left(-\frac{\pi}{2}, 0\right)$.
6. Let $f(x) = \begin{cases} 0 & \text{for } x = 0 \\ x^2 \sin(\pi / x) & \text{for } -1 < x < 1 \ (x \neq 0) \\ x|x| & \text{for } x \geq 1 \text{ or } x \leq -1 \end{cases}$ then show that f(x) is an even function.
7. If $f(x) = \sin(\cos x) - x + \tan(\sin x)\ \forall\ x \in (-\infty, 0)$ then find the odd extension of f(x) in $(0, \infty)$.
8. Prove that $f(x) = \sin(\tan^{-1}(\tan(\cot^{-1}x)) + \cos(\tan{-1}x). \tan(\cos{-1}x)$ is an odd function.
9. If graph of the function y = f(x) is symmetrical about the line x = 1, then show that

   (i) $f(1 + \alpha) = f(1 - \alpha)$ (ii) $f(2 - \alpha) = f(\alpha)$

10. Let $g(x) = f(x) - 1$. If $f(x) - f(1 - x) = 0\ \ x \in R$, then find the line about which the graph of g(x) is symmetrical.
11. Let $f(x) = |x - 2| + |x - 3| + |x - 4|$ and $g(x) = f(x + c)$. Then find c so that g(x) is an even function.

12. If the function f(x) satisfies the relation $f(x + y) + f(x - y) = 2 f(x) . f(y)$ $\forall$ x, y $\in$ R and $f(0) \neq 0$, then prove that f(x) is an even function.

13. If f(x) is a polynomial satisfying $f(x) . f(y) = f(x) + f(y) + f(xy) - 2$ $\forall$ x, y $\in$ R and $f(2) = 5$, then show that f(x) is an even function.

14. Determine whether $f(x) = x^2 - |x| \left[(x+1)^2\right]^{1/3} + \left[(x-1)^2\right]^{1/3}$ is even or odd?

15. If f is odd, find whether f(x) = cos + f(|sin x|) is even or odd.

16. Check whether the function $f(x) = \begin{cases} x|x|, & x \leq -1 \\ [1+x]+[1-x], & -1 < x < 1 \\ -x|x|, & x \geq 1 \end{cases}$ is even, odd or neither, where [x] denotes the greatest integer function.

17. If f is an even function defined over all real numbers and g is an odd function defined over all real numbers, how many of the following functions are even ?
    $a(x) = (fog)(x)$; $b(x) = f(|x|) + g(|x|)$;
    $c(x) = f(x^3) + g(x^2)$; $d(x) = xf(x)g(x)$.

18. f and g are odd functions defined on all real numbers and satisfying $f(-2) = 1$ and $g(-1) = -2$. What is the value of (fog) (1) ?

19. Prove that in the product $(1 - x + x^2 - x^3 + ... - x^{99} + x^{100})$ $(1 + x + x^2 + x^3 + ... + x^{99} + x^{100})$ after multiplying and collecting terms, there does not appear a term in x of odd degree.

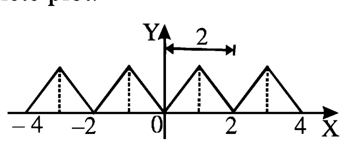
20. Let $f : R - \left\{-\frac{3}{2}\right\} \to R - \left\{\frac{c}{2}\right\}$ where $f(x) = \frac{cx}{2x+3}$ be such that $(fof)(x) = x$. Find the value of c.

21. If $f(x + y) = f(x) . f(y)$ for all real x, y and $f(0) \neq 0$ then prove that the function,
    $g(x) = \frac{f(x)}{1+f^2(x)}$ is an even function.

4.10 | PERIODIC FUNCTIONS

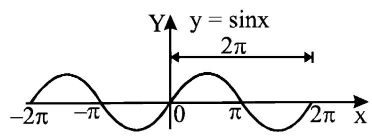
In the world around us, we encounter many phenomena which repeat after certain interval of time. In mathematics, the notion of periodicity remains same. The periodicity of a function is not limited to time. We look for repetition of function values with respect to the independent variable. Time could be just one such independent variable. For example, we have seen that in trigonometric functions we get the same value of trigonometric function for different angles. This many one relation is the basic requirement for a function to be periodic. In addition, these same values of the function should appear at regular intervals for the values of independent variables in the domain.

We can visualize periodic nature of a function by observing its graph in which a particular smallest segment of the plot can be repeated to construct the complete plot.



In the graph shown above, we have considered four segments corresponding to intervals of "2" unit length. The graph is constructed by repeating the segments one after another. It is clear from the figure that we need the smallest segment of length "2" to construct the curve.

Now consider the graph of the sine function:



This graph is constructed by repeating segments of length 2π, one after another.

Since, the graph of a periodic function with period T repeats after every interval of length T, a periodic function can be analyzed over an interval of length T anywhere in the domain, as the same analysis repeats over the entire domain.

Definition of periodic function

A function f(x) is said to be periodic if there exist a positive constant T independent of x such that $f(x + T) = f(x)$ for all such $x \in D_f$ for which $x + T \in D_f$, where D_f is the domain of the function f.

The least positive real number T ($T > 0$) is known as the fundamental period or simply the period of the function. It is also called as primitive or principal period.

Clearly $f(x - T) = f(x) = f(x + T) = f(x + 2T) = \ldots$

Hence,

$$f(x + nT) = f(x);\ n \in I, \text{ if } x \text{ and } x + nT \in D_f.$$

T is not a unique positive number. Any positive integral multiple of T within the domain of the function is also the period of the function since,

$$f(x + nT) = f(x),\ n \in N.$$

An aperiodic function is one, which in not periodic. It is also called non-periodic. On the other hand, a function is said to be anti-periodic if $f(x + T) = -f(x)$ for all x.

Basic periodic functions

Not many of the functions that we encounter are periodic. There are few functions, which are periodic by their very definition. We are, so far, familiar with the following periodic functions in this course :

(i) Constant function, $f(x) = c$.

For constant function to be periodic function,

$$f(x + T) = f(x)$$

By definition of constant function,

$$f(x + T) = f(x) = c$$

Clearly, constant function meets the requirement of a periodic function, but there is no least period. The relation of periodicity, here, holds for any change in x. We, therefore, conclude that constant function is a periodic function without any fundamental period. Any positive real number can be taken as the period of the function.

(ii) Trigonometric functions

The six trigonometric functions are the most commonly used periodic functions. They are used in various combination to generate other periodic functions. Graphs of trigonometric functions clearly show that periods of sinx, cosx, cosecx and secx are "2π" and that of tanx and cotx are "π".

(iii) Fraction part function, {x}

Fraction part function (FPF) is related to real number x and greatest integer function (GIF) as $\{x\} = x - [x]$. For understanding the nature of function, let us compute few function values as here :

$\{1.35\} = 1.35 - [1.35] = 1.35 - 1 = 0.35$

$\{2.35\} = 2.35 - [2.35] = 2.35 - 2 = 0.35$

$\{3.35\} = 3.35 - [3.35] = 3.35 - 3 = 0.35$

From the data in table, we infer that the function {x} is periodic with a period of "1". Note that function value repeats for an increment of "1" in the value of x. We, now, proceed to prove this analytically. Here,

$$f(x) = x - [x]$$

$$\Rightarrow f(x + T) = x + T - [x + T]$$

Let us assume that the given function is indeed a periodic function. Then by definition,

$$f(x + T) = f(x)$$

$$\Rightarrow x + T - [x + T] = x - [x]$$

$$\Rightarrow T = [x + T] - [x]$$

$$\Rightarrow T \in I$$

Clearly, T is an integer as both greatest integer functions return integers. There exists $T > 0$,

which satisfies the equation f(x + T) = f(x). The least positive integer is 1. Hence, the period of the function is 1.

(iv) The Dirichlet function $f(x) = \begin{cases} 1 & , \; x \in Q \\ 0 & , \; x \notin Q \end{cases}$ is also periodic with any positive rational number as its period.

Note that sin x and cos x are transcendental functions. We can prove the more general result that no periodic function can be either a rational function or an algebraic function (excluding constant function).

Note

No periodic function (unless it be a mere constant) can be a rational function.

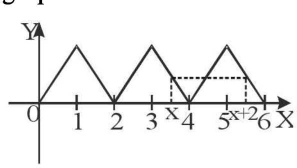
Suppose that $f(x) = \dfrac{P(x)}{Q(x)}$, $(Q(0) \neq 0)$, where P and Q are polynomials, and that $f(x) = f(x + T)$ for all values of x. Let $f(0) = a$, then the equation of the nth degree

$P(x) - aQ(x) = 0$

is satisfied by $n + 1$ values of x, viz. 0, T, 2T, ... nT. Hence, by the fundamental theorem of algebra, f(x) must be identically equal to a for all values of x. Hence proved.

Similarly, no periodic function can be an algebraic function.

Now, consider the function $y = f(x)$, $x \in [0, 6]$ shown in the graph :



Here $f(x + 2) = f(x)$ for all such $x \in [0, 6]$ for which $x + 2 \in [0, 6]$. Hence, f is periodic with period 2.

The domain of a periodic function may be bounded. In fact, we can discuss periodicity on an interval. For example, consider the function

$$f(x) = \begin{cases} \dfrac{1}{x}, & x < 0, \\ \sin x, & x \geq 0 \end{cases}.$$

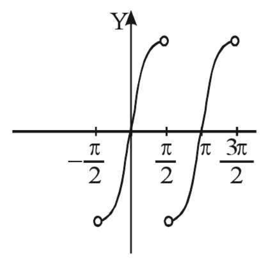
It is non-periodic in its domain. But, it is periodic for $x \geq 0$.

EXAMPLE 4.1 Using graphs, find whether the following functions are periodic, and find the period :

(i) $y = \dfrac{\sin 2x}{|\cos x|}$ (ii) y = sin x cosec x

SOLUTION

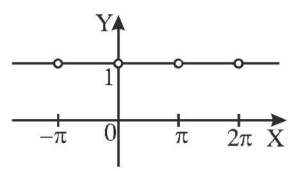
(i) $y = \dfrac{\sin 2x}{|\cos x|} = \begin{cases} 2\sin x & , \; \cos x > 0 \\ -2\sin x & , \; \cos x < 0 \end{cases}$.



From the graph, we see that the period is $T = \pi$.

(ii) y = sin x cosec x $\Rightarrow y = 1$, $x \neq n\pi$.

We see that the period is $T = \pi$.



EXAMPLE 4.2 Let $g(x) = \sqrt{x - 2k}$ where $2k \leq x < 2(k+1)$, $k \in I$. Check whether g(x) is periodic or not.

SOLUTION $g(x) = \sqrt{x - 2k}$ where $2k \leq x < 2(k+1)$, $k \in I$.

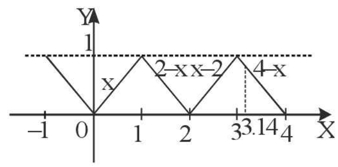
$$g(x) = \begin{cases} \sqrt{x+2}, & -2 \leq x < 0 \\ \sqrt{x}, & 0 \leq x < 2 \\ \sqrt{x-2}, & 2 \leq x < 4 \\ \sqrt{x-4}, & 4 \leq x < 6, \ldots \end{cases}$$

We can sketch the graph of g, and find that it repeats at regular intervals of 2 unit length.
$\Rightarrow$ g is periodic with period 2.

EXAMPLE 4.3 Suppose that f is an even, periodic function with period 2, and that f (x) = x for all x in the interval [0, 1]. Find the value of f (3.14).

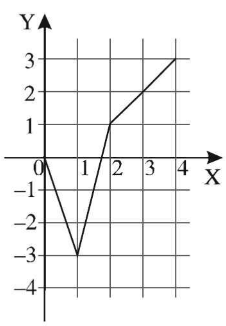
SOLUTION Since f (x) is even , the graph is symmetry about y-axis. Given that $f(x) = x \ \forall\ x \in [0, 1]$, we have $f(x) = -x \ \forall\ x \in [-1, 0)$. Now f is periodic with period 2.

$\therefore$ The graph of f (x) is as shown



$f(3.14) = 4 - 3.14 = 0.86.$

EXAMPLE 4.4 Given below is a partial graph of an even periodic function f whose period is 8. If [.] denotes greatest integer function then find the value of the expression $f(-3) + 2\,|f(-1)| + \left[f\left(\frac{7}{8}\right)\right] + f(0) + \cos^{-1}(f(-2)) + f(-7) + f(20)$



SOLUTION

$f(-3) = f(3) = 2$ [f (x) is an even function]

Again $f(-1) = f(1) = -3$

$\therefore\ 2\,|f(-1)| = 2\,|f(1)| = 2\,|-3| = 6$

From the graph, $-3 < f\left(\frac{7}{8}\right) < -2$

$\therefore\ \left[f\left(\frac{7}{8}\right)\right] = -3.$

$f(0) = 0$ (from the graph)

$\cos^{-1}(f(-2)) = \cos^{-1}(f(2)) = \cos^{-1}(1) = 0.$

$f(-7) = f(-7 + 8) = f(1) = -3.$ [f (x) has period 8]

$f(20) = f(4 + 16) = f(4) = 3$ [f(nT + x) = f(x)]

The expression= 2 + 6 – 3 + 0 + 0 – 3 + 3 = 5.

Once, we know periods of standard functions, we use different rules, properties and results of periodic functions to determine periods of other functions, which are formed as composition or combination of standard periodic functions.

Periodicity using basic definition

The definition of periodicity is helpful in determining period of some functions. It is also used to determine whether a given function is periodic or not in the first place.

In order to determine periodicity and period of a function, we can use the following steps:

1. Put f(x + T) = f(x).
2. If there exists a positive number T satisfying the above equation independent of x, then f(x) is periodic. Otherwise, the function is aperiodic.
3. The least value of T is the period of the periodic function.

Consider f(x) = x + sin x.

Let for some T, f(x) = f(x + T)

$\Rightarrow x + T + \sin(x + T) = x + \sin x$

$\Rightarrow T + \sin(x + T) = \sin x$

$\Rightarrow \sin(x + T) - \sin x = -T$

$\Rightarrow \cos\left(x + \frac{T}{2}\right)\sin\frac{T}{2} = -T$

Here ‘T, cannot be made independent of x. So f(x) is not a periodic function.

Note

Strictly increasing and strictly decreasing functions are non-periodic.

f(x) = x + sin x

$f'(x) = 1 + \cos x \geq 0 \ \forall \ x.$

The function is strictly increasing and hence, non-periodic.

EXAMPLE 4.5 Determine whether the function $f(x) = \cos\sqrt{x}$ is periodic ?

SOLUTION We check periodicity by applying the definition of periodic function. According to the definition of periodic function,

$f(x + T) = f(x)$

$\Rightarrow \cos\sqrt{(x+T)} = \cos\sqrt{x}$

$\Rightarrow \sqrt{(x+T)} = 2n\pi \pm \sqrt{x}, \ n \in I$

Squaring both sides and solving for T, we have:

$\Rightarrow (x + T) = (2n\pi \pm \sqrt{x})^2$

$\Rightarrow T = (2n\pi \pm \sqrt{x})^2 - x$

If we expand the square term, then we find that the expression of T is not independent of x. Hence, the given function is not a periodic function.

EXAMPLE 4.6 Prove that $\sin(1/x)$, $(x \neq 0)$ is a non-periodic function

SOLUTION Let $f(x) = \sin(1/x)$ be periodic with period T, $T > 0$.

$\therefore f(x + T) = f(x)$

$\Rightarrow \sin\left(\frac{1}{x+T}\right) = \sin\left(\frac{1}{x}\right)$

$$\Rightarrow \frac{1}{x+T} = n\pi + (-1)^n \quad (1)$$

Put $x = T$ and $x = 2T$ in (1), then

$$\frac{1}{2T} = n\pi + (-1)^n \frac{1}{T} \quad (2)$$

$$\text{and } \frac{1}{3T} = n\pi + (-1)^n \frac{1}{2T} \quad (3)$$

Subtracting (3) from (2), we get $\frac{1}{6T} = (-1)^n . \frac{1}{2T}$

or $\frac{1}{3} = (-1)^n$ which is impossible.

Hence $\sin(1/x)$ is a non-periodic function.

EXAMPLE 4.7 Determine whether the function $f(x) = x \cos x$ is periodic ?

SOLUTION We check periodicity applying definition of a periodic function. Let the function be periodic. Then,

$f(x + T) = f(x)$

$\Rightarrow (x + T)\cos(x + T) = x \cos x$

$\Rightarrow T\cos(x + T) = x\cos x - x\cos(x + T)$

$= x\{\cos x - \cos(x + T)\}$

We see that the right hand side is a product of algebraic function and trigonometric function. On the left hand side, there is only trigonometric function (apart from T, which is a constant). There is no algebraic function in x on the left which can cancel x on the right. Thus, we conclude that T is not independent of x and as such, the given function is not periodic.

EXAMPLE 4.8 If the function

$f(x) = \sin x + \cos ax$ be periodic, then prove that a is rational.

SOLUTION The function is sum of two trigonometric functions. It is given that the function is a periodic function.

Let T be the period, then according to definition:

$\sin(x + T) + \cos a(x + T) = \sin x + \cos ax$

Putting, $x = 0$, we have :

$\sin T + \cos aT = 1$

Putting, $x = -T$, we have :

$\sin 0 + \cos 0 = -\sin T + \cos T$

$\Rightarrow -\sin T + \cos aT = 1$

Subtracting one equation from another,

$\sin T = 0$

$\Rightarrow T = n\pi, \ n \in I$

and from the above equations :

$\cos aT = 1$

$\Rightarrow aT = 2m\pi, \ m \in I$

Combining the two results:

$\Rightarrow a = .$

Hence, a is a rational number.

Arithmetic operations on periodic functions

A periodic function can be modified by arithmetic operations on the independent variable of the

function or function itself. The arithmetic operations involved here are addition, subtraction, multiplication, and division. We have earlier studied that these operations have different effects on the graph of the core function.

Arithmetic operations on independent variable change input to the function and the graph of core function is transformed horizontally (along x-axis). On the other hand, operations on the function itself change output and the graph of core function is transformed vertically (along, y-axis). The combined input/output arithmetic operations related to function are symbolically represented as:

$$y = pf(ax + b) + q; \quad a, b, p, q \in R$$

Important thing to understand here is that periodicity is defined in terms of independent variable, x. A periodic function repeats a set of its values after regular interval of the independent variable i.e. x., Clearly, periodicity of a periodic function is not affected by transformations in vertical direction. Hence, arithmetic operations with function involving constants "p" and "q" do not affect periodicity of a periodic function.

Not all arithmetic operations on independent variable will change or affect periodicity. Shifting of core graph due to addition or subtraction results in shifting of the graph as a whole either to the left or right. This operation does not change size and shape of the graph. Thus, addition and subtraction operation involving constant "b" does not affect periodicity of a function.

Negation of independent variable, when "a" is negative, results in flipping of the graph without any change in size and shape of the graph. As such, negation of independent variable does not change periodicity either.

It is only the multiplication or division of independent variable x by a positive constant, "a", results in change in size with respect to origin in the horizontal direction. The graph shrinks horizontally when independent variable is multiplied by a positive constant greater than 1 by the factor which is equal to the multiplier. This means periodicity of graph decreases by the same factor i.e. $|a|$.

The graph stretches horizontally when the independent variable is divided by positive constant greater than 1 by the factor which is equal to the divisor. This means periodicity of graph increases by the same factor i.e.$|a|$.

We combine these two observations by saying that period of graph decrease by a factor $|a|$. Note that magnitude of constant "a" more than 1 represents multiplication and less than 1 represents division.

In the nutshell, if T is the period of f(x), then period of function of the form given below is $T/|a|$:

$$y = pf(ax + b) + q;\ a, b, p, q \in R.$$

EXAMPLE 4.9 What is the period of function

$$f(x) = 3 + 2\sin\left(\frac{\pi x + 2}{3}\right)\ ?$$

SOLUTION Rearranging, we have :

$$f(x) = 3 + 2\sin\left(\frac{\pi}{3}x + \frac{2}{3}\right)$$

The period of sine function is 2p. Comparing with the function form " pf (ax + b) + q", magnitude of a i.e. $|a|$ is $\pi/3$. Hence, period of the given function is :

$$T' = \frac{T}{|a|} = \frac{2\pi}{\frac{\pi}{3}} = 6.$$

EXAMPLE 4.10 Find the period of the following functions :

(i) $f(x) = \tan 2x$

(ii) $f(x) = \sin^4 x + \cos^4 x$

SOLUTION (i) $f(x) = \tan 2x$ has period $\frac{\pi}{2}$ as $\tan x$ has period π

(ii) $f(x) = \sin^4 x + \cos^4 x$

$$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos 2x$$

$$= 1 - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{4}(1 - \cos 4x)$$

$$= \frac{3}{4} + \frac{1}{4}\cos 4x$$

Since cos x has period 2π, cos 4x has period $\frac{2\pi}{4} = \frac{\pi}{2}$.

$\therefore$ f (x) has period.

EXAMPLE 4.11 Determine period of the function

$f(x) = a \sin k x + b \cos k x$

SOLUTION The function is sum of two trigonometric functions. We can reduce this function in terms of a single trigonometric function to determine its periodic nature. Let

$a = r \cos \theta; \quad b = r \sin \theta$

$\Rightarrow r = \sqrt{(a^2 + b^2)}$

Substituting in the function, we have

$f(x) = r \cos \theta \sin kx + r \sin \theta \cos k x$

$= r \sin (kx + \theta)$

This is a periodic function. Also, period of ag(x) is same as that of g(x). Therefore, period of r sin (kx + θ) is same as that of sin (kx + θ). On the other hand, period of g(ax + b) is equal to the period of g(x), divided by |a|. Now, period of sinx is 2π. Hence, period of the given function is :

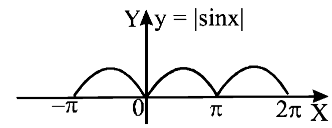
$$T = \frac{2\pi}{|k|}$$

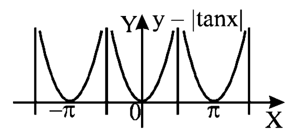
Alternatively, we can treat given function as addition of two functions. The period of each term is $2\pi/|k|$. Applying LCM rule (to be dealt later), the period of given function is equal to LCM of two periods, which is $2\pi/|k|$.

Modulus of periodic functions

Let us determine the periods of modulus of trigonometric functions like |sinx|, |cosx|, |tanx| etc. We know that modulus operation on function converts negative function values to positive function values with equal magnitude.

From the graphs, we observe that periods of |sinx| and |tan x| are π. Similarly, we find that periods of modulus of all six trigonometric functions are π. Thus, |sin x|, |cos x|, |cosec x|, |sec x|, |tan x| and |cot x| are periodic functions with period π.

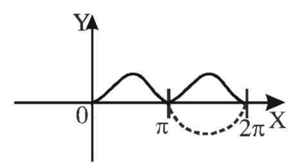




Integral powers of periodic functions

The periods of trigonometric functions which are raised to integral powers, depend on the nature of exponents. The periods of trigonometric exponentiations are different for even and odd powers.

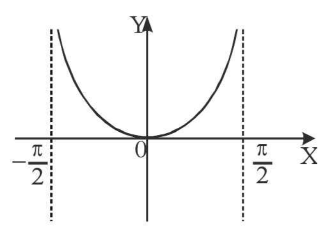
Consider the function $y = \sin^2 x$. On raising sinx to even power the portion of the graph below x-axis gets reflected above the x-axis and the period reduces from 2π to π. See figure.



We can also verify this using transformation formula $y = \sin^2 x = \frac{1 - \cos 2x}{2}$.

However, the period of $y = \sin^3 x$ remains 2π since raising sinx to odd power does not reflect the portion of the graph below x-axis, above the x-axis.

For the function $y = \tan^2 x$, the period is same as y = tanx i.e. π, because the graph, even when it is reflected above, does not repeat at a smaller length. See figure.



Similarly, the period of $y = (\{x\})^n$ is 1, irrespective of any power n.

The following results with respect these exponentiated trigonometric functions are useful :

Functions $\sin^n x$; $\cos^n x$; $\text{cosec}^n x$ and $\sec^n x$ are periodic with period "π" when n is even and "2π" when n is odd.

On the other hand, tannx and $\cot^n x$ are periodic with period π whether n is even or odd.

EXAMPLE 4.12 Find the period of the function

SOLUTION $f(x) = \sin^2 x + (1 - \sin^2 x)^2 + 2$

$= 3 - \sin^2 x + \sin^4 x = 3 - \sin^2 x \cos^2 x$

$= 3 - \dfrac{\sin^2 2x}{4}$.

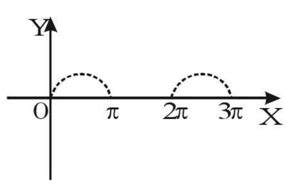
The period of sin 2x is π. Because of even power of sin 2x, the period of f(x) is $\dfrac{\pi}{2}$.

Note

1. If f is periodic with period T, then $\dfrac{1}{f(x)}$ is also periodic with period T.
2. If f is periodic with period T then $(f(x))^{1/n}$ is also periodic with period T.

Thus, if f has period T then, $\sqrt[3]{f(x)}$ and $\sqrt{f(x)}$ also have period T.

For example, the period of $y = \sqrt{\sin x}$ is 2π.



Period of sum of functions

LCM rule

When two periodic functions are added or subtracted, the resulting function is a periodic function, if the LCM of the periods of the given functions exists. Consider a function,

$f(x) = \sin x + \sin(x/2)$

The period of sinx is 2π, whereas period of $\sin(x/2)$ is 4π. The LCM (least common multiple) of 2π and 4π is 4π. The graph constructed by adding two periodic lengths of sin x and one periodic length of $\sin(x/2)$, over a length of 4π, repeats. Hence, the period of the function f(x) is 4π, which is equal to LCM of the two periods.

If a and b are non-zero numbers and functions g(x) and h(x) are periodic functions having periods, T_1 and T_2, then the function $f(x) = ag(x) \pm bh(x)$ is also a periodic function. The period of f(x) is LCM of T_1 and T_2.

EXAMPLE 4.13 Find the period of

$f(x) = \sin 2x + \cos 3x$.

SOLUTION Let P be the period of f(x). The period of sin 2x is π and the period of cos 3x is $\dfrac{2\pi}{3}$. In one full period of length P, both sin 2x and cos 3x must go through an integral number of periods.

Thus $P = s\pi = \dfrac{2\pi t}{3}$, for some positive integers s and t. But then $3s = 2t$. The smallest positive solution of this is $s = 2$, $t = 3$.

The period sought is then $P = s\pi = 2\pi$.

EXAMPLE 4.14 Find period of the function

$f(x) = 2\sin x - 3\tan(x/2)$.

SOLUTION The function is difference of two trigonometric functions. Each of the functions is periodic. Therefore, we apply LCM rule to find the period of the function.

The period of sin x is 2π. Hence,

$T_1 = 2\pi$

We also know that period of $g(ax + b)$ is equal to the period of g(x), divided by |a|. The period of second term of f(x), therefore, is equal to the period of tanx, divided by 1/2.

The period of tan x is π. Hence, period of tan (x/2) is $T_2 = \dfrac{\pi}{\frac{1}{2}} = 2\pi$.

Now, LCM of 2π and 2π is 2π. Hence, $T = 2\pi$.

Finding LCM

LCM of integral numbers is obtained easily. There is, however, difficulty in finding LCM when numbers are fractions (like 3/4, 1/3 etc.) or irrational numbers (like π, $2\sqrt{2}$ etc.).

For rational fraction, we can find LCM using the following formula :

$$\text{LCM} = \frac{\text{LCM of numerators}}{\text{HCF of denominators}}$$

Consider fractions 3/5 and 2/3. The LCM of numerators 3 and 2 is 6. The HCF of denominators is 1. Hence, LCM of the two fractions is 6/1 i.e. 6.

This rule also works for irrational numbers like $2\sqrt{2}/3$, $3\sqrt{2}/5$ etc. or $\pi/2$, $3\pi/2$ etc.

For example, LCM of $\pi/3$ and $3\pi/2$ is :

$$\text{LCM} = \frac{\text{LCM of numerators}}{\text{HCF of denominators}}$$

$$= \frac{\text{LCM of } (\pi, 3\pi)}{\text{HCF of } (2,3)}$$

$$= \frac{3\pi}{1} = 3\pi.$$

However, we cannot find LCM of irrational numbers of different kinds like $2\sqrt{2}$ and π.

If individual periods are rational and irrational numbers respectively, then LCM is not defined.

Let f(x) = sin x + {x}

Here period of fractional part {x} is 1 and that of sin x is 2π. As the L.C.M. of 2π and 1 does not exist, (2π is an irrational quantity while 1 is a rational quantity) f(x) is a non-periodic function.

EXAMPLE 4.15 Find period of the function $f(x) = \sin^3 x$. Writing identity for sin3x, we have

SOLUTION

$$f(x) = \frac{3\sin x - \sin 3x}{4} = \frac{3}{4}\sin x - \frac{3}{4}\sin 3x.$$

We know that period of ag(x) is same as that of g(x). The period of first term of f(x), therefore, is equal to the period of sinx i.e. $T_1 = 2\pi$.

Now, period of sin 3x, is equal to the period of sin x i.e. 2π, divided by 3. Hence, $T_2 = \frac{2\pi}{3}$.

Applying LCM rule,

$$T = \text{LCM}\left(2\pi, \frac{2\pi}{3}\right)$$

$$= \frac{\text{LCM of } 2\pi \text{ and } 2\pi}{\text{HCF of 1 and 3}} = \frac{2\pi}{1} = \pi.$$

EXAMPLE 4.16 Find period of

$$f(x) = \sin\left(2\pi x + \frac{\pi}{8}\right) + 2\sin\left(3\pi x + \frac{\pi}{3}\right)$$

SOLUTION Period of $\sin(2\pi x + \pi/8)$ is :

$$T_1 = \frac{2\pi}{2\pi} = 1$$

Period of $2\sin(3\pi x + \pi/3)$ is :

$$T_2 = \frac{2\pi}{3\pi} = \frac{2}{3}$$

LCM of numbers involving fraction is equal to the ratio of LCM of numerators and HCF of denominators. Hence,

$$\text{LCM} = \frac{\text{LCM of numerators}}{\text{HCF of denominators}} = \frac{\text{LCM of } (1,2)}{\text{HCF of } (1,3)}$$

$$= \frac{2}{1} = 2.$$

EXAMPLE 4.17 Find period of the function:

f(x) = 2 sin 2{3x} + 5cos3{2x}

where {} denotes fraction part function.

SOLUTION The period of {x} is 1. Recall that only the coefficient of x changes the period of FPF. Therefore, period of {3x} is 1/3. Hence, period of 3sin2{3x} is 1/3. On the other hand, period of {2x} is 1/2. Hence, period of 5cos3{2x} is 1/2.

LCM of 1/2 and 1/3

$$= \frac{\text{LCM of } (1,1)}{\text{HCF of } (2,3)} = \frac{1}{1} = 1.$$

Hence, the period is 1.

Determination of periods involving radicals is evaluated applying LCM rule. We can determine LCM of radicals, if they are of same kind. If radicals involved are of different kinds, then we cannot determine LCM. In that case, the given function is not periodic.

EXAMPLE 4.18 Find period of the function $f(x) = 3\sin 2\sqrt{3}\,x + 2\cos 5\sqrt{3}\,x$.

SOLUTION Period of $3\sin 2\sqrt{3}\,x$ is $T_1 = \dfrac{2\pi}{2\sqrt{3}}$.

Period of $2\cos 5\sqrt{3}\,x$ is $T_2 = \dfrac{2\pi}{5\sqrt{3}}$.

The two irrational periods are of same kind. Hence, the period of the given function is :

$$\text{LCM} = \frac{\text{LCM of } (2\pi, 2\pi)}{\text{HCF of } (2\sqrt{3}, 5\sqrt{3})} = \frac{2\pi}{\sqrt{3}}.$$

EXAMPLE 4.19 Find period of the function $f(x) = \sin\sqrt{2}\,x + \cos\sqrt{3}\,x$

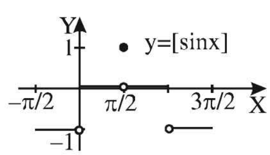
SOLUTION Period of $\sin\sqrt{2}\,x$ is $T_1 = \dfrac{2\pi}{\sqrt{2}}$

Period of $\cos\sqrt{3}\,x$ is $T_2 = \dfrac{2\pi}{\sqrt{3}}$.

The two irrational periods are not of same kind. Hence, we cannot determine LCM.
Thus, the function is not periodic.

EXAMPLE 4.20 Find the period of the function, $f(x) = [\sin 3x] + |\cos 6x|$, [.] denotes the greatest integer function.

SOLUTION Let $f_1(x) = [\sin 3x]$.
$[\sin x]$ is periodic with period 2π.



$\therefore$ The period of $f_1(x) = \dfrac{2\pi}{3}$.

Let $f_2(x) = |\cos 6x| = \sqrt{\left(\dfrac{1+\cos 12x}{2}\right)}$

$\therefore$ period of $f_2(x) = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$.

Hence period of f(x) = L. C. M. of $\left(\dfrac{2\pi}{3}, \dfrac{\pi}{6}\right)$

$$= \frac{\text{L.C.M. of } \{2\pi, \pi\}}{\text{H.C.F. of } \{3, 6\}} = \frac{2\pi}{3}.$$

Exception to LCM rule

LCM rule may not always provide the fundamental period of the function. There are exceptions to this rule. We do not apply this rule, when functions are co-functions of each other or when functions are even functions.

Two functions f(x) and g(y) are co-functions if x and y are complimentary inputs. The functions sinx and cosx are co-functions as :

$$\sin x = \cos\frac{\pi}{2} - x$$

Similarly, $|\cos x|$ and $|\sin x|$ are cofunctions as :

$$|\cos x| = |\sin\frac{\pi}{2} - x|$$

Let us find the period of $f(x) = |\cos x| + |\sin x|$

We know that $|\cos x|$ and $|\sin x|$ are co-functions. This suggests that the function may have the period $\pi/2$. We check this as:

$$f\left(x+\frac{\pi}{2}\right) = \left|\cos\left(x+\frac{\pi}{2}\right)\right| + \left|\sin\left(x+\frac{\pi}{2}\right)\right|$$

$$\Rightarrow f\left(x+\frac{\pi}{2}\right) = |-\cos x| + |\sin x| = |\cos x| + |\sin x| = f(x)$$

Hence, period is $\pi/2$.

Let us work with this problem using LCM rule and compare the result. The periods of $|\cos x|$ and $|\sin x|$ are π. Now, applying LCM rule, the period of the given function is LCM of π and π, which is π. Thus the LCM rule does not provide the fundamental period.

If f(x) is a periodic function with period T_1 and g(x) is a periodic function with period T_2, then the function f(x) + g(x) is also periodic with period T which is the L. C. M. of T_1 & T_2, provided f(x) and g(x) can not be interchanged by adding a positive number in x less than L.C.M. of T_1 & T_2. In the case when f(x) and g(x) get interchanged by adding a positive number in x less than L.C.M. of T_1 & T_2, then this number is the period of f(x) + g(x).

Consider the function

$$y = \{x\} + \left\{x + \frac{1}{2}\right\}.$$

The period of both the individual functions is 1 and the LCM is also 1. However, the period of the function is 1/2 because, the two functions gets interchanged when we replace x by x + 1/2.

$$f = \left(x+\frac{1}{2}\right) + \left(x+\frac{1}{2}\right) + \{x+1\}$$

$$= \left(x+\frac{1}{2}\right) + \{x\} = f(x)$$

$$\therefore\ T = \frac{1}{2}.$$

Note

1. The period of a |sin x| + b |cos x| = π if a ≠ b $= \frac{\pi}{2}$ if a = b.
2. The period of |sin mx| + |cos nx| $= \text{LCM of } \left(\frac{\pi}{m}, \frac{\pi}{n}\right)$ if m ≠ n $= \frac{1}{2} \text{ LCM of } \left(\frac{\pi}{m}, \frac{\pi}{n}\right)$ if m = n.

EXAMPLE 4.21 Find the period of the following functions : (i) |sin2x| + |cos 2x|. (ii) |sin2x| + |sin 3x|.

SOLUTION (i) The period of |sin 2x| is $\frac{1}{2}\left(\frac{2\pi}{2}\right) = \frac{\pi}{2}$

and the period of |cos 2x| is $\frac{1}{2}\left(\frac{2\pi}{2}\right) = \frac{\pi}{2}$.

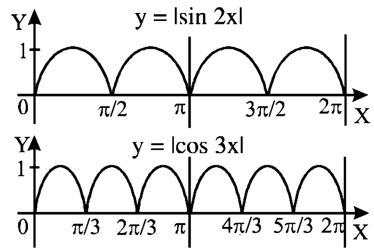
Hence, the period of the given function is

$$\frac{1}{2} \text{ LCM of } \left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{\pi}{4}.$$

(ii) The period of |sin 2x| is $\frac{1}{2}\left(\frac{2\pi}{2}\right) = \frac{\pi}{2}$ and the period of |sin 3x| is $\frac{1}{2}\left(\frac{2\pi}{3}\right) = \frac{\pi}{3}$.

Hence, period of the given function is

$$\text{LCM of } \left(\frac{\pi}{2}, \frac{\pi}{3}\right) = \frac{\text{LCM of } (\pi, \pi)}{\text{HCF of } (2,3)} = \pi.$$



From the plot of y = |sin 2x| and y = |sin 3x|, we can see that both the curves taken together repeat simultaneously at intervals of π.
Hence, the fundamental period is π.

EXAMPLE 4.22 Find period of
$f(x) = \sin^2 x + \cos^4 x$

SOLUTION Here, we see that f(–x) = f(x)

$$f(-x) = \sin^2(-x) + \cos^4(-x) = \sin^2 x + \cos^4 x = f(x)$$

This means that given function is even function. As such, we can apply LCM rule. We, therefore, proceed to reduce the given function in terms of one trigonometric function type.

$$f(x) = \sin^2 x + \cos^2 x\,(1 - \sin^2 x)$$

$$= \sin^2 x + \cos^2 x - \sin^2 x \cos^2 x$$

$$\Rightarrow f(x) = 1 - \frac{1}{4}\sin^2 2x = 1 - \frac{1}{4}\cdot\frac{(1-\cos 4x)}{2}$$

$$\Rightarrow f(x) = 1 - \frac{\cos 4x}{8} \quad \Rightarrow \quad T = \frac{2\pi}{4} = \pi/2$$

Note that if we apply the LCM rule, then period evaluates to π.

Note

1. The sum of a periodic and a non-periodic function may be periodic.
For example, f(x) = cot x is periodic and g(x) = $\frac{x}{x}$ is non-periodic, but their sum (f + g) (x) = cot x + $\frac{x}{x}$ = cot x + 1, x ≠ nπ, n ∈ I is periodic with period π.
However, y = sin x + $\frac{1}{x^2+1}$ is a non-periodic function.

2. The sum of two non-periodic functions may be periodic.
For example, the sum of two non-periodic functions y = sin x + x and y = cos x – x
i.e. y = sin x + cos x is a periodic function.

Period of product / division of functions

Consider the function $h(x) = \dfrac{f_1(x)..........f_n(x)}{g_1(x)..........g_m(x)}$,
where f_i and g_j are periodic for all i and j. Let their periods be T_i and T_j.
If the LCM of all T_i and T_j exists, then the function h(x) is periodic. However, the LCM need not be the fundamental period. To get the fundamental period, we need to check the LCM using:
(i) graph of h(x), or,
(ii) transformation of h(x) into sum of functions, or,
(iii) trial of (LCM)/n as period using definition
For example, the function y = {x}sin x is non-periodic as the LCM of {1, 2π} does not exist.
The function y = sin x. cos x is periodic because the LCM of (2π, 2π) is 2π. However, we need to examine whether 2π is the fundamental period or not.
We know that y = sin x. cos x = $\frac{1}{2}$ sin 2x. By our previous concepts, the period is π instead of 2π.
Similarly, the LCM for the function
y = cos x . cos 2x . cos 3x is 2π.
On trial of 2π/2 = π,
f(x + π) = – cos x cos 2x (–cos 3x) = f(x),
the function is having a period of π.
Further trials of π/2, π/3, etc. are unsuccessful.
We can also look for transformation into sum of functions :

$$f(x) = \frac{1}{2}\{\cos 4x + \cos 2x\}\cos 2x$$

$$= \frac{1}{4}\{\cos 6x + \cos 2x + 1 + \cos 4x\}$$

LCM of $\{\frac{\pi}{3}, \pi, \frac{\pi}{2}\}$ is π.

Thus, the period of the function is π.

Consider the function $f(x) = \dfrac{\tan x - \cot x}{|\sin x| - |\cos x|}$

The periods of all the four functions involved are π. Their LCM is π.

We try $T = \frac{\pi}{2} : f\left(x + \frac{\pi}{2}\right) = f(x)$, which holds.

Hence, the period is $\frac{\pi}{2}$.

Now, consider the function $f(x) = \dfrac{\sin x + \sin 3x}{\cos x + \cos 3x}$.

The LCM of periods of all the four functions involved is 2π. We now try to reduce the period.
If we simplify the function we get

$$f(x) = \frac{\sin 2x.\cos x}{\cos 2x.\cos x} = \tan 2x,\ \cos x \neq 0.$$

The period of tan 2x is $\frac{\pi}{2}$.

However, the period of f(x) is not .
We must take care of the loss of domain in tan 2x due to cos x ≠ 0. Taking this into account, the period of f(x) is π.

EXAMPLE 4.23 Find period of the function
f(x) = cos x . cos 3x

SOLUTION f(x) = cos x . cos 3x
Period of f(x) is L.C.M. of $\left(2\pi, \frac{2\pi}{3}\right) = 2\pi$.

But 2π may or may not be the fundamental period.
The fundamental period can be $\frac{2\pi}{n}$, where $n \in N$.
Hence cross-checking for n = 1, 2, 3..... we find π to be the fundamental period :
f(π + x) = (–cos x) (– cos 3x) = f(x).

EXAMPLE 4.24 Find value of n, if period of the function $f(x) = \dfrac{4\sin 2nx}{1+\cos^2 nx}$, n ∈ N, is π/4.

SOLUTION The numerator and denominator are periodic functions. Hence, period of the given function is LCM of the periods of numerator and denominator.

Now, period of 4 sin 2nx is $T_1 = \frac{2\pi}{2n} = \frac{\pi}{n}$.

Using trigonometric identity, the denominator of the given function is :

$$1 + \cos^2 nx = 1 + \frac{1+\cos 2nx}{2}$$

The period of denominator is $T_2 = \frac{2\pi}{2n} = \frac{\pi}{n}$.

Hence, the period of given function is LCM of π/n and π/n, which is π/n. According to the question,

$$\frac{\pi}{n} = \frac{\pi}{4} \quad \Rightarrow \quad n = 4.$$

EXAMPLE 4.25 Find period of the function:

$$f(x) = \sin^3 x + \sin^2\left(x+\frac{\pi}{3}\right) - \cos x \cos\left(x+\frac{\pi}{3}\right)$$

SOLUTION The given function is not an even function as $\sin^3(-x) = -\sin^3 x$. We can use LCM rule to determine period of the given function. The periods of $\sin^3 x$ and $\sin^2\left(x+\frac{\pi}{3}\right)$ are 2π and π respectively. Using trigonometric identity,

$$\cos x \cos\left(x+\frac{\pi}{3}\right) = \frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right) + \frac{1}{2}$$

The period of cos (2x + π/3) is 2π/2 = π. Now, periods of three terms of the given function are 2π, π and π. Hence, period of the given function is LCM of the three terms i.e. 2π.

Period of composite functions

If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods. Further if g is a strictly increasing or decreasing function in the range of f, then T is the fundamental period of gof.

For example, let f(x) = sin x and $g(x) = 2^x$.

$gof(x) = 2^{\sin x}$ has period 2π.

Now, let f(x) = sin x and $g(x) = x^2$

$gof(x) = (\sin x)^2$ has period π.

The period of f is 2π. The period of gof in this case reduces because $g(x) = x^2$ is non-monotonous.

Note

1. The composition of a non-periodic function with a periodic function may be periodic.

For example, composition of the non-periodic function f(x) = [x] with the periodic g(x) = sin πx i.e. gof (x) = sin π [x] is periodic.

We can check that its period is 2.

Similarly, the function y = tan (x + sin x) has period T = 2π.

f(x + 2π) = tan (x + 2π + sin (x + 2π))

= tan (x + 2π + sin x)

= tan (x + sin x).

2. The composition of two non-periodic functions may be periodic.

Consider, f(x) = [x] and $g(x) = \begin{cases} x^2, & x \notin I \\ 2, & x \in I \end{cases}$

We have gof (x) = 2 for all x. Now this is a periodic function.

EXAMPLE 4.26 Find period of the function f(x) = sin {x}, where {.} denotes fraction part function.

SOLUTION The fraction part function {x} is a periodic function with period 1. and sin x is strictly increasing in [0, 1). Hence, period otf sin{x} is 1.

EXAMPLE 4.27 Find period of function $\tan^{-1}(\tan x)$.

SOLUTION The function $\tan^{-1}x$ is strictly increasing and tanx is a periodic function with period π. Hence, the function $\tan^{-1}(\tan x)$ is a periodic function with period π.

Periodicity involving functional equation

We are sometimes required to determine period of a function f(x) based on certain conditional relation. In such a situation, we should manipulate the given condition in such a manner that we ultimately get a relation of the type f(x+T) = f(x).

Generally, this requires substitution of independent variable with expression which results in existing expressions. This enables us to use the given relation repeatedly. This method is better understood by working with examples.

EXAMPLE 4.28 Let f(x) be a function and k be a positive real number such that

$f(x + k) + f(x) = 0$ for all $x \in R$.

Prove that f(x) is periodic. Also determine a period.

SOLUTION The given equation can be re-written as

$f(x + k) = -f(x)$ for all $x \in R$.

Here, our objective is to convert RHS of the equation as f(x). Replacing x by x + k, we have

$f(x + 2k) = -f(x + k)$ for all $x \in R$

Combining the two equations,

$f(x + 2k) = -(-f(x))$

$\Rightarrow f(x + 2k) = f(x)$ for all $x \in R$.

It means that f(x) is a periodic function and one of its period is 2k.

EXAMPLE 4.29 Consider those functions f that satisfy $f(x + 4) + f(x - 4) = f(x)$ for all real x. Prove that any such function is periodic, and there is a common positive period π for all of them. Find the value of p.

SOLUTION $f(x + 4) + f(x - 4) = f(x)$ (1)

Replacing x by x + 4

$f(x + 8) + f(x) = f(x + 4)$ (2)

Substituting the value of f (x + 4) from (1) in (2),

$f(x + 8) + f(x) = f(x) - f(x - 4)$

$\Rightarrow f(x + 8) = -f(x - 4)$

Replacing x by x + 4

$f(x + 12) = -f(x)$

Replacing x by x + 12

$f(x + 24) = -f(x + 12)$

$f(x + 24) = f(x)$

$\Rightarrow$ f (x) is periodic with period 24.

EXAMPLE 4.30 Prove that f is periodic if it satisfies the condition

$$f(x + p) = 1 + \{1 - 3f(x) + 3f^2(x) - f^3(x)\}^{1/3}.$$

SOLUTION We have

$\{f(x+p) - 1\}^3 = 1 - 3f(x) + 3f^2(x) - f^3(x) = \{1 - f(x)\}^3$

$\Rightarrow f(x + p) - 1 = 1 - f(x)$

$\Rightarrow f(x + p) + f(x) = 2$ (1)

Replacing x by x + p, equation (1) reduces to

$f(x + 2p) + f(x + p) = 2$ (2)

Subtracing equation (1) from equation (2), we have

$f(x + 2p) - f(x) = 0$

$\Rightarrow f(x + 2p) = f(x)$

Hence, f is periodic, having a period 2p.

EXAMPLE 4.31 Consider $y = f(x)$, $x \in R$. It is given that the graph of f(x) is symmetrical about the lines x = a and x = b, where a < b. Prove that f(x) is periodic.

SOLUTION Since the function f(x) is symmetrical about lines x = a and x = b, therefore we have

$f(a - x) = f(a + x)$

$\Rightarrow f(x) = f(2a - x)$ [replacing x by a – x] (1)

and $f(b - x) = f(b + x)$

$\Rightarrow f(x) = f(2b - x)$ [replacing x by b – x] (2)

From equation (1), we have

$f(x) = f(2a - x)$

$= f\{2b - (2a - x)\}$

$= f\{x + 2(b - a)\}$

Hence, f(x) is periodic, having a period 2(b – a).

EXAMPLE 4.32 A function f(x) satisfies the equation $f(x+1) + f(x - 1) = \sqrt{3} f(x)$ for all real x. Prove that f is periodic.

SOLUTION $f(x+1) + f(x - 1) = \sqrt{3} f(x)$ (1)

Here, the main strategy is to replace x such that we get expression on RHS which is same as the expression on LHS. Replacing x by x +1 and replacing x by x – 1 separately in the given equation, we get two equations :

$f(x + 2) + f(x) = \sqrt{3} f(x + 1)$

$f(x) + f(x - 2) = \sqrt{3} f(x - 1)$

Adding these two equations, we get term on RHS, which is same as LHS:

$$f(x+2) + f(x-2) + 2f(x) = \sqrt{3}\{f(x+1) + f(x-1)\}$$

$= \sqrt{3}\{\sqrt{3}f(x)\}$ using(1)

$= 3f(x).$

$\Rightarrow f(x+2) + f(x-2) = f(x)$ (2)

Replacing x by x + 2 in (2), we have

$f(x+4) + f(x) = f(x+2)$ (3)

Note that we have not replaced x by x – 2, because that yields a relation which has argument form of x – 4. As definition of periodic function involves addition of a positive constant being added to independent variable, we opt to replace x by x + 2.

Now, adding (2) and (3),

$f(x+4) + f(x-2) = 0$ ...(4)

Replacing x by x + 2 in (4) so that one of the two terms becomes f(x), we have

$f(x+6) + f(x) = 0$

$\Rightarrow f(x+6) = -f(x)$ ...(5)

Replacing x by x + 6 in (5), we have

$f(x+12) = -f(x+6)$

$\Rightarrow f(x+12) = f(x)$ using (5)

Hence, the function is periodic and one of the periods is T = 12.

CONCEPT PROBLEMS [D]

1. Find the period, if any, of the following functions :

   (i) $f(x) = \left(\{x\} - \frac{1}{2}\right)^2$, {.} is FPF.

   (ii) $f(x) = \left(3x - \frac{\pi}{4}\right) 2\sin + 1$

   (iii) $f(x) = \left(\cos^{1/3}\left(4x - \frac{\pi}{3}\right)\right)^2 + 1$

   (iv) $f(x) = \sec(\text{cosec } x) + \sec(\sec x)$.

2. Prove that the function of the form $f(x) = \cos(ax + \sin(bx))$, where a and b are real nonzero numbers, is periodic if and only if the number a/b is rational.

3. Let $f(x) = \sin^2 x + \cos^4 x + 2$ and $g(x) = \cos(\cos x) + \cos(\sin x)$. Also let period of f (x) and g (x) be T_1 and T_2 respectively then show that $T_1 = T_2$.

4. Is the product and division of two non-periodic function is always non-periodic ?

5. If $f : R \to R$ is a function satisfying the property $f(x+1) + f(x+3) = 2\ \forall\ x \in R$ then find a period of f(x).

6. Let $||x|| = \min_{n \in I}|x - n|$. Prove that $f(x) = ||x||$ is periodic and find its period. Also, graph this function. Notice that this function measures the distance of a real number to its nearest integer.

7. Prove that the following function are not periodic:

   (i) $f(x) = x + x \sin x$
   (ii) $f(x) = \cos x^2$
   (iii) $f(x) = \cos\left(\sqrt[3]{|x|^2}\right)$
   (iv) $f(x) = \cos\sqrt{2}\,x + \cos 2x$

8. Discuss whether the function $f(x) = \sin(\cos x + x)$ is perodic or not , if yes then what is its period.

PRACTICE PROBLEMS [J]

9. Find the period, if any, of the following functions :

 (i) $f(x) = \sin x + \tan\frac{x}{2} + \sin\frac{x}{2^2} + \tan\frac{x}{2^3} + \ldots\ldots\ldots + \sin\frac{x}{2^{n-1}} + \tan\frac{x}{2^n}$

 (ii) $f(x) = \tan(\sqrt{2}x) + \cot(\sqrt{3}x)$

 (iii) $f(x) = \sin^3 x \sin 3x$

 (iv) $f(x) = \sin(\cos(\sin(\cos x))) + \tan^{-1}(\tan x)$

10. Find the period, if any, of the following functions :

 (i) $f(x) = \cos(\cos x) + \cos(\sin x)$ (ii) $f(x) = x + \cos(\pi x) - [x]$

 (iii) $f(x) = e^{\ln(\sin x)} + \tan^3 x - \operatorname{cosec}(3x - 5)$ (iv) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

11. Find the period, if any, of the following functions :

 (i) $f(x) = \cos 2\pi + \{2x\}$, $\{.\}$ is FPF. (ii) $f(x) = \dfrac{\sin x + \sin 7x}{\cos x + \cos 7x}$

 (iii) $f(x) = \dfrac{1}{2}\left(\dfrac{|\sin x|}{\cos x} + \dfrac{\sin x}{|\cos x|}\right)$ (iv) $f(x) = \sin(x + \sin x)$

12. Find the period, if any, of the following functions :

 (i) $f(x) = \left|\sin^3\frac{x}{2}\right| + \left|\cos^5\frac{x}{5}\right|$ (ii) $f(x) = \sin^3\frac{\pi x}{2} + 2\cos\frac{\pi|x|}{3} - \tan^2\frac{\pi x}{4}$

 (iii) $f(x) = \left|\cos^5\left(\frac{x}{2}\right)\right| + \sqrt{\sin^{10}\left(\frac{x}{2}\right)}$ (iv) $f(x) = \dfrac{|\sin x| + |\cos x|}{|\sin x - \cos x|}$.

13. Prove that the function of the form $f(x) = \cos(ax) + \tan(bx)$, where a and b are real nonzero numbers, is periodic if and only if the ratio a/b is a rational number.

14. Let $f(x) = 2\lambda|\cos x| + \lambda^2|\sin x| + 3$. If $\frac{\pi}{2}$ is the fundamental period of f(x), find λ.

15. Find the period of the functions :

 (i) $f(x) = \sin(\cos x) + \cos(\sin x)$ (ii) $f(x) = \sin\left(\cos\frac{x}{2}\right) + \cos(\sin x)$

16. If f(x) is of period 3 and g(x) is of period 2, then find a period of the function

$$F(x) = \begin{vmatrix} f(x) & f\left(\frac{x}{3}\right) \\ g(x) & g\left(\frac{x}{2}\right) \end{vmatrix}.$$

17. Let $f(x + 1) + f(x + 5) = f(x + 3) + f(x + 7)\ \forall\ x \in R$. Prove that f(x) is periodic.

18. Let $f(x) + f(x + 3) = 5$ for all $x \in R$. Prove that $f(x + 12) = f(x)$ for all $x \in R$.

19. If f is a function satisfying $f(x - 1) + f(x + 1) = \sqrt{2}\, f(x)$, prove that f(x) is periodic.

4.11 | INVERSE OF A FUNCTION

Inverse of a function

Inverse relation is like looking a relation in the opposite direction. For example, consider the relation "husband of". The inverse to this relation is "wife of". This is an explicit relation very easily conceivable. In other situations involving functions, inverse relations may not be so explicit. We shall, therefore, develop mathematical technique to obtain inverse function (relation) for a given function (relation).

We use the concept of pre-image and image to connect the elements of a function in the direction from domain A to co-domain B. The related elements are connected by a rule f(x) such that :

Image = f (x) = f(pre-image)

Clearly, x is the pre-image and f(x) is image. Now, we want to derive a similar rule, $f^{-1}(x)$, which evaluates to pre-image like:

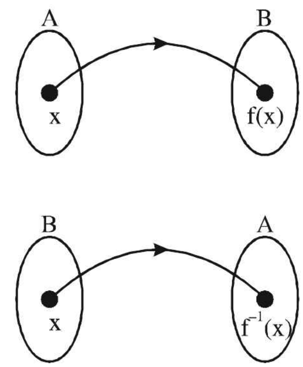
Pre-image(s) = $f^{-1}(x) = f^{-1}$(image)

Clearly, x is now the image and $f^{-1}(x)$ is pre-image.

Once the inverse rule is constructed, it is easy to define inverse function. However, we should be careful in one important aspect. An inverse function, f^{-1} is a function ultimately. This puts the requirement that every element of the domain of the new function f^{-1} should be related to exactly one element to its co-domain set.

We must also understand that this new function, f^{-1}, gives the perspective of relation from co-domain to domain of the given function f. However, new function f^{-1} is read from its new domain to its new co-domain. After all this is how a function is read. This simply means that domain and co-domain of the function f is exchanged for f^{-1}.

By definition, every element of domain set of the given function f is also related to exactly one element of in its co-domain. Thus, there is bidirectional requirement that elements of one set are related to exactly one element of other set.



The process of inversion can be applied to any function f having the property that for each y in the range of f, there is exactly one x in the domain of f such that f(x) = y. In particular, a function that is continuous and strictly monotonic on an interval [a, b] has this property.

In other words, we can define inverse function, f^{-1}, only if the given function is an injection and surjection function at the same time. Hence, the inverse of function f is defined as :

if f : A → B is a bijective function, then the function, f–1 : B → A which associates each element b ∈ B and a ∈ A such that f(a) = b is called the inverse of the function f.

Let A = {1, 2, 3, 4} and B = {3, 6, 9, 12}

f : A → B by f (x) = 3x for all x ∈ A

Then, the function set in the roaster form is :

f = {(1, 3) , (2, 6) , (3, 9) , (4, 12)}

This function is clearly a bijection as only distinct elements of two sets are paired. Its domain and co-domains are :

Domain of f = {1, 2, 3, 4}

Co-domain of f = {3, 6, 9, 12}

Now, the inverse function is given by :

f^{-1} : A → B by f(x) = for all x ∈ A

where A = {3, 6, 9, 12}

In the roaster form, the inverse function is :

f^{-1} = {(3, 1), (6, 2), (9, 3), (12, 4)}

Note that we can find inverse relation by merely exchanging positions of elements in the ordered pairs.

The domain and co-domain of new function f^{-1} are:

Domain of $f^{-1} = \{3, 6, 9, 12\}$

Co–domain of $f^{-1} = \{1, 2, 3, 4\}$

Thus, we see that the domain of inverse function f^{-1} is co-domain of the function f and co-domain of inverse function f^{-1} is domain of the function f.

Inverse function is unique function

Let $f : A \to B$ be a one-one and onto function, then their exists a unique function

$f^{-1} : B \to A$ such that

$f(x) = y \Leftrightarrow f^{-1}(y) = x, \forall x \in A, y \in B$.

This means that there is only one inverse function for a given function. The inverse of a bijective function is unique.

Method of finding inverse function

(i) Check whether the function is invertible i.e. bijective.

(ii) Solve the given function for x in terms of y.

(iii) Substitute x by the inverse symbol $f^{-1}(x)$ and y by x to find the rule of inverse function.

(iv) Change the domain and range of the given function with that of inverse function.

EXAMPLE 4.1 A function is given as

$f : R \to R$ by $f(x) = 2x + 5$ for all $x \in R$

Construct the inverse rule. Determine f(x) for first 5 natural numbers. Check validity of inverse rule with the values of images so obtained. Find inverse function, if it exists.

SOLUTION Now, in order to find inverse function, we need to determine that the given function is an injection and surjection.

For injection, let us assume that x_1 and x_2 be two different elements such that $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 + 5 = 2x_2 + 5$

$x_1 = x_2$

This means that given function is an injection. Now, to prove surjection, we solve the rule for x as:

$$x = \frac{y-5}{2}$$

We see that this equation is valid for all values of y i.e. all values in the co-domain of the given function. This means that every element of the co-domain is related. Hence, the given function is a surjection.

From, $y = 2x + 5$, we have

$$x = \frac{y-5}{2}$$

Changing notations,

$$f^{-1}(x) = \frac{x-5}{2}.$$

The images i.e. corresponding f(x), for first five natural numbers are : $f(1) = 2x + 5 = 7$; $f(2) = 9$; $f(3) = 11$; $f(4) = 13$ and $f(5) = 15$.

Now, the corresponding pre-images, using inverse rule for the two values of images are :

$$\Rightarrow f^{-1}(7) = \frac{7-5}{2} = 1$$

$$\Rightarrow f^{-1}(11) = \frac{11-5}{2} = 3.$$

Thus, we see that the inverse rule correctly determines the pre-images as intended.

The inverse function, therefore, is given as:

$$f^{-1} : R \to R \text{ by } f^{-1}(x) = \frac{(x-5)}{2}.$$

EXAMPLE 4.2 Let $f : D - \{0\} \to R - \{0\}$ be defined as

$f(x) = \ln\left(1+\frac{1}{x_1}\right)$. Find whether the function is invertible or not.

SOLUTION $f(x) = y = \ln\left(1+\frac{1}{x}\right)$

Let $f(x_1) = f(x_2)$

$$\Rightarrow \ln\left(1+\frac{1}{x_1}\right) = \ln\left(1+\frac{1}{x_2}\right)$$

$\Rightarrow x_1 = x_2$

Hence, f is one - one.

Now, $e^y = 1 + \frac{1}{x}$

$\Rightarrow \frac{1}{x} = e^y - 1$

$\Rightarrow x = \frac{1}{e^y - 1}, y \neq 0$

The range is R – {0}, which is same as co-domain. Hence, f is onto.
Finally, the function is bijective, and hence invertible.

EXAMPLE 4.3 Assume that the function $f: R - \{-1\} \to R - \{1\}, f(x) = \frac{x-1}{x+1}$ is a bijection. Determine its inverse.

SOLUTION Put $\frac{x-1}{x+1} = y$ and solve for x:

$\frac{x-1}{x+1} = y \Rightarrow x - 1 = yx + y \Rightarrow x - yx = 1 + y$

$\Rightarrow x(1 - y) = 1 + y$

$\Rightarrow x = \frac{1+y}{1-y}$

Now, exchange x and y :

$y = \frac{1+x}{1-x}$.

The desired inverse is

$f^{-1} : R - \{1\} \to R - \{-1\}$

$f^{-1}(x) = \frac{1+x}{1-x}$

EXAMPLE 4.4 Let $f: \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \to [0, 4]$ be a function defined as f(x) = sin x – cosx + 2. Then find f^{-1} (x).

SOLUTION $f(x) = \sqrt{3}\ \sin x - \cos x + 2$

$= 2 \sin\left(x - \frac{\pi}{6}\right) + 2$

Since f(x) is one-one and onto, f is invertible.

Now, $fof^{-1}(x) = x \Rightarrow 2 \sin\left(f^{-1}(x) - \frac{\pi}{6}\right) + 2 = x$

$\Rightarrow \sin\left(f^{-1}(x) - \frac{\pi}{6}\right) = \frac{x}{2} - 1$.

Note that we have ≤ 1 for all $x \in [0, 4]$.

$\Rightarrow f^{-1} : [0, 4] \to \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$,

$f^{-1}(x) = \sin^{-1}\left(\frac{x}{2} - 1\right) + \frac{\pi}{6}$.

EXAMPLE 4.5 Given f : [0, a] → S, such that $f(x) = 3 \cos\frac{x}{2}$. Find the largest value of 'a', for which f has an inverse function f^{-1}. Find f^{-1}. State the domain and the range of f and f^{-1}.

SOLUTION From the graph of $y = 3 \cos\frac{x}{2}$, we notice that the function is one-one in the largest interval of [0, 2π], and the range in that case is [– 3, 3].

$x = f^{-1}(y)$ and $f^{-1} : [-3, 3] \to [0, 2\pi]$

$\therefore \cos\frac{x}{2} = \frac{y}{3}$

$\Rightarrow \frac{x}{2} = \cos^{-1}\frac{y}{3}$

$\Rightarrow x = 2\cos^{-1}\frac{y}{3}$

$\Rightarrow f^{-1}(y) = 2\cos^{-1}\frac{y}{3}$

$\Rightarrow f^{-1}(x) = 2\cos^{-1}\frac{x}{3}$.

EXAMPLE 4.6 Find the inverse of the function $f(x) = \ln(x^2 + 3x + 1), x \in [1, 3]$, assuming it to be an onto function.

SOLUTION Given $f(x) = \ln(x^2 + 3x + 1)$

$\therefore f'(x) = \frac{2x+3}{(x^2+3x+1)} > 0\ \forall\ x \in [1, 3]$

which is a strictly increasing function. Thus f(x) is injective, and it is given that f(x) is onto. Hence the given function f(x) is invertible.

The range of the function is [ln5, ln19].

Now let $y = f(x) = \ln(x^2 + 3x + 1)$

then $x = f^{-1}(y)$ (1)

$y = \ln(x^2 + 3x + 1)$

$\Rightarrow e^y = x^2 + 3x + 1$

$\Rightarrow x^2 + 3x + 1 - e^y = 0$

$$\therefore\ x = \frac{-3 \pm \sqrt{9 - 4.1.(1 - e^y)}}{2} = \frac{-3 \pm \sqrt{(5 + 4e^y)}}{2}$$

$$= \frac{-3 + \sqrt{(5 + 4e^y)}}{2} \quad \{\text{since } x \in [1, 3]\} \qquad (2)$$

From (1) and (2), we get

$$f^{-1}(y) = \frac{-3 + \sqrt{(5 + 4e^y)}}{2}$$

Replacing y with x, we get

$$f^{-1} : [\ln 5, \ln 19] \to [1, 3]$$

$$f^{-1}(x) = \frac{-3 + \sqrt{(5 + 4e^x)}}{2}.$$

EXAMPLE 4.7 Is the function

$f : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where $f(x) = \sin^{-1}\left(2x\sqrt{1 - x^2}\right)$ invertible?

SOLUTION f is continuous

$$f'(x) = \frac{2(1 - 2x^2)}{|1 - 2x^2|\sqrt{1 - x^2}}$$

$$= \begin{cases} \dfrac{2}{\sqrt{1 - x^2}} & \text{if } \dfrac{-1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}} \\ \dfrac{-2}{\sqrt{1 - x^2}} & \text{if } x < \dfrac{-1}{\sqrt{2}} \text{ or } x > \dfrac{1}{\sqrt{2}} \end{cases}$$

$\therefore$ f(x) is increasing in $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and is decreasing in each of the intervals $\left(-1, \frac{-1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, 1\right)$.

$\therefore$ f(x) is not one-one, so it is not invertible.

EXAMPLE 4.8 A function f: $[1, \infty) \to [1, \infty)$ is given by $f(x) = 2^{x(x-1)}$. Find $f^{-1}(x)$.

SOLUTION To check for one-one function, we determine the derivative of the function as :

$$f(x) = 2^{x(x-1)}$$

Taking logarithm on both sides,

$$\log_e(f(x)) = x(x - 1)\log_e 2$$

Taking derivative on either side of the equation, we have :

$$\frac{f'(x)}{f(x)} = (2x - 1)\log_e 2$$

$$\Rightarrow f'(x) = f(x)(2x - 1)\log_e 2$$

f(x) is a positive number. Also, $\log_e 2$ is a positive number.

$\therefore$ $f'(x)$ = positive number $\times$ $(2x - 1)$.

Now, x lies in the interval $[1, \infty)$. The derivative is positive in the interval. We, therefore, conclude that the function is increasing in the interval given by $[1, \infty)$. The least value of the function is given as :

$$f(1) = 2^{x(x-1)} = 2^0 = 1.$$

We interpret the function, when x tends to positive infinity.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 2^{x(x-1)} = 2^\infty = \infty.$$

Thus, range of the given function is $[1, \infty)$. Clearly, Range = Co-domain = $[1, \infty)$.

Therefore, the function is onto and hence bijection. It means that function is invertible.

Now, we solve the function for x for finding the inverse function. Taking log on the base of 2 on either side of the equation,

$$\log_2 y = \log_2 x(x - 1) = x(x - 1) = x^2 - x$$

$$\Rightarrow x^2 - x - \log_2 y = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{(1 + 4\log_2 y)}}{2}.$$

But, we know that domain of the function is $x \geq 1$. Also, $\sqrt{(1 + 4\log_2 y)} \geq 1$. It means that only positive sign in the expression is valid.

$$\Rightarrow x = \frac{1 + \sqrt{(1 + 4\log_2 y)}}{2}$$

Substituting x by $f^{-1}(x)$ and y by x, we have the rule of inverse function as :

$$f^{-1}(x) = \frac{1 + \sqrt{(1 + 4\log_2 x)}}{2}.$$

EXAMPLE 4.9 Find the inverse of the function

$$f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$$

SOLUTION Given $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

Let $f(x) = y \quad \Rightarrow \quad x = f^{-1}(y)$ (1)

$$\therefore \; x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le \sqrt{y} \le 4 \\ y^2/64, & y^2/64 > 4 \end{cases}$$

$$= \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y > 16 \end{cases}$$

$$f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y > 16 \end{cases}$$

Hence $f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ x^2/64, & x > 16 \end{cases}$.

EXAMPLE 4.10 Let $g : R \to R$ be given by $g(x) = 3 + 4x$. If $g^n(x) = gogo....og(x)$, show that $g^n(x) = (4^n - 1) + 4^n x$. If $g^{-n}(x)$ denotes the inverse of $g^n(x)$, find $g^{-n}(x)$.

SOLUTION Since $g(x) = 3 + 4x$

$\therefore \; g^2(x) = (gog)(x) = g\{g(x)\} = g(3 + 4x)$

$= 3 + 4(3 + 4x)$

or $g^2(x) = 15 + 4^2x = (4^2 - 1) + 4^2x$

Now $g^3(x) = (gogog)x = g\{g^2(x)\} = g(15 + 4^2x)$

$= 3 + 4(15 + 4^2x) = 63 + 4^3x = (4^3 - 1) + 4^3x$

Similarly we get $g^n(x) = (4^n - 1) + 4^n x$

Now leg $g^n(x) = y \quad \Rightarrow \quad x = g–n(y)$ (1)

$\therefore \; y = (4^n - 1) + 4^n x$

or $x = (y + 1 - 4^n)4^{-n}$ (2)

From (1) and (2) we get $g^{-n}(y) = (y + 1 - 4^n)\,4^{-n}$

Hence $g^{-n}(x) = (x + 1 - 4^n)\,4^{-n}$.

Inverse as composition of functions

The notion of an inverse function also can be expressed in terms of composition of functions. Suppose two functions f and g have the following composition behaviour $g(f(x)) = x$.

We see that x is not only the initial input but the final output. The input x is transformed into output f(x) by f. The output f (x) now becomes input for g .The function g transforms f(x) back into the original input x.

Definition. Let f be any function and suppose there is a function g with the property that $g(f(x)) = x_1$. we then call g the inverse function of f and give it the notation f^{-1} . Hence $g = f^{-1}$ by definition, and $f^{-1}(f(x)) = x$.

EXAMPLE 4.11 Let $g(x) = \dfrac{3x+1}{x-2}$ and $f(x) = \dfrac{2x+1}{x-3}$.

(i) Show $g(f(4)) = 4$. (ii) Show $g = f^{-1}$

(iii) Show $f(f^{-1}(x)) = x$

SOLUTION

(i) $g(f(4)) = g\left(\dfrac{2(4)+1}{4-3}\right) = g(9) = \dfrac{3(9)+1}{9-2} = 4$

(ii) $g(f(x)) = g\left(\dfrac{2x+1}{x-3}\right) = \dfrac{3\left(\dfrac{2x+1}{x-3}\right)+1}{\left(\dfrac{2x+1}{x-3}\right)-2} = \dfrac{7x}{7} = x$

$\Rightarrow g = f^{-1}$

(iii) $f(f^{-1}(x)) = f\left(\dfrac{3x+1}{x-2}\right) = \dfrac{2\left(\dfrac{3x+1}{x-2}\right)+1}{\left(\dfrac{3x+1}{x-2}\right)-3} = \dfrac{7x}{7}$

$= x.$

Note

(ii) and (iii) together imply $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

EXAMPLE 4.12 A function f(x) is given as :

$f(x) = \{a - x\}^{1/n}$, where $a > 0$, $x > 0$ and n is a positive integer. Find f{f(x)}. Hence show that f is inverse of itself.

SOLUTION The domain of the given function is $x > 0$. In order to find, the composition, we evaluate f(y), where y = f(x).

$\Rightarrow f\{f(x)\} = f(y) = (a - y^n)^{1/n} = [a - \{(a - x^n)^{1/n}\}^n]^{1/n}$

$\Rightarrow f\{f(x)\} = (a - a - x^n)^{1/n} = (x^n)^{1/n} = x;\ x > 0$

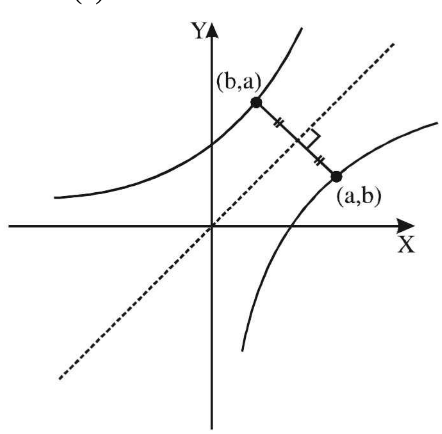
Here, composition is that of function with itself. As such, domain of composition is equal to intersection of domain of the given function with itself. But, the intersection of an interval with itself is same interval. Hence, we have retained the domain interval of the composition same as that of given function. Since f(f(x)) = x, f is the inverse of itself.

Graph of inverse function

In order to investigate the nature of the inverse graph, let us consider a plot of an invertible function, f(x). Let (a,b) be a point on the plot. Then, by definition of an inverse function, the point (b,a) is a point on the plot of inverse function, if plotted on the same coordinate system.

$y = f(x)$

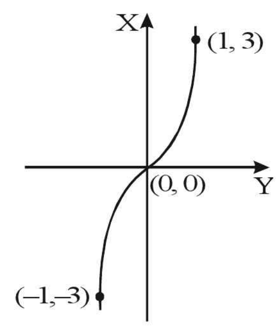
$y = f^{-1}(x)$



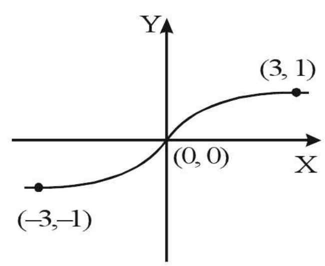
By geometry, the line joining points (a,b) and (b,a) is bisected at right angles by the line y = x. It means that the two points under consideration are object and image for the mirror defined by y = x.

Thus, the graphs of y = f(x) and $y = f^{-1}(x)$ are the mirror images of each other in the line y = x.

Consider the graph of $f(x) = x^5 + x^3 + x$.



Observe the graph of $y = f^{-1}(x)$ shown below which is obtained by reflecting the graph of $y = f^{-1}(x)$ about the line y = x.

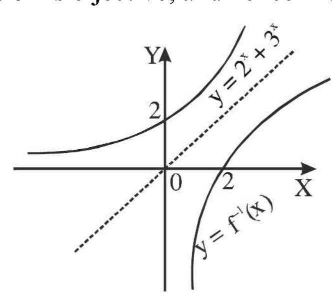


Note

Consider the function $f : R \to R+$, where

$f(x) = 2^x + 3^x$.

The function is bijective, and hence invertible.



However, the inverse function cannot be expressed explicitly. We can write:

$f^{-1} : R^+ \to R$

$y = f^{-1}(x)$, where $x = 2^y + 3^y$.

Note that we can draw the graph of the inverse function.

To evaluate $f^{-1}(x)$ we can sometimes use trial and error. For example, to find $f^{-1}(5)$, we guess that $y = 1$ satisfies the equation $5 = 2^y + 3^y$.

Hence, $f^{-1}(5) = 1$.

EXAMPLE 4.13 The lines L and K are symmetric to each other with respect to the line $y = x$. If the equation of the line L is $y = ax + b$ where a and b are non zero, then find the equation of K.

SOLUTION What is being asked for, is the inverse of the function $f(x) = ax + b$.

Hence if $y = ax + b$, we solve for x in terms of y : namely, $x = \dfrac{y-b}{a}$.

Hence $f^{-1}(y) = \dfrac{y-b}{a} \Rightarrow f^{-1}(x) = \dfrac{x-b}{a}$.

Hence, the equation of K is $y = \dfrac{x}{a} - \dfrac{b}{a}$.

Inverses of non-monotonic functions

Suppose we try to apply the process of inversion to a function that is not monotonic on [a, b]. For example, suppose that $f(x) = x^2$, $x \in (-\infty, \infty)$.

We can solve equation $y = x^2$ for x in terms of y, but there are two values of x corresponding to each y in $[0, \infty)$, namely,

$x = \sqrt{y}$ and $x = -\sqrt{y}$

In a case like this we say that the process of inversion gives rise to two new functions, say g_1 and g_2, where

$g_1(y) = \sqrt{y}$ and

$g_2(y) = -\sqrt{y}$ for each y in $[0, \infty)$.

To fit this in with the notion of inverse as explained above, we can look upon the equation $y = x^2$ as defining not one function f but two functions f_1 and f_2, say, where

$f_1(x) = x^2$ if $0 \le x < \infty$ and

$f_2(x) = x^2$ if $-\infty < x \le 0$

These may be considered as distinct functions because they have different domains. Each function is monotonic in its domain and each has an inverse, the inverse of f_1 being g_1 and the inverse of f_2 being g_2, where g_1 and g_2 are given above.

This illustrates how the process of inversion can be applied to non-monotonic functions in piecewise form. We simply consider such a function as a union of several monotonic functions and invert each monotonic piece.

EXAMPLE 4.14 A quadratic function is given as:

$f(x) = x^2 + x + 1$.

Is the function invertible? If not, then find the intervals in which it is invertible. Also find the corresponding inverse functions.

SOLUTION The given function is a continuous function valid for all values of x.

We need to analyze whether function is bijective.

Let x_1 and x_2 be two values. Then,

$$x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$$

$$\Rightarrow x_1^2 = x_2^2 \quad \Rightarrow \quad x_1 = \pm x_2$$

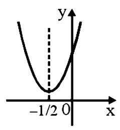
It means that the function is not one-one, but many one function. For determining whether function is onto function, we investigate the nature of its derivative,

$f'(x) = 2x + 1$

Its root , when equated to zero, is $-1/2$. The function value at $x = -1/2$ corresponds to the least value of the function.

$$\Rightarrow f\left(-\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}.$$

The function is strictly decreasing in the interval $(-\infty, -1/2]$ and strictly increasing in the interval $[-1/2, \infty)$. Further, we also observe from the graph as shown in the figure below that function is one-one in the individual intervals.



Now, for determining the inverse functions, we solve the function for x.

$y = f(x) = x^2 + x + 1$

$\Rightarrow x^2 + x + 1 - y = 0$

Solving for x, we have :

$$x = \frac{-1 \pm \sqrt{(4y-3)}}{2}$$

Substituting x by f–1(x) and y by x, we have the rule of inverse function as :

For the interval $\left(-\infty, -\frac{1}{2}\right]$,

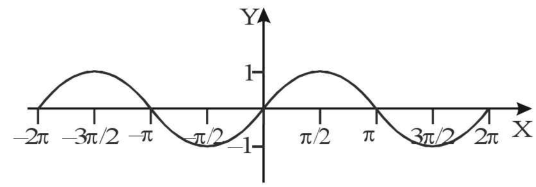
$$f^{-1}(x) = \frac{-1 - \sqrt{(4x-3)}}{2}$$

For the interval $\left[-\frac{1}{2}, \infty\right)$,

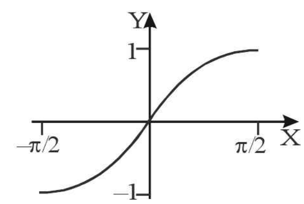
$$f^{-1}(x) = \frac{-1 + \sqrt{(4x-3)}}{2}.$$

Trigonometric functions frequently arise in problems, and often it is necessary to invert these functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn't actually have an inverse that reliably "undoes" the sine function. If you know that sin x = 1/2, you can't reverse this to discover x, that is, you can't solve for x, as there are infinitely many angles with sine 1/2. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we "discard" all other angles, the resulting function does have a proper inverse. The sine takes on all values between –1 and 1 exactly once on the interval [–π/2, π/2]. If we truncate the sine, keeping only the interval [–π/2, π/2], as shown in the figure below, then this truncated sine has an inverse function. We call this the inverse sine or the arcsine, and write y = arcsin(x) = $\sin^{-1}x$.

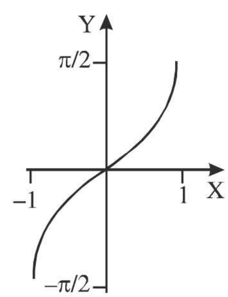
Graph of y = sin x



Graph of y = sin x, x ∈ [–π/2, π/2]



Graph of y = $\sin^{-1}$ x, x ∈ [–1, 1]

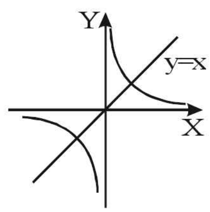


Recall that a function and its inverse undo each other in either order, for example, $(\sqrt[3]{x})^3 = x$ and $\sqrt[3]{x^3} = x$. This does not work with the sine and the "inverse sine" because the inverse sine is the inverse of the truncated sine function, not the real sine function. It is true that sin(arcsin(x)) = x, that is, the sine undoes the arcsine. It is not true that the arcsine undoes the sine, for example, sin(5π/6) = 1/2 and arcsin(1/2) = π/6, so doing first the sine then the arcsine does not get us back where we started. This is because 5π/6 is not in the domain of the truncated sine. If we start with an angle between –π/2 and π/2 then the arcsine does reverse the sine: sin(π/6) = 1/2 and arcsin(1/2) = π/6.

Function inverse of itself

A function is identical with its inverse, i.e. $f(x) \equiv f^{-1}(x)$ if $fof\,(x) \equiv x$. In such a case, the graph of f^{-1} matches with the graph of f. For this to happen, the graph of f must be symmetrical w.r.t. the line y = x.

For example, if $f(x) = \frac{1}{x}$, we also have $f^{-1}(x) = \frac{1}{x}$ and the graph of $y = f(x) = \frac{1}{x}$ is symmetrical w.r.t. the line y = x.



A function f is said to be an involution if for all x for which f(x) and f(f(x)) are defined we have f(f(x)) = x.

For example, $f(x) = \sqrt[3]{1-x^3}$ is an involution.

EXAMPLE 4.15 If the function f (x) = ax + b is inverse of itself then find the ordered pair (a, b).

SOLUTION $y = f(x) \Rightarrow x = f^{-1}(x)$

Now y = ax + b

$$x = \frac{y}{a} - \frac{b}{a} \quad \Rightarrow f^{-1}(y) = \frac{y}{a} - \frac{b}{a}$$

$$f^{-1}(x) = \frac{x}{a} - \frac{b}{a} \qquad (1)$$

and f (x) = ax + b (2)

Now in order that (1) and (2) coincide

$$a = \frac{1}{a} \qquad (1)$$

$$\frac{b}{a} = -b \qquad (2)$$

from (1), $a^2 = 1 \;\Rightarrow\; a = 1$ or -1

if $a = -1$, then $b = b \;\Rightarrow\; b \in R$

if $a = 1$, then $2b = 0 \;\Rightarrow\; b = 0$

Hence, the ordered pairs (a, b) are $(-1, b)$, $(1, 0)$ where $b \in R$.

EXAMPLE 4.16 Find the value of a so that $f(x) = \frac{ax+1}{x+3}$ is identical with $f^{-1}(x)$.

SOLUTION We must have $fof \equiv x$

$$\Rightarrow \frac{a\left(\frac{ax+1}{x+3}\right)+1}{\frac{ax+1}{x+3}+3} \equiv x$$

$\Rightarrow a^2x + a + x + 3 \equiv x(ax + 1 + 3x + a)$

$\Rightarrow (a^2 + 1)x + a + 3 = (a + 3)x^2 + 10x$

$\Rightarrow a + 3 = 0$ and $a^2 + 1 = 10$

$\Rightarrow a = -3$ and $a = \pm 3$

$\Rightarrow a = -3$.

Note

In general, the function $f(x) = \frac{ax+b}{cx+d}$ is identical with its inverse if a + d = 0.

Equation $f(x) = f^{-1}(x)$

EXAMPLE 4.17 Find the solution of the equation:

$$x^2 - x + 1 = \frac{1}{2} + \sqrt{\left(x - \frac{3}{4}\right)}$$

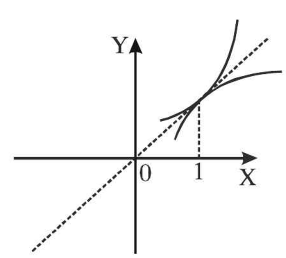
SOLUTION Consider the function

$$f : \left[\frac{1}{2}, \infty\right) \to \left[\frac{3}{4}, \infty\right) \text{ where } f(x) = x^2 - x + 1.$$

We note that f is one-one and onto.

Its inverse is $f^{-1} : \left[\frac{3}{4}, \infty\right) \to \left[\frac{1}{2}, \infty\right)$,

$$f^{-1}(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}.$$



Hence the two sides of the given equation are inverse of each other. We draw the graphs of the given functions and observe that they intersect on line y = x only.

Clearly, the intersection point is the solution of the equation,

$f(x) = x$

$\Rightarrow x^2 - x + 1 = x \qquad \Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0 \qquad \Rightarrow x = 1.$

This is the answer. It is interesting to know that we can also proceed to find the solution by working on the inverse function. This should also give the same result as given functions are inverse to each other.

$f^{-1}(x) = x$

$$\Rightarrow \frac{1}{2} + \sqrt{\left(x - \frac{3}{4}\right)} = x$$

$$\Rightarrow \sqrt{\left(x - \frac{3}{4}\right)} = x - \frac{1}{2}$$

Squaring both sides,

$$\Rightarrow \left(x - \frac{3}{4}\right) = \left(x - \frac{1}{2}\right)^2 = x^2 + \frac{1}{4} - x$$

$$\Rightarrow x^2 + \frac{1}{4} - x - x + \frac{3}{4} = 0$$

$\Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow x = 1.$

Note

Some points of intersection of f and f–1 lie on the line y = x. However, it must be noted that f(x) and f–1(x) may intersect elsewhere also.

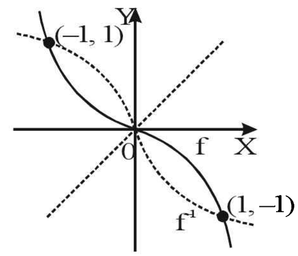
EXAMPLE 4.18 If $f(x) = -x^3$, solve the equation $f(x) = f^{-1}(x)$.

SOLUTION $f(x) = f^{-1}(x) \quad \Rightarrow -x^3 = -x^{1/3}$

$x^9 = x \quad \Rightarrow x = 0, 1, -1.$

This can be observed in the figure given below.

The graphs of y = f(x) and y = f–1(x) intersect at three points (0, 0), (–1, 1) and (1, –1).



The point (0, 0) lies on the line y = x, while the other two points (–1, 1) and (1, –1) lie at equal distances from the line y = x in a direction perpendicular to the line y = x.

Solving f(x) = x, we have

$-x^3 = x \qquad \Rightarrow x^3 + x = 0$

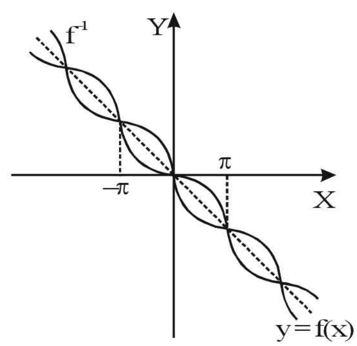
$\Rightarrow x(x^2 + 1) = 0 \Rightarrow x = 0.$

Here, we see that solving f(x) = x is not sufficient for finding all the solutions of the equation

$f(x) = f^{-1}(x).$

EXAMPLE 4.19 If f(x) = –x + sin x, solve the equation $f(x) = f^{-1}(x)$.

SOLUTION To solve the equation $f(x) = f^{-1}(x)$, we first draw the graphs of both the functions to observe their points of intersection.



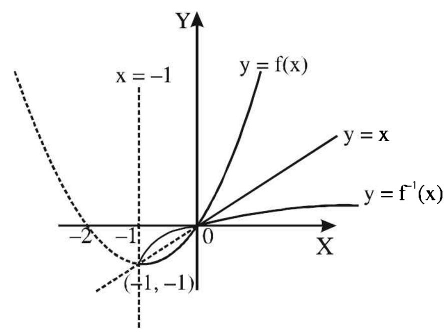
From, the graph, we realize that only the point (0, 0) lies on the line y = x. There are other points of intersection (nπ, nπ), n ∈ I which lie on the line y = – x.

Hence, the solution set is {nπ, n ∈ I}.

EXAMPLE 4.20 Let $f(x) = x^2 + 2x;\ x \geq -1.$

Draw graph of $f^{-1}(x)$ also find the number of solutions of the equation, $f(x) = f^{-1}(x)$.

SOLUTION We draw the graphs of the given function and its inverse, and observe that they intersect on line $y = x$ only.



So, $f(x) = f^{-1}(x)$ is equivalent to solving $f(x) = x$

$\Rightarrow x^2 + 2x = x$

$\Rightarrow x(x + 1) = 0$

$\Rightarrow x = 0, -1.$

Hence, $x = 0, -1$ are solutions of the equation $f(x) = f–1(x)$.

Properties of inverse function

There are few properties of inverse function that result from the fact that it is inverse of a bijection. We can check the validity of these properties in terms of the example given earlier.

1. If f is an invertible function then $(f–1)^{-1} = f$.
2. If $f : A \to B$ be a one-one and onto mapping, then $f^{-1} \text{ o } f = I_A$ and $f \text{ o } f^{-1} = I_B$,

where I_A and I_B are the identity mappings of the sets A and B respectively.

Proof : We shall first prove that $f^{-1} \text{ o } f = I_A$.

Since $f : A \to B$ and $f^{-1} : B \to A$, therefore $(f^{-1} \text{ o } f) : A \to A$.

Let x be any arbitrary element $\in$ A and let $f(x) = y$, where $y \in B$. Then by the definition of the mapping f^{-1}, $f^{-1}(y) = x$.

We have $(f^{-1} \text{ o } f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$. Thus the function $f^{-1} \text{ o } f$ maps every element $x \in A$ onto itself. Therefore $f^{-1} \text{ o } f = I_A$.

Now we shall prove that $f \text{ o } f^{-1} = I_B$. Since $f^{-1} : B \to A$ and $f : A \to B$, therefore $(f \text{ o } f^{-1}): B \to B$.

Let y be any arbitrary element $\in$ B and let $f^{-1}(y) = x$, where $x \in A$. Then $f(x) = y$.

We have $(f \text{ o } f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$. Thus the function $f \text{ o } f^{-1}$ maps every element $y \in B$ onto itself. Therefore $f \text{ o } f^{-1} = I_B$.

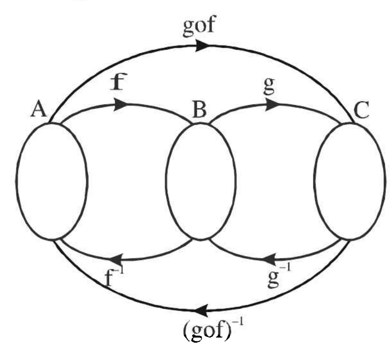
EXAMPLE 4.21 Let $f : [-1, 1] \to R$,

$f(x) = 3 + x + \tan\frac{\pi}{2} x$. Find $\left(f^{-1}\left(\frac{9}{2}\right)\right)$.

SOLUTION Since f is invertible and 9/2 lies in the range of the function, we have $\left(f^{-1}\left(\frac{9}{2}\right)\right) = \frac{9}{2}$.

Note that $f^{-1}(f(2)) = 2$ is not true since 2 does not lie in the domain of the function.

3. Let $f : A \to B$, $g : B \to C$ be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$. This is called reversal law.



Proof : To show that gof is invertible with $(gof)^{-1} = f^{-1}og^{-1}$, it is enough to show that $(f^{-1}og^{-1})o(gof) = I_A$ and $(gof)o(f^{-1}og^{-1}) = I_C$.

Now $(f^{-1}og^{-1})o(gof) = ((f^{-1}og^{-1}) \text{ og}) \text{ of}$,

$= (f^{-1}o(g^{-1}og)) \text{ of}$

$= (f^{-1}oI_B) \text{ of}$, by definition of g^{-1}

$= I_A$

Similarly, it can be shown that $(gof)o(f^{-1}og^{-1}) = I_B$.

CONCEPT PROBLEMS [E]

1. Assume that $f : R - \{1\} \to R - \{1\}$ is a bijection, where $f(x) = \sqrt[3]{\frac{x+2}{x-1}}$. Find f^{-1}.
2. Let f and g be invertible functions satisfying $f(1) = 2$, $f(2) = 3$, $f(3) = 1$
   $g(1) = -1$, $g(2) = 3$, $g(4) = -2$, find $(f \circ g)^{-1}(1)$.
3. Prove that the inverse of the fractional function ,
   $f : R - \left[-\frac{d}{c}\right] \to R - \left[\frac{a}{c}\right], f(x) = \frac{ax+b}{cx+d}, (ad - bc \neq 0)$,
   is also a linear fractional function. Under what conditions $f(x)$ is identical with its inverse.
4. A function $f : [2, \infty) \to [5, \infty)$ is defined as $f(x) = x^2 - 4x + 9$. Find its inverse.
5. Find all the real solutions to the equation $x^2 - \frac{1}{4} = \sqrt{x + \frac{1}{4}}$
6. Given $g : R \to R$, $g(x) = 2x + 8$ and $f : R - \{-2\} \to R - \{0\}$, $f(x) = \frac{1}{x+2}$ find $(g \circ f^{-1})(-2)$.
7. Prove that if $ab \neq -4$ and $f : R - \{2/b\} \to R - \{2/b\}$, $f(x) = \frac{2x+a}{bx-2}$ then $f = f^{-1}$.

PRACTICE PROBLEMS [K]

8. For what values of m is $f(x) = (m + 2)x^3 - 3\,mx^2 + 9mx - 1$ invertible ?
9. Let $f : [0, \infty) \to [0, \infty)$ be given by $f(x) = \sqrt{x + \sqrt{x}}$. Show that f is bijective and that its inverse is $f^{-1} : [0, \infty) \to f^{-1}(x) = \frac{1 - \sqrt{1+4x^2}}{2} + x^2$.
10. Determine f–1(x), if the given function is invertible
    (i) $f : (-\infty, -1)\ (-\infty, -2)$, $f(x) = -(x + 1)^2 - 2$ (ii) $f : \left[\frac{\pi}{6}, \frac{7\pi}{6}\right]\ [-1, 1]$, $f(x) = \sin\left(x + \frac{\pi}{3}\right)$.
11. Determine the values of m for which the function $f(x) = (m^2 - 4)x + |m|$, $x \in (-\infty, \infty)$ has an inverse. Find the inverse function.
12. Let $f : N \to N$ defined by $f(x) = x + (-1)^{x+1}$. Then prove that the function is inverse of itself.
13. If $f(x) = \begin{cases} x^3 & , \quad x \le 0 \\ x^2 & , \quad 0 \le x < 1 \\ 2x - 1 & , \quad 1 \le x \le 2 \\ \frac{x^3}{9} & , \quad x > 3 \end{cases}$ then show that
    (i) $f^{-1}(-1) = -1$ (ii) $f^{-1}(-8) = -2$ (iii) $f^{-1}(1) = 1$.

14. Let $f : [1/2, 1] \to [-1, 1]$ is given by $f(x) = 4x^3 - 3x$, then show that $f^{-1}(x) = \cos(1/3 \cos^{-1} x)$.

15. The function $f(x) = \dfrac{3x-2}{x+4}$ has an inverse that can be written in the form $f^{-1}(x) = \dfrac{x+b}{cx+d}$. Find the value of $(b + c + d)$.

16. Find the inverse of the function $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} + 2$.

17. If the functions f and g are defined from R to R such that $f(x) = e^x$, $g(x) = 3x - 2$, then find functions fog and gof. Also find the domain of functions $(fog)^{-1}$ and $(gof)^{-1}$.

18. If $f(x) = (ax^2 + b)^3$, then find a function g such that $f(g(x)) = g(f(x))$.

19. Let $f(x) = \sqrt[3]{a - x^3 + 3x^2 - 3bx + b^3} + b$. Find b if f(x) is inverse of itself.

20. A function $f : \left[\frac{3}{2}, \infty\right) \to \left[\frac{7}{4}, \infty\right)$ is defined as $f(x) = x^2 - 3x + 4$. Then compute $f^{-1}(x)$ and find the solution of the equation, $f(x) = f^{-1}(x)$.

4.12 | FUNCTIONAL EQUATION

Definition. Functional equation is an equation where the unknown is a function. A functional equation asks for a formula satisfying certain features. On solving such an equation we obtain one or more functions as solutions.

EXAMPLE 4.1 Let $f : R - \{0\} \to R$ be any function such that $f(x) + 2f\left(\frac{1}{x}\right) = 3x$. Find the sum of the values of x for which $f(x) = 1$.

SOLUTION $f(x) + 2f\left(\frac{1}{x}\right) = 3x \quad (1)$

Replacing x with $\left(\frac{1}{x}\right)$,

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x} \quad (2)$$

Multiplying (2) by 2, $2f\left(\frac{1}{x}\right) + 4f(x) = \frac{6}{x} \quad (3)$

Substracting (3) – (1)

$$3f(x) = \frac{6}{x} - 3x$$

$$\Rightarrow f(x) = \frac{2}{x} - x$$

Solving, $f(x) = 1$, we get

$$2 - x^2 = x$$

$$\Rightarrow x^2 + x - 2 = 0$$

$$\Rightarrow x_1 + x_2 = -1.$$

Thus the sum of values of x for which $f(x) = 1$ is -1.

EXAMPLE 4.2 Let $f(\sin x) + 2f(\cos x) = 3 \ \forall x \in$. Find f(x).

SOLUTION $f(\sin x) + 2f(\cos x) = 3 \quad (1)$

Replace x by $\frac{\pi}{2} - x$.

$$f(\cos x) + 2f(\sin x) = 3 \quad (2)$$

Putting the value of f(cos x) from (1) in (2)

$$\frac{3 - f(\sin x)}{2} + 2f(\sin x) = 3$$

$$\Rightarrow 3 + 3f(\sin x) = 6$$

$\Rightarrow$ f(sinx) = 1.

Let u = sinx. Then $0 < u < 1$ for $0 < x < \frac{\pi}{2}$.

Hence, f(u) = 1, $u \in (0, 1)$
i.e. f(x) = 1, $x \in (0, 1)$.

EXAMPLE 4.3 Find all functions $f : R \to R$ such that

$$x^2 f(x) + f(1 - x) = 2x - x^4$$

SOLUTION From the given equation,

$$f(1 - x) = 2x - x^4 - x^2 f(x)$$

Replacing x by 1 – x, we obtain

$$(1 - x)^2 f(1 - x) + f(x) = 2(1 - x) - (1 - x)^4$$

This implies that

$$f(x) = 2(1 - x) - (1 - x)^4 - (1 - x)2\, f(1 - x)$$
$$= 2(1 - x) - (1 - x)^4 - (1 - x)^2(2x - x^4 - x^2 f(x))$$

which in turn, gives

$$f(x) = 2(1-x) - (1 - x)^4 - 2x(1 - x)^2 + x^4(1 - x)^2x^2 f(x)$$

Solving now for f(x), we get

$$f(x) = \frac{2(1-x)-(1-x)^4-2x(1-x)^2+x^4(1-x)^2}{1-(1-x)^2x^2}$$

$$= \frac{(1-x)(2-(1-x)^3-2x(1-x)+x^4(1-x))}{(1-(1-x)x)(1+(1-x)x)}$$

$$= \frac{(1-x)(2-(1-3x+3x^2-x^3)-2x+2x^2+x^4-x^5)}{(1-x+x^2)(1+x-x^2)}$$

$$= 1 - x^2$$

We now check. If $f(x) = 1 - x^2$ then

$$x^2 f(x) + f(1 - x) = x^2(1 - x^2) + 1 - (1 - x)^2$$
$$= x^2 - x^4 + 1 - 1 + 2x - x^2 = 2x - x^4.$$

Hence, $f(x) = 1 - x^2$ is the only solution.

EXAMPLE 4.4 Determine all functions f satisfying the functional relation,

$$f(x) + f = \left(\frac{1}{1-x}\right) \text{ where } \frac{2(1-2x)}{x(1-x)} \quad x \in R - \{0, 1\}.$$

SOLUTION Given

$$f(x) + f = \left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)} = \frac{2}{x} - \frac{2}{1-x} \quad (1)$$

Replacing x by $\frac{1}{1-x}$ we obtain

$$f\left(\frac{1}{1-x}\right) + f\left(\frac{1}{1-\frac{1}{1-x}}\right) = 2(1-x) - \frac{2}{1-\frac{1}{1-x}}$$

$$\Rightarrow f\left(\frac{1}{1-x}\right) + f\left(1-\frac{1}{x}\right) = -2x + \frac{2}{x} \quad (2)$$

Again replacing x by $\left(1-\frac{1}{x}\right)$ in (1) we obtain

$$f\left(1-\frac{1}{x}\right) + f\left(\frac{1}{1-\left(1-\frac{1}{x}\right)}\right) = \frac{2}{1-\frac{1}{x}} - \frac{2}{1-\left(1-\frac{1}{x}\right)}$$

$$\Rightarrow f\left(1-\frac{1}{x}\right) + f(x) = \frac{2x}{x-1} - 2x \quad (3)$$

Substracting (2) from (1), we get

$$f(x) - f\left(1-\frac{1}{x}\right) = 2x - \frac{2}{1-x} \quad (4)$$

Now adding (3) and (4) we get

$$2f(x) = \frac{2x}{x-1} - \frac{2}{1-x}$$

$$\Rightarrow f(x) = \frac{x+1}{x-1}.$$

EXAMPLE 4.5 Determine all functions $f : R \to R$ such that $f(x)\, f(y) - f(xy) = x + y$ for all $x, y \in R$. (1)

SOLUTION Put x = y = 0 in (1).

Hence f(0) f(0) – f(0) = 0.
This implies that f(0) = 0 or f(0) = 1.
If f(0) = 0 then f(x) f(0) – f(0) = x + 0.
Hence, x = 0 for all $x \in R$, a contradiction.
Hence, f(0) = 1.
Substituting y = 0, in (1) we get f(x) – 1 = x.
Hence, f(x) = x + 1 is the only solution of (1).

EXAMPLE 4.6 Let f be a function defined from $R^+ \to R^+$. If $(f(xy))^2 = x$ for all positive numbers x and y and f (2) = 6 then find f (50).

SOLUTION Putting x = 25 and y = 2

$(f(50))^2 = 25(f(2))^2$

$\Rightarrow (f(50))^2 = 25(6)^2$

$\Rightarrow f(50) = 30$ or -30 (rejected)

Alternative : Putting x = 1/y, we get

$$f^2(1) = \frac{1}{y} f^2(y)$$

Also f (2) = 6, so putting y = 2, we have

$$f^2(1) = \frac{1}{2} f^2(2) = \frac{1}{2} \times 36 = 18$$

$\Rightarrow f(1) = \sqrt{18} = 3\sqrt{2}$

$\therefore f2(y) = 18y$

$\Rightarrow f(y) = 3\sqrt{2y}$

$\Rightarrow f(50) = 3\sqrt{100} = 30.$

EXAMPLE 4.7 Let f, g : R → R be functions satisfying for all real numbers x and y the equality f(x + g(y)) = 2x + y + 5. Find an expression for g(x + f(y)).

SOLUTION If y = 0 then f(x + g(0)) = 2x + 5.

Hence, $f(x) = f[(x - g(0)) + g(0)] = 2(x - g(0)) + 5$
$= 2x - 2g(0) + 5$

We deduce that $f(0) = -2g(0) + 5$ and hence,

$-2g(0) + 5 = f[(-g(y) + g(y)] = 2(-g(y)) + y + 5$

$$\Rightarrow g(y) = g(0) + \frac{y}{2}$$

This gives g(x + f(y))

$$= g(0) + \frac{x + 2y - 2g(0) + 5}{2} = \frac{x + 2y + 5}{2}.$$

EXAMPLE 4.8 Find all the functions g : R → R satisfying $g(x + y) + g(x - y) = 2x^2 + 2y^2$ for all x and y.

SOLUTION If y = 0, then $2g(x) = 2x^2$, that is, $g(x) = x^2$. Let us verify that $g(x) = x^2$ works.

We have $g(x + y) + g(x - y) = (x + y)^2 + (x - y)^2$
$= x^2 + 2xy + y^2 + x^2 - 2xy + y^2 = 2x^2 + 2y^2$,
from where the only solution is $g(x) = x^2$.

EXAMPLE 4.9 If

$f(x + y + 1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$ and

$f(0) = 1 \ \forall\ x, y \in W$. Determine f(x).

SOLUTION Given

$$f(x + y + 1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$$

Putting x = y = 0,

$$f(1) = (\sqrt{f(0)} + \sqrt{f(0)})^2 = (1 + 1)^2 = 2^2$$

Again putting x = 0, y = 1

$$f(2) = (\sqrt{f(0)} + \sqrt{f(1)})^2 = (1 + 2)^2 = 3^2$$

and for x = 1, y = 1

$$f(3) = (\sqrt{f(1)} + \sqrt{f(1)})^2 = (2 + 2)^2 = 4^2.$$

We guess that $f(x) = (x + 1)^2$ and prove this using induction.

The statement is true for x = 1.

Assume that it is true for x = n.

We now prove that it is true for x = n + 1 also.

$$f(n + 1) = f(n + 0 + 1) = (\sqrt{f(n)} + \sqrt{f(0)})^2$$

$$= (\sqrt{(n+1)^2} + \sqrt{1})^2$$

$= (n + 1 + 1)^2$, which is true.

Hence, $f(x) = (x + 1)^2$ for $x \in W$.

EXAMPLE 4.10 Let a function f satisfy $f(x + 1) = f(x) + x \ \forall\ x \in N$ where f(1) = 0. Find f(3) and a formula for f(x).

SOLUTION $x = 1 \Rightarrow f(2) = f(1) + 1 \Rightarrow f(2) = 0 + 1$

$x = 2 \Rightarrow f(3) = f(2) + 2 \Rightarrow f(3) = 3.$

Now, $f(x + 1) - f(x) = x$

We put x = 1, 2, 3, ..., n –1 and the resulting equations

$x = 1 \Rightarrow f(2) - f(1) = 1$

$x = 2 \Rightarrow f(3) - f(2) = 2$

$x = 3 \Rightarrow f(4) - f(3) = 3$

....

$x = n - 1 \Rightarrow f(n) - f(n-1) = n - 1$

On addition, we get

$f(n) - f(1) = 1 + 2 + 3 + ... + n - 1$

$\Rightarrow f(n) = \dfrac{n(n-1)}{2}$

$\Rightarrow f(x) = \dfrac{x(x-1)}{2}, x \in N.$

EXAMPLE 4.11 Let f be a function from the set of positive integers to the set of real numbers i.e.

$f : N \to R$, such that.

(i) $f(1) = 1$

(ii) $f(1) + 2f(2) + 3f(3) + ... + nf(n) = n(n+1) f(n)$ for $n \geq 2$ then find the value of $f(1994)$.

SOLUTION

Given $f(1) + 2f(2) + 3f(3) + ... + nf(n)$

$= n(n+1) f(n)$ (1)

Replacing n by (n + 1) then

$f(1) + 2f(2) + 3f(3) + ... + nf(n) + n(n+1) f(n+1)$

$= (n+1)(n+2) f(n+1)$ (2)

Subtracting (1) from (2) then we get

$(n+1) f(n+1)$

$= (n+1)(n+2) f(n+1) - n(n+1) f(n)$

$\Rightarrow nf(n) = (n+1) f(n+1)$

From which we conclude that

$2f(2) = 3f(3) = 4f(4) = ... = nf(n)$

Substituting the value of 2f(2), 3f(3), ... in terms of nf(n) in (1), we have

$f(1) + (n-1) nf(n) = n(n+1) f(n)$

$\Rightarrow f(1) = 2n f(n)$

$\therefore f(n) = \dfrac{f(1)}{2n} = \dfrac{1}{2n}$ $(\because f(1) = 1)$

$\Rightarrow f(1994) = \dfrac{1}{2.1994} = \dfrac{1}{3988}.$

EXAMPLE 4.12 Let f be a function satisfying

$2f(xy) = \{f(x)\}y + \{f(y)\}x$ and $f(1) = k \neq 1$.

Prove that $(k-1)\sum_{r=1}^{n} f(r) = kn + 1 - k$

SOLUTION Given that

$2f(xy) = f(x))^y + (f(y))^x$ ...(1)

Replacing y by 1, we get

$2f(x) = f(x) + (f(1))^x$ ...(2)

or, $f(x) = f(1)^x = k^x$ $\{\because f(1) = k\}$

$\therefore \sum_{r=1}^{n} f(r) = f(1) + f(2) + ...+ f(2)$

$= k^1 + k^2 + k^3 + ... + k^n$

$= \dfrac{k(k^n - 1)}{(k-1)} = \dfrac{k^{n+1} - k}{(k-1)}$

or $(k-1)\sum_{r=1}^{n} f(r) = (k^{n+1} - k).$

Note

Here, we have some important functional equations and their solutions. These equations can be solved using some advanced techniques. However, at this stage one can guess these solutions and verify them to be true.

If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\, ln\, x, k \in R.$

(ii) $f(xy) = f(x) . f(y) \Rightarrow f(x) = 0$ or $x^n, n \in R$

(iii) $f(x+y) = f(x) . f(y) \Rightarrow f(x) = 0$ or $a^x, a > 0$

(iv) $f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx, k \in R.$

EXAMPLE 4.13 Find the natural number 'a' for which

$\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$ where f satisfies

$f(x+y) = f(x). f(y)\ \forall\ x, y \in N$, and $f(1) = 2$.

SOLUTION Given $f(1) = 2$.

We put x = 1, y = 1 in the given equation :

$f(1+1) = f(1) . f(1) = 2^2$ $\Rightarrow f(2) = 2^2$

Now, $f(3) = f(2+1) = f(2).f(1) = 2^2.2 = 2^3$.

We predict that $f(n) = 2^n, n \in N$ and prove this using induction.

Now, $\sum_{k=1}^{n} f(a).f(k) = 16(2n - 1)$

$\Rightarrow$ f(a) $\sum_{k=1}^{n} f(k) = 16(2^n - 1)$

$\Rightarrow 2^a(2^1 + 2^2 + ... + 2^n) = 16(2^n - 1)$

$\Rightarrow 2^a . \dfrac{2(2^n - 1)}{2 - 1} = 16(2^n - 1) \quad \Rightarrow a = 3.$

Polynomial Functional Equations

Consider an equation such as

$f(x - 1) f(x + 1) = f(f(x)),$ (1)

where f is a polynomial with real coefficients. Let d = deg f. The left hand side of the equation is of degree 2d, whereas the right hand side is of degree d2. Thus $2d = d^2$, implying that d = 0 or d = 2. A polynomial of zero degree is constant, and the only constant f(x) = c satisfying this equation is c = 0 or c = 1.

The case d = 2 can now be considered. We can verify that $f(x) = (x - 1)^2$ in this case.

EXAMPLE 4.14 Let a polynomial function

$f(x^2) = x^3 f(x) + x^3 - 1 \; \forall x \in R$ where f(2) = 7, find f(x).

SOLUTION $f(x^2) = x^3 f(x) + x^3 - 1$ ...(1)

Let the degree of the polynomial f(x) be n. Then the degree of L.H.S. of (1) is 2n and the degree of R.H.S. is 0 or n + 3.

The degree is 0 when f(x) is –1. Since, f(2) = 7, this is not acceptable.

If the degree of R.H.S. is n + 3, then

$2n = n + 3 \Rightarrow n = 3.$

Let $f(x) = ax^3 + bx^2 + cx + d$.

Placing f(x) in (1), we get

$ax^6 + bx^4 + cx^2 + d = ax^6 + bx^5 + cx^4 + dx^3 + x^3 - 1.$

Comparing the coefficients,

b = 0

$b = c \Rightarrow c = 0$

$0 = d + 1 \Rightarrow d = -1$

Hence, $f(x) = ax^3 - 1$.

Since, f(2) = 7, 7 = 8a – 1

$\Rightarrow 8a = 8 \Rightarrow a = 1$ $f(x) = x^3 - 1$.

EXAMPLE 4.15 Let a polynomial function f satisfy

$f(x) + f\left(\dfrac{1}{x}\right) = f(x) . f\left(\dfrac{1}{x}\right) \; \forall x \in R - \{0\}$

Prove that $f(x) = \pm x^n + 1$, where $n \in W$.

SOLUTION

Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a^n$, $a_0 \neq 0$.

We place this into the given equation,

$(a_0x^n + a_1x^{n-1} + ... + a_n)$

$+ \left(\dfrac{a_0}{x^n} + \dfrac{a_1}{x^{n-1}} + ... + a_n\right)$

$= (a_0x^n + ax^{n-1} + ... + a_n)$

$\times \left(\dfrac{a_0}{x^n} + \dfrac{a_1}{x^{n-1}} + ... + a_n\right)$

$(a_0x^{2n} + a_1x^{2n-1} + ... + a^nx^n)$

$+ (a_0 + a_1x + ...a_{n-1}x^{n-1} + a_nx^n)$

$= (a_0x^n + a_1x^{n-1} + ... + a_n)$

$\times (a_0 + a_1x + ...a_{n-1}x^{n-1} + a_nx^n)$

Since, the equation is valid for all x except 0, we compare the coefficients of various powers of x on both sides :

x^{2n}: $a_0 = a_0a_n \quad \Rightarrow a_n = 1$ as $a_0 \neq 0$.

x^{2n-1}: $a_1 = a_0a_{n-1} + a_1a_n \Rightarrow a_{n-1} = 0$

x^{2n-2}: $a_{n-2} = 0$.

Similarly we prove that $a_{n-2} = 0 , ..., a_1 = 0$.

Finally comparing the coefficient of x^n:

$2a_n = a_0^2 + a_n^2$

$\Rightarrow 2 = a_n^2 + 1 \Rightarrow a_n = \pm 1$

Placing these coefficients in f(x), we get

$f(x) = \pm x^n + 1$.

EXAMPLE 4.16 If f(x) is a polynomial function satisfying $f(x). f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right) \; \forall x \in R - \{0\}$ and f(2) = 9, then find f (3).

SOLUTION

$f(x) = 1 \pm x^n$ using the previous example

As f(2) = 9, we have $f(x) = 1 + x^3$

Hence, $f(3) = 1 + 3^3 = 28$.

EXAMPLE 4.17 Given f (x) is a polynomial function of x, satisfying f(x). f(y) = f(x) + f(y) + f(xy) – 2 and that f (2) = 5. Then find f (3).

SOLUTION

Given $f(x) . f(y) = f(x) + f(y) + f(xy) - 2$

Put $y = 1/x$,

$$f(x) . f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + f(1) - 2$$

Put $x = 1 , y = 1$

$f(1) . f(1) = 2f(1) + f(1) - 2$

$f^2(1) - 3f(1) + 2 = 0$

$(f(1) - 1)(f(1) - 2) = 0 \Rightarrow f(1) = 1$ or $f(1) = 2$

But f (1) 1, as in this case $f(x) = 1$ for all x.

$f(1) = 2.$

Hence $f(x). f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

using the previous example

$f(x) = \pm x^n + 1$

$f(2) = \pm 2^n + 1 = 5 \Rightarrow n = 2$ using (+) sign

$\therefore f(x) = x^2 + 1 \Rightarrow f(3) = 10.$

The next method for solving polynomial equations involves the use of the following principle. Suppose f(x) is a polynomial which is periodic in the sense that there exists some $a \neq 0$ for which $f(x + a) = f(x)$, for all real x. Then $f(x) = c$ for all x.

Now let us consider how this can be applied to functional equations. Suppose a solution, say $f_{0}(x)$ has been found to a given polynomial equation. We must ask whether this is the only solution, or whether there are other. We can write a general solution in the form

$f(x) = f_0(x) + g(x),$

where g(x) is some polynomial whose form is to be determined. It may be possible to show that g(x) satisfies the conditions of the above principle, and thereby to conclude that all solutions of the equation have the form $f_0(x) + c$.

To illustrate this, consider all polynomials satisfying the equation

$f(x + 1) - f(x - 1) = 6x^2 + 2.$

By inspection, we can observe that $f_0(x) = x^3$ is a solution But is it the only one?

Letting a general solution have the form

$f(x) = x^3 + g(x)$, we plug into our equation to obtain

$g(x + 1) - g(x - 1) = 0$ for all x.

So g(x) is a polynomial satisfying the conditions of the above principle with $a = 2$.

Therefore $f(x) = x^3 + c$. It is immediately seen that any value of c will work.

EXAMPLE 4.18 Find all polynomials P(x) such that

$xP(x - 1) = (x - 15) P(x).$ ...(1)

SOLUTION We observe that x divides P(x). Thus we can find a polynomial $P_1(x)$ such that $P(x) = xP_1(x)$. Putting this in (1) and effecting suitable cancellations, we obtain

$(x - 1) P_1(x - 1) = (x - 15) P_1(x)$ ...(2)

This shows that $(x - 1)$ divides $P_1(x)$. If we introduce $P_1(x) = (x - 1) P_2(x)$ and substitute this in (2), we obtain

$(x - 2) P_2(x - 1) = (x - 15) P_2(x).$

We can also write this in the form

$x(x - 1)(x - 2) P_2(x - 1) = (x - 15) P(x).$

Continuing this process, we obtain a polynomial $P_{15}(x)$ such that

$x(x - 1)(x - 2) .... (x - 15) P_{15}(x - 1)$

$= (x - 15) P(x) = x(x - 1)(x - 2) ... (x - 15) P_{15}(x).$

Thus we arrive at the conclusion that

$P_{15}(x - 1) = P_{15}(x).$

If $P_{15}(x)$ is not a constant polynomial, then $P_{15}(x) = 0$ has a root α in C. But then $P_{15}(\alpha - 1) = P_{15}(x) = 0$ so that $\alpha - 1$ is also a root of $P_{15}(x) = 0$. Continuting this argument, we see that α, $\alpha - 1$, $\alpha - 2$,.. are all roots of $P_{15}(x) = 0$. This is clearly impossible since the equation $P_{15}(x) = 0$ can have at most finitely many roots. We conclude that $P_{15}(x)$ is a constant and hence.

$P(x) = cx(x - 1)(x - 2)(x - 3) ... (x - 14),$

for some constant c.

EXAMPLE 4.19 Find all polynomials P(x) such that

$P(x) P(x + 1) = P(x^2).$

SOLUTION Suppose α is a root of $P(x) = 0$. Then the given relations shows that $\alpha^2, \alpha^4, \alpha^8, ...$ are also roots of $P(x) = 0$.

It follows that $|\alpha| = 0$ or $|\alpha| = 1$, for otherwise we get an infinite set of roots of $P(x) = 0$.

Similarly $\alpha - 1$ is a root of $P(x) = 0$ and hence each of $(\alpha - 1)^2$. $(\alpha - 1)^4$,... is a root. We conclude again that $|\alpha - 1| = 0$ or 1.

Suppose $|\alpha| = 1$ and $|\alpha - 1| = 1$. Writing $\alpha = \cos\theta + i \sin \theta$, we see that $2 \cos \theta = 1$. Thus $\cos \theta = 1/2$, giving us $\theta = \pi/3$ or $5\pi/3$. If $\theta = \pi/3$, consider α^2 which is also a root of $P(x) = 0$.

Then $\alpha^2 - 1$ is also a root of $P(x) = 0$ and

$$|\alpha^2 - 1|^2 = \left(\cos\frac{2\pi}{3} - 1\right)^2 + \sin^2\frac{2\pi}{3} = 3.$$

Thus we have a root $\alpha^2 - 1$ of $P(x) = 0$ which is of absolute value > 1.

But then this leads to an infinite set of roots of $P(x) = 0$. Similarly is the case when $\theta = 5\pi/3$. We conclude that $\alpha = 0$ or $\alpha - 1 = 0$. This simplies that $P(x)$ is of the form $cx^m(1 - x)^n$, for some constant c and non-negative integers m, n.

Substituting this in the given equation, it is easy to check that $c = 1$ and $m = n$. Thus the class of polynomials satisfying the given relation is $P(x) = x^m (1 - x)^m$ where $m \geq 0$ is an integer.

EXAMPLE 4.20 Find all functions $f: N \to N$ such that

(a) $f(2) = 2$;

(b) $f(mn) = f(m) f(n)$ for all m, n in N;

(c) $f(m) < f(n)$ whenever $m < n$.

(The property (b) is often referred to as multiplicativity condition. The condition (c) simply says that f is strictly increasing on N).

SOLUTION One important tool we have while solving equations on N is the principle of mathematical induction. We use it here to solve our problem.

We see from (b) that $f(1) = f(1\cdot1) = f(1)^2$.

Since we are in N, we conclude that $f(1) = 1$.

Similarly, taking $m = n = 2$ in (b), we obtain $f(4) = f(2)2 = 4$.

Now using (c) we can fix $f(3)$.

Because $2 < 3 < 4$, we known from (c) that $f(2) < f(3) < f(4)$.

But $f(2) = 2$ and $f(4) = 4$, and 3 is the only natural number between 2 and 4. We conclude that $f(3) = 3$, thus getting the value of $f(3)$.

Now we use this information to conclude that

$$f(6) = f(2.3) = f(2)\, f(3) = 6.$$

We use the fact that $4 < 5 < 6$ and (c) to conclude that $r = f(4) < f(5) < f(6) = 6$. Hence, $f(5) = 5$. We now know how to proceed in order to complete the proof by induction. Suppose we have proved that $f(1) = 1$, $f(2) = 2$,..., $f(2k) = 2k$, for some natural number k. Using (b), we have

$$f(2k + 2) = f(2)\, f(k + 1) = 2(k + 1) = 2k + 2.$$

We have used the fact that $k + 1 \leq 2k$ and the induction hypothesis. Since (c) implies that $2k = f(2k) < f(2k + 1) < f(2k + 2) = 2k + 2$, we conclude $f(2k + 1) = 2k + 1$. It follows by principle of induction that $f(n) = n$ for all natural numbers.

EXAMPLE 4.21 Find all functions $f : R \to R$ such that

(a) $f(-x) = - f(x)$ for all real x;

(b) $f(x + 1) = f(x) + 1$, for all real x;

(c) $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$ for all $x \neq 0$

SOLUTION Putting $x = 0$ in (a), we obtain $f(0) = 0$. Using (b), we see that $f(1) = 1$. An easy induction using (b) shows that $f(n) = n$ for all natural numbers n. Another application of (a) now implies that $f(n) = n$ for all integers n.

Consider $1 + \frac{1}{x}$, for all $x \neq 0$ and $x \neq -1$. Using (b) and (c), we obtain

$$f\left(1 + \frac{1}{x}\right) = 1 + f\left(\frac{1}{x}\right) = 1 + \frac{f(x)}{x^2}.$$

On the other hand, we write

$$1 + \frac{1}{x} = \frac{x+1}{x} = \frac{1}{(x/(x+1))},$$

and this gives in view of (c) another expression

$$f\left(1 + \frac{1}{x}\right) = f\left(\frac{1}{x/(x+1)}\right) = \frac{f(x/(x+1))}{(x/(x+1))^2}$$

But we also have

$$f\left(\frac{x}{x+1}\right) = f\left(1-\frac{1}{x+1}\right) = 1 - f\left(\frac{1}{x+1}\right)$$

$$= 1 - \frac{f(x+1)}{(x+1)^2} = \frac{(x+1)^2 - 1 - f(x)}{(x+1)^2}.$$

Using this expression, we obtain

$$f\left(1+\frac{1}{x}\right) = \frac{(x+1)^2 - 1 - f(x)}{(x+1)^2}$$

Now comparing two expressions for, $f\left(1+\frac{1}{x}\right)$ we see that

$x^2 + f(x) = x^2 + 2x - f(x)$, for all $x \neq 0$, $x \neq -1$.

But we know that $f(0) = 0$ and $f(-1) = -f(1) = -1$.

Thus $f(x) = x$ holds good for all real numbers x.

EXAMPLE 4.22 Find all functions $f: Q \to Q$ such that

$f(x + y) + f(x - y) = 2f(x) + 2f(y)$, ...(1)

for all rational x, y.

SOLUTION If we set $x = y = 0$ in (1), we see that $f(0) = 0$ and hence $f(2x) = 4f(x)$.

By an easy induction, we can show $f(nx) = n^2f(x)$ for all natural numbers n and rational numbers x.

Taking $x = 0$, we also observe that $f(-y) = f(y)$ so that $f(nx) = n^2f(x)$ for all integers n.

If $x = p/q$ is a rational number, then

$q^2f(x) = f(qx) = f(p) = p^2f(1)$.

We thus obtain $f(x) = cx^2$ for all rationals x, where $c = f(1)$.

EXAMPLE 4.23 Let $f : [1, \infty) \to [1, \infty)$ satisfy

(a) $f(x) \leq 2(1 + x)$ for all $x \in [1, \infty)$

(b) $xf(x + 1) = f(x)2 - 1$ for all $x \in [1, \infty)$

Prove that $f(x) \leq x + 1$.

SOLUTION We have

$$f(x)^2 = xf(x + 1) + 1 \leq x (2(x + 1)) + 1$$
$$= 1 + 4x + 2x^2$$
$$< 2(1 + 2x + x^2) = 2(1+ x)^2.$$

It follows that $f(x) < (1 + x)$. Using this fresh bound, we obtain

$$f(x)^2 = xf(x + 1) + 1 < \sqrt{2}\, x (2 + x) + 1$$
$$= \sqrt{2}\, x^2 + 2\sqrt{2}\, x + 1$$
$$< \sqrt{2}\, (x^2 + 2x + 1)$$
$$= \sqrt{2}\, (x + 1)^2.$$

Thus we obtain another bound: $f(x) < 2^{1/4}(x + 1)$. Continuing by induction, we arrive at

$f(x) < 2^{1/2^k}\ (1 + x)$

for all $k \in N$, and $x \in [1, \infty)$. It follows that $f(x) \leq 1 + x$ for all $x \in [1, \infty)$.

EXAMPLE 4.24 Let $f : I \to I$ satisfy the equations

$f(x^2) = [f(x)]^2$, ...(1)

$f(x + 1) = f(x) + 1$. ...(2)

Prove that $f(x) = x$ for all integral x.

SOLUTION From (1) we see that $f(x) \geq 0$ for all $x \geq 0$. In addition, by mathematical induction on (2) we get

$f(x + n) = f(x) + n$ ...(3)

for all positive integers n. Substituting $x - 1$ for x in (2) and applying mathematical induction again, we see that (3) holds for all integers n and all real x.

Now applying (1) we get

$[f(x - 1)]^2 = [f(1 - x)]^2$.

Using (3) on this identity yields.

$[1 + f(-x)]^2 = [f(x) - 1]^2$.

We can expand this, noting in passing that $[f(-x)]^2 = [f(x)]^2$ to obtain the fact that

$f(-x) = -f(x)$ ...(4)

for all real values of x. Equation (4) immediately implies that $f(0) = 0$. So, this fact and (3) together imply that $f(n) = n$ for all integers n.

EXAMPLE 4.25 If a function f satisfies

$f(x + y) \leq f(x) + f(y)$ for all real x and y and $f(x) \leq x$ for all real x, find f.

SOLUTION First, we get x = y = 0 in

$f(x + y) \le f(x) + f(y)$ ...(1)

to obtain

$f(0) \le 2f(0)$ or $f(0) \ge 0$. ...(2)

Because we also have $f(0) \le 0$ by $f(x) \le x$, it follows that $f(0) = 0$. Next set $y = -x$ in (1). We get

$f(0) \le f(x) + f(-x)$

or $f(x) \ge -f(-x)$ ...(3)

using $f(0) = 0$.

Combining f(x) £ x with (3) we have

$f(x) \ge -f(-x) \ge -(-x) = x$.

Combined with the inequality $f(x) \le x$, we can conclude that $f(x) = x$.

CONCEPT PROBLEMS [F]

1. A certain function f (x) satisfies $f(x) + 2f(6 - x) = x$ for all real numbers x. Find the value of f (1).
2. If $2f(x^2) + 3f(1/x^2) = x^2 - 1$ $(x \ne 0)$ then find $f(x^2)$.
3. If function f(x) is satisfying $2f(\sin x) + f(\cos x) = x$ for all $x \in R$ then express f(sin x) as a polynomial in x.
4. Let f be a real valued function such that $f(x) + 2f\left(\frac{2002}{x}\right) = 3x$ for all $x > 0$. Find f (2).
5. Check that a solution to $f^2(x) = x^{-1}$, $x \ne 0$, is given by $f(x) = -x^{\text{sgn}(x)}$, where sgn(x) denotes the signum function.
6. Find all polynomials P(x) which satisfy the relation $P(x + 1) = P(x) + 2x + 1$.
7. If f(x) is a polynomial function satisfying $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ $\forall x \in R - \{0\}$ and $f(3) = -8$, then find f(4).

PRACTICE PROBLEMS [L]

8. Let $f(x + y) = x + f(y)$ for any two real number x and y, and $f(0) = 2$. Find the value of f (18)
9. Let f be a function satisfying $f(xy) = \frac{f(x)}{y}$ for all positive real numbers x and y. If $f(30) = 20$, then find the value of f (40).
10. If a function F is such that $F(0) = 2$, $F(1) = 3$, $F(x + 2) = 2F(x) - F(x + 1)$ for $x \ge 0$, then find F(5).
11. Let $f\left(x + \frac{1}{y}\right) + f\left(x - \frac{1}{y}\right) = 2f(x) \cdot f\left(\frac{1}{y}\right)$ $\forall x, y \in R$ and $y \ne 0$. If $f(0) = 0$, then show that $f(1) = f(2) = 0$.
12. Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. Then find the value of f (5).
13. Find all the functions f that satisfy $f(xy) = yf(x)$.

14. Find all functions f: R–{–1} → R such that
$(f(x))^2 \cdot f\left(\dfrac{1-x}{1+x}\right) = 64x.$

15. Let f satisfy $f(n + 1) = (-1)^{n+1}n - 2f(n)$, $n \geq 1$ If $f(1) = f(1001)$ find
$f(1) + f(2) + f(3) + ... f(1000)$.

16. In the equation below f is a function from R to R. Find f:
   (i) $f(x + y) - 2f(x - y) + f(x) - 2f(y) = y - 2$
   (ii) $f(x + y) + 2f(x - y) + f(x) + 2f(y) = 4x + y$
   (iii) $f(x)\, f(x + y) = f(y)^2 f(x - y)^2 e^{y+4}$
   (iv) $f(x + y) + f(x - y) - (y + 2) f(x) + y(x^2 - 2y) = 0$

17. Suppose $f : R \to R$ is such that $f(xy) = xf(x) + yf(y)$, for all $x, y \in R$. Prove that $f(x) = 0$ for all $x \in R$.

18. Suppose $f : R \to R$ is such that $f(xf(z) + f(y)) = zf(x) + y$, for all real numbers x, y, z. Prove that $f(x) = x$ for all real x.

19. Find all pairs of functions $f, g : R \to R$ which satisfy
   (a) g is an one-one function;
   (b) $f(g(x) + y) = g(x + f(y))$, for all $x, y \in R$.

20. Let $f : R \to R$ satisfy $f(3) = 1$, $\forall\, x \in R$ $f(x + 3) \geq f(x) + 3$, $f(x + 1) \leq f(x) + 1$. Put
$g(x) = f(x) - x + 1$. Determine g(2008)

21. Prove that $f(n) = 1 - n$ is the only integer valued function defined on the integers that satisfies the following conditions.
   (a) $f(f(n)) = n$, for all integers n;
   (b) $f(f(n + 2) + 2) = n$, for all integers n;
   (c) $f(0) = 1$.

TARGET PROBLEMS for JEE ADVANCED

PROBLEM 4.1 Discuss the domain of definition of the function, $f(x) = \log(ax^2 + bx + c)$ for different values of $a, b, c \in R$.

SOLUTION If $b^2 - 4ax > 0$ & $a > 0$, then the function is defined throughout the number scale except for the interval $x_1 \leq x \leq x_2$, where x_1 & x_2 are the roots of the trinomial.

If $b^2 - 4ac > 0$ & $a < 0$, then the function is defined only for $x_1 < x < x_2$. If $b^2 - 4ac < 0$ & $a > 0$, then the function is defined through the number scale. If $b^2 - 4ac < 0$ & $a < 0$, then the function is defined nowhere. Finally, if $b^2 - 4ac = 0$, then the function will be defined throughout the number scale, except for one of its point : $x = -b/2a$ if $a > 0$ and is defined nowhere if $a < 0$.

PROBLEM 4.2 Find the domain of definition of the function,

$$f(x) = [2\tan \pi x]^{\log_{[2\tan \pi x]}\left(\frac{x^2 + 2x - 3}{4x^2 - 4x - 3}\right)}$$

where [ ] denotes the greatest integer function.

SOLUTION $f(x) = a^{\log_a N}N$

$\Rightarrow\ 0 < [2 \tan \pi x]$ or $[2 \tan \pi x] > 1$

and $\dfrac{x^2+2x-3}{4x^2-4x-3}>0$ i.e. $\dfrac{(x+3)(x-1)}{(2x-3)(2x+1)}>0$

$\Rightarrow x \in (-\infty, -3) \cup (-1/2, 1) \cup (3/2, \infty)$ ...(1)

Now $0 < [2 \tan \pi x] \le 1$ not possible.

$\therefore$ $[2 \tan \pi x] > 1 \Rightarrow 2 \tan \pi x \ge 2 \Rightarrow \tan \pi x \ge 1$

$\Rightarrow n\pi + \dfrac{\pi}{4} \le \pi x < n\pi + \dfrac{\pi}{2}$, $n \in I$.

$\Rightarrow n + \dfrac{1}{4} \le x < n + 1/2$ ...(2)

Common solution of (1) and (2) is

$\left[n+\dfrac{1}{4}, n+\dfrac{1}{2}\right)$ where $n = 0, n \ge 2$ or $n \le -4$.

PROBLEM 4.3 Find the domain of the function $f(x) = \dfrac{1}{[|x-2|]+[|x-10|]-8}$ where [ ] represents greatest integer function.

SOLUTION

Here, $f(x) = \dfrac{1}{[|x-2|]+[|x-10|]-8}$ is defined when

$[|x-2|] + [|x-10|] - 8 \ne 0$ ...(1)

Here we have three cases:

Case 1: $x \le 2$

$[2-x] + [10-x] - 8 \ne 0,$

$\Rightarrow 2 + [-x] + 10 + [-x] - 8 \ne 0,$

$\Rightarrow 2[-x] + 4 \ne 0$

$\Rightarrow [-x] \ne -2$

$\Rightarrow -x \notin [-2, -1)$

or $x \notin (1, 2]$ ...(2)

Case 2: $2 \le x \le 10$

$\Rightarrow [x-2] + [10-x] - 8 \ne 0$

$\Rightarrow [x] - 2 + 10 + [-x] - 8 \ne 0$

$\Rightarrow [x] + [-x] \ne 0$

$\Rightarrow x \notin \{2, 3, 4, \ldots\ldots, 10\}$ ...(3)

Case 3: $x \ge 10$

$\Rightarrow [x-2] + [x-10] - 8 \ne 0$

$\Rightarrow 2[x] \ne 20$

$\Rightarrow [x] \ne 10$

$\Rightarrow x \notin [10, 11)$ ...(4)

Thus for f(x) to be defined

$x \in R - \{(1, 2) \cup \{2, 3, 4, 5, 6, 7, 8, 10\} \cup (10, 11)\}$

PROBLEM 4.4 If $f(x) = 4x^3 - x^2 - 2x + 1$ and

$$g(x) = \begin{cases} \text{Min } \{f(t) : 0 \le t \le x\} & ; 0 \le x \le 1 \\ 3 - x & ; 1 < x \le 2 \end{cases}$$

then evaluate $g\left(\dfrac{1}{4}\right) + g\left(\dfrac{3}{4}\right) + g\left(\dfrac{5}{4}\right)$.

SOLUTION $f(x) = 4x^3 - x^2 - 2x + 1$

$f'(x) = 12x^2 - 2x - 2 = 2(6x^2 - x - 1)$

$x = \dfrac{1 \pm \sqrt{1 + 4 \times 6}}{2 \times 6}$

$x = \dfrac{1 \pm 5}{12} = \dfrac{6}{12}, \dfrac{4}{12} - = \dfrac{1}{2}, -\dfrac{1}{3}$

$f'(x) = 2(2x-1)(3x+1)$

$f'(x) > 0$ for $x > \dfrac{1}{2}$

$f'(x) < 0$ for $0 < x < \dfrac{1}{2}$

$\therefore$ f(x) is minimum at $x = \dfrac{1}{2}$ in (0, 1)

Also $\left[\dfrac{1}{2}, 1\right]$ in f(x) is decreasing

in $\left[\dfrac{1}{2}, 1\right]$ f(x) is increasing

$g\left(\dfrac{1}{4}\right) = f\left(\dfrac{1}{4}\right)$

$= 4 \times \left(\dfrac{1}{4}\right)^3 - \left(\dfrac{1}{4}\right)^2 - 2 \times \dfrac{1}{4} + 1 = \dfrac{1}{2}$

$g\left(\dfrac{3}{4}\right) = f\left(\dfrac{1}{2}\right)$

$= 4 \times \left(\dfrac{1}{2}\right)^3 - \left(\dfrac{1}{2}\right)^2 - 2 \times \dfrac{1}{2} + 1 = \dfrac{1}{4}$

$g\left(\dfrac{5}{4}\right) = f\left(\dfrac{5}{4}\right) = 3 - \dfrac{5}{4} = \dfrac{7}{4}$.

PROBLEM 4.5 Find the range of the function $f(x) = \cos x \left\{ \sin x + \sqrt{(\sin^2 x + \sin^2 \alpha)} \right\}$.

SOLUTION

$y = \cos x \left\{ \sin x + \sqrt{(\sin^2 x + \sin^2 \alpha)} \right\}$

Dividing by $\cos^2 x$ we get,

$y \sec^2 x = \tan x + \sqrt{\tan^2 x + \sec^2 x \sin^2 \alpha}$

$\Rightarrow y^2 \sec^4 x - 2y \tan x \sec^2 x - \sin^2\alpha \sec^2 x = 0$

$\Rightarrow y^2 (1 + \tan^2 x) - 2y \tan x - \sin^2 \alpha = 0$ [$\sec^2 x \neq 0$]

$y^2 \tan^2 x - 2y \tan x + y^2 - \sin^2\alpha = 0$

tan x is real we have $D \geq 0$

$4y^2 - 4y^2 (y^2 - \sin^2 \alpha) \geq 0$

$4y^2 (1 - y^2 + \sin^2\alpha) \geq 0 \Rightarrow y^2 \leq 1 + \sin^2 \alpha$

$-\sqrt{1+\sin^2\alpha} \leq y \leq \sqrt{1+\sin^2\alpha}$

Range of $f(x) = \left[-\sqrt{1+\sin^2\alpha}, \sqrt{1+\sin^2\alpha}\right]$.

PROBLEM 4.6 Find the range of the expression

$$y = \frac{\tan^2\theta - 2\tan\theta - 8}{\tan^2\theta - 4\tan\theta - 5},$$

for all permissible values of θ.

SOLUTION Let $x = \tan\theta$, where $x \in R$ for all permissible values of θ.

$$\Rightarrow y = \frac{x^2 - 2x - 8}{x^2 - 4x - 5}$$

$\Rightarrow x^2y - 4xy - 5y = x^2 - 2x - 8$

$\Rightarrow (y - 1)x^2 + 2x(1 - 2y) + 8 - 5y = 0$

$\therefore$ $x \in R$ hence $D \geq 0$

$\Rightarrow 4(1 - 2y)^2 - 4(y - 1)(8 - 5y) \geq 0$

$\Rightarrow (4y^2 - 4y + 1) - (13y - 8 - 5y^2) \geq 0$

$\Rightarrow 9y^2 - 17y + 9 \geq 0$ ....(1)

since coefficient of $y^2 > 0$

and $D = 289 - 324 < 0$

Hence (1) is always true.

Therefore range of y is $(-\infty, \infty)$

PROBLEM 4.7 Find the range of the expression

$$y = \frac{(\cot^2\theta + 5)(\cot^2\theta + 10)}{\cot^2\theta + 1},$$

for all permissible values of θ.

SOLUTION Let $x = \cot^2\theta + 1$, where $x \in [1, \infty)$ for all permissible values of θ.

Then $y = \dfrac{(x+4)(x+9)}{x} = x + 13 + \dfrac{36}{x}$

$= \left(\sqrt{x} - \dfrac{6}{\sqrt{x}}\right)^2 + 25$

Hence the range of y is $[25, \infty)$.

PROBLEM 4.8 Find the range of the following functions

(i) $y = \cos^{-1}\left(\dfrac{\sqrt{2x^2+1}}{x^2+1}\right)$

(ii) $y = \log_2 (2 - \log_{\sqrt{2}} (16 \sin^2 x + 1))$

SOLUTION (i) $y = \cos^{-1}\left(\dfrac{\sqrt{2x^2+1}}{x^2+1}\right)$

$= \sin^{-1}\left(\sqrt{1 - \dfrac{2x^2+1}{(x^2+1)^2}}\right)$

$= \sin^{-1}\left(\sqrt{\dfrac{x^4}{(x^2+1)^2}}\right) = \sin^{-1}\left(\dfrac{x^2}{1+x^2}\right)$

We know that $0 \leq \dfrac{x^2}{x^2+1} < 1$

$\Rightarrow 0 \leq \sin^{-1}\left(\dfrac{x^2}{x^2+1}\right) < \dfrac{\pi}{2} \Rightarrow y \in \left[0, \dfrac{\pi}{2}\right)$.

$\therefore$ The range of y is $[0, \pi/2)$.

(ii) $y = \log_2 (2 - \log_{\sqrt{2}} (16 \sin^2 x + 1))$

Now for y to be well defined

$2 - \log_{\sqrt{2}} (16 \sin^2 x + 1) > 0$ and $16 \sin^2 x + 1 > 0$

The second inequality is true for all $x \in R$.

$\Rightarrow$ y is well defined for $16 \sin^2 x + 1 < 2$

i.e. for $0 \le 16 \sin^2 x < 1$

i.e. $1 \le 16 \sin^2 x + 1 < 2$.

$\Rightarrow 0 \le \log_{\sqrt{2}} (16 \sin^2 x + 1) < 2$

$\Rightarrow 0 < 2 - \log_{\sqrt{2}} (16 \sin^2 x + 1) \le 2$

$\Rightarrow -\infty < y \le 1$

$\therefore$ Range of y is $(-\infty, 1]$.

PROBLEM 4.9 Find the range of the function

$$f(x) = \sin^{-1} x^2 + \left[\left\{ \ln \sqrt{x - [x]} \right\}\right] + \cot^{-1}\left(\frac{1}{1+\sqrt{2}\,x^2}\right),$$

where {.} and [.] are fractional part function and greatest integer function respectively.

SOLUTION The domain of function is $(-1, 1) - \{0\}$ because $x - [x] = 0$ for integral value of x, hence the middle term will not be defined.

Also $[\{k\}] = 0$, whenever k is meaningful.

$\therefore f(x) = \sin^{-1} x^2 + \tan^{-1} (1 + \sqrt{2}\, x^2)$,

$$\text{since} \cot^{-1} x = \tan^{-1}\frac{1}{x} \text{ when } x > 0.$$

The function is continuous and is even.

The least value of the function will occur when $x \to 0$ and is $\frac{\pi}{4}$.

$\therefore$ The maximum value $= \lim\limits_{x \to \pm 1} f(x)$

$= \sin^{-1} 1 + \tan^{-1} (1 + \sqrt{2})$

$= \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}$.

$\therefore$ The range of f (x) is $\left(\frac{\pi}{4}, \frac{7\pi}{8}\right)$.

PROBLEM 4.10 Find the range of the function

$$f(x) = \sin^{-1}\frac{\sqrt{1+x^4}}{1+5x^{10}}.$$

SOLUTION Consider $g(x) = \frac{\sqrt{1+x^4}}{1+5x^{10}}$

Also g(x) is positive $\forall\, x \in R$ and g(x) is continuous $\forall\, x \in R$ and $g(0) = 1$ and $\lim\limits_{x \to \infty} g(x) = 0$.

$\Rightarrow$ g(x) can take all values from (0, 1]

$\Rightarrow$ Range of f(x) i.e. sin–1 (g(x)) is $\left(0, \frac{\pi}{2}\right]$.

PROBLEM 4.11 Find the range of the following functions

(i) $f(x) = \log_e (\sin x^{\sin x} + 1)$ where $0 < x < \pi/2$.

(ii) $f(x) = \log_e (2 \sin x + \tan x - 3x + 1)$ where $\frac{\pi}{6} \le x \le \frac{\pi}{3}$.

SOLUTION (i) $0 < x < \pi/2 \Rightarrow 0 < \sin x < 1$

$\therefore$ Range of $\log_e (\sin x^{\sin x} + 1)$ for $0 < x < \pi/2$

$=$ Range of $\log_e (x^x + 1)$ for $0 < x < 1$.

Let $h(x) = x^x + 1 = e^{x \log_e x} + 1$

$\therefore h'(x) = e^{x \log_e x} (1 + \log_e x)$

$\Rightarrow h'(x) > 0$ for $x > 1/e$ and $h'(x) < 0$ for $x < 1/e$

$\therefore$ h(x) has a minima at $x = 1/e$.

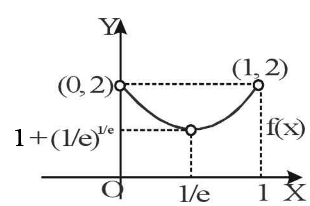
Also $\lim\limits_{x \to 0^+} h(x) = 1 + e^{\lim\limits_{x\to 0}\left(\frac{\ln x}{1/x}\right)}$

$= 1 + e^{\lim\limits_{x\to 0}\left(\frac{1/x}{-1/x^2}\right)}$

$= 1 + e^0 = 2$ and $\lim\limits_{x \to 1^-} h(x) = 2$.

$\therefore 0 < x < 1 \Rightarrow 1 + \left(\frac{1}{e}\right)^{\frac{1}{e}} < (x^x + 1) < 2$

$\Rightarrow \log_e \left(1 + \left(\frac{1}{e}\right)^{\frac{1}{e}}\right) < \log_e (x^x + 1) < \log_e 2$.



$\therefore$ Range of $f(x) = \left(\log_e\left(1 + e^{\frac{-1}{e}}\right), \log_e 2\right)$.

(ii) Let h(x) = (2 sin x + tan x – 3x +1)

$\Rightarrow$ h′(x) = (2 cos x + sec2 x – 3)

$$= \frac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x}$$

$\therefore$ $h'(x) > 0 \Rightarrow 2\cos^3 x - 3\cos^2 x + 1 > 0$

$(\cos x - 1)^2 \left(\cos x + \frac{1}{2}\right) > 0 \ \forall\ x \in [\pi/6, \pi/3]$

$\Rightarrow$ h(x) is an increasing function of x

$\Rightarrow$ $h(\pi/6) \le h(x) \le h(\pi 3)$

$\Rightarrow$ $\le \log_e h(x) \le \log_e (1 + 2\sqrt{3} - \pi)$.

$\therefore$ Range of f(x) is

$$\left[\log_e\left(2 + \frac{1}{\sqrt{3}} - \frac{\pi}{2}\right), \log_e(1 + 2\sqrt{3} - \pi)\right]$$

PROBLEM 4.12 If

$$g(x) = \left(4\cos^4 x - 2\cos 2x - \frac{1}{2}\cos 4x - x^7\right)^{\frac{1}{7}},$$

then find the value of .

SOLUTION We have

$= 4\cos^4 x - 2\cos 2x - \frac{1}{2}\cos 4x - x^7$

$= 4\cos^4 x - 2(2\cos^2 x - 1) - \frac{1}{2}(2\cos^2 2x - 1) - x^7$

$= 4\cos^4 x - 4\cos^2 x + 2 - (2\cos^2 x - 1)^2 + \frac{1}{2} - x^7$

$= \left(\frac{3}{2} - x^7\right)$. We get $g(x) = \left(\frac{3}{2} - x^7\right)^{\frac{1}{7}}$.

$$\Rightarrow g(g(x)) = \left(\frac{3}{2} - (g(x))^7\right)^{\frac{1}{7}} = \left(\frac{3}{2} - \left(\frac{3}{2} - x^7\right)\right)^{\frac{1}{7}}$$

$= x$.

Hence g(g(100)) = 100.

PROBLEM 4.13 Let $f : R^+ \to R$, $f(x) = e^x$ and $g : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $g(x) = \sin^{-1} x$. Find the domain and range of fog (x).

SOLUTION Domain of f(x) is $(0, \infty)$.

Range of g(x) is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The values in range of g(x) which are accepted by f(x) are $\left(0, \frac{\pi}{2}\right]$.

$\Rightarrow\ 0 < g(x) \le \frac{\pi}{2} \Rightarrow 0 < \sin^{-1} x \le \frac{\pi}{2}$

$\Rightarrow\ < x \le 0$

Hence domain of fog(x) is $x \in (0, 1]$.

The range of fog(x) is the range of $f(x) = e^x$ under the domain $\left(0, \frac{\pi}{2}\right]$, which is $(1/ e^{\pi/2}]$.

PROBLEM 4.14 Let $f(x) = x^2 - 1$ and

$$g(x) = \begin{cases} [|f(|x|)|], & x \in (-1, 0) \cup (0, 1) \\ 1, & \text{otherwise} \end{cases}.$$

Then find the range of ℓn ([|g(x)|]), where [.] denotes the greatest integer function.

SOLUTION $|f(|x|)| = 1 - x^2$, $x \in (-1, 0) \cup (0, 1)$

$\Rightarrow$ $[\,|f(|x|)|\,] = 0$, $x \in (-1, 0) \cup (0, 1)$

Hence $g(x) = 0$, $x \in (-1, 0) \cup (0, 1)$

$\Rightarrow$ $[\,|g(x)|\,] = 0$, $x \in (-1, 0) \cup (0, 1)$

and $[\,|g(x)|\,] = 1$, $x \in R - \{(-1, 0) \cup (0, 1)\}$

$\Rightarrow$ Range of ℓn $([\,|g(x)|\,])$ is {0}.

PROBLEM 4.15 If $f(x) = \begin{cases} 2 + x, & \text{if } x \ge 0 \\ 2 - x, & \text{if } x < 0 \end{cases}$, then find (fof) (x).

SOLUTION (fof) (x) = f{f(x)}

$$= \begin{cases} 2 + f(x), & f(x) \ge 0 \\ 2 - f(x), & f(x) < 0 \end{cases}$$

$$= \begin{cases} 2 + 2 + x, & 2 + x \ge 0 \text{ and } x \ge 0 \\ 2 - (2 + x), & 2 + x < 0 \text{ and } x \ge 0 \\ 2 + 2 - x, & 2 - x \ge 0 \text{ and } x < 0 \\ 2 - (2 - x), & 2 - x < 0 \text{ and } x < 0 \end{cases}$$

$$= \begin{cases} 4+x, & x \geq 0 \\ -x, & x \in \phi \\ 4-x, & x < 0 \\ x, & x \in \phi \end{cases}. \text{ Hence (fof) (x)}$$

$$= \begin{cases} 4+x, & x \geq 0 \\ 4-x, & x < 0 \end{cases}.$$

PROBLEM 4.16 If f and g be two linear functions from [–1, 1] onto [0, 2] and $\phi : R^+ - \{-1, 1\} \to R$ be defined by $\phi(x) = \dfrac{f(x)}{g(x)}$, then show that $\left|\phi(\phi(x)) + \phi\left(\phi\left(\dfrac{1}{2}\right)\right)\right| \geq 2$.

SOLUTION Let h be a linear function from [–1, 1] onto [0, 2].

Let h(x) = ax + b. If a > 0, then h(x) is an increasing function and h(–1) = 0 and h(1) = 2

$\Rightarrow -a + b = 0$ and $a + b = 2$

$\Rightarrow a = 1$ and $b = 1$

Hence h(x) = x + 1.

If a < 0, then h(x) is a decreasing function and h(–1) = 2 and h(1) = 0

$\Rightarrow -a + b$ and $a + b = 0$

$\Rightarrow a = -1$ and $b = 1$

Hence h(x) = 1 – x

Now according to the question

f(x) = 1 + x and g(x) = 1 – x

or f(x) = 1 – x and g(x) = 1 + x

$$\therefore \ \phi(x) = \frac{f(x)}{g(x)} = \frac{1-x}{1+x} \text{ or } \frac{1+x}{1-x}$$

Case 1 : When $\phi(x) = \left(\dfrac{1-x}{1+x}\right)$, $x \neq -1$.

$$\phi\left(\phi\left(\frac{1}{x}\right)\right) = \frac{1}{x}$$

Case 2 : When $\phi(x) = \left(\dfrac{1+x}{1-x}\right)$, $x \neq 1$

$$\phi\left(\phi\left(\frac{1}{x}\right)\right) = -x.$$

In both cases, $|\phi(f(x)) + \phi(\phi(1/x))|$

$$= \left|x + \frac{1}{x}\right| \text{ (where } x > 0) = \left|\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2\right| \geq 2.$$

PROBLEM 4.17 If an even function f is defined on the interval (–5, 5), then find the real values for which $f(x) = f\left(\dfrac{x+1}{x+2}\right)$.

SOLUTION It is given that function f is even. Hence, arguments of the functions on the two sides are related either as $x = \dfrac{x+1}{x+2}$ ...(1)

or as $-x = \dfrac{x+1}{x+2}$ ...(2)

From the first relation, $x^2 + x - 1 = 0$

$$\Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$$

From the second relation, $x^2 + 3x + 1 = 0$

$$\Rightarrow x = \frac{-3 \pm \sqrt{5}}{2}$$

We see that values are within the specified domain. Hence, all the four solutions satisfy the given equation.

PROBLEM 4.18 Let $f(x) = e^{\{e^{|x|} \operatorname{sgn} x\}}$ and g (x) = $e^{[e^{|x|} \operatorname{sgn} x]}$, $x \in R$ where { } and [ ] denotes the fractional part and integral part functions respectively. Also h (x) = *l*n + *l*n then for all real x, show that h(x) is an odd function.

SOLUTION Let $e^{|x|} \operatorname{sgn} x = y$

$$h(x) = ln\ (f(x) \cdot g(x)) = ln\ e^{\{y\}+[y]}$$

$$= \{ y \} + [y] = y = e^{|x|} \operatorname{sgn} x$$

$$\therefore \ h(x) = e\,|\,x\,|\operatorname{sgn} x = \begin{cases} e^x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -e^{-x} & \text{if } x < 0 \end{cases}$$

$$h(-x) = \begin{cases} e^{-x} & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ -e^{x} & \text{if } x > 0 \end{cases} = -h(x) \text{ for all } x.$$

Hence, h(x) is an odd function.

PROBLEM 4.19 If f(x) satisfies the relation $f(x + y) = f(x) + f(y)$ for all $x, y \in R$ and $f(1) = 5$ then find $\sum_{n=1}^{m} f(n)$. Also prove that f(x) is odd.

SOLUTION Here $f(r) = f\,(\overline{r-1+1})$

$= f(r-1) + f(1)$ {using the given property}

$\therefore\ f(r) = f(r-1) + 5$ ...(1)

$= \{f(r-2) + 5\} + 5$, using (1)

$= f(r-2) + 2\,.\,5 = \{f(r-3) + 5\} + 2\,.\,3$

$= f(r-3) + 3\,.\,5$

....

$= f(\overline{r-1}) + (r-1)\,.\,5$

$= f(1) + 5(r-1) = 5 + 5(r-1) = 5r$

$\therefore\ = \sum_{n=1}^{m} f(n) = \sum_{n=1}^{m} 5n = 5(1 + 2 + 3 + ... + m)$

$= \dfrac{5m(m+1)}{2}$.

Now, putting x = 0 and y = 0 in the given relation,

$f(0 + 0) = f(0) + f(0)$; $\therefore$ $f(0) = 0$.

Also, putting – x for y in the given relation

$f(x - x) = f(x) + f(-x)$

$\therefore\ f(0) = f(x) + f(-x)$

$\therefore\ 0 = f(x) + f(-x)$, i.e., $f(-x) = -f(x)$.

So, the function is odd.

PROBLEM 4.20 Find period of the following functions:

(i) $f(x) = \sin \dfrac{x}{2} + \cos \dfrac{x}{3}$

(ii) $f(x) = \sin \dfrac{3x}{2} - \cos \dfrac{x}{3} - \tan \dfrac{2x}{3}$.

(iii) $f(x) = x - [x] + |\cos \pi x| + |\cos 2\pi x| + ... + |\cos n\pi x|$.

SOLUTION

(i) Period of sin x/2 is 4π while period of cos x/3 is 6π. L.C.M. of 4 and 6 is 12. Hence period of sin x/2 + cos x/3 is 12π.

(iv) Period of f(x) is L.C.M. of $\dfrac{2\pi}{3/2}, \dfrac{2\pi}{1/3}, \dfrac{\pi}{3/2}$ = L.C.M. of $\dfrac{4\pi}{3}, \dfrac{2\pi}{3}$ $6\pi, = 12\pi$

(iii) $x - [x]$ has period 1

$|\cos x|$ has period π

$|\cos \pi x|$ has period $\dfrac{\pi}{\pi} = 1$

$|\cos 2\pi x|$ has period $\dfrac{\pi}{2\pi} = \dfrac{1}{2}$

$|\cos n\pi x|$ has period $\dfrac{\pi}{n\pi} = \dfrac{1}{n}$

L.C.M. $\left\{1, 1, \dfrac{1}{2}, \dfrac{1}{3}, ..., \dfrac{1}{n}\right\} = 1$

$\therefore$ f (x) has period 1.

PROBLEM 4.21 Find the least and the greatest values of the function defined by $f(x + \lambda) = 1 + \sqrt{2f(x) - f^2(x)}$ $\forall x \in R$, and find whether it is periodic or not.

SOLUTION The given function is defined if

$2f(x) - f^2(x) \ge 0 \Rightarrow f(x)[f(x) - 2] \le 0$

$\Rightarrow\ 0 \le f(x) \le 2$ ...(1)

Also from the given function, it is clear that

$f(x + \lambda) \ge 1 \Rightarrow f(x) \ge 1$ ...(2)

From (1) and (2), we conclude that $1 \le f(x) \le 2$.

Again, we have $\{f(x + \lambda) - 1\}^2 = 2f(x) - f^2(x)$

$\Rightarrow\ \{f(x + \lambda) - 1\}^2 = -\{(f(x) - \lambda)^2\} + 1$ ...(3)

Replacing x by x + l in above equation, we get

$\{f(x + 2\lambda) - 1\}^2 = -\{f(x + \lambda) - 1\}^2 + 1$ ...(4)

From (4) – (3), we get

$\{f(x + 2\lambda) - 1\}^2 = \{f(x) - 1)\}^2$

$\Rightarrow\ f(x + 2\lambda) = f(x)$ using $1 \le f(x) \le 2$.

$\Rightarrow$ f is a periodic function with period 2λ.

PROBLEM 4.22 Let f (x, y) be a periodic function satisfying f(x, y) = f(2x + 2y, 2y − 2x) for all x, y. Show that $g(x) = f(2^x, 0)$ is a periodic function with period 12.

SOLUTION Since f(x, y) = f(2x + 2y, 2y − 2x)(1)

or f(I, II) = f (2 I + 2 II, 2II − 2I)

$\therefore$ f(2x + 2y, 2y − 2x)

= f(2(2x + 2y) + 2(2y − 2x), 2(2y − 2x) − 2 (2x + 2y)) = f(8y, − 8x) ...(2)

From (1) and (2) we get f(x, y) = f(8y, − 8x) ...(3)

or f(I, II) = f(8II, − 8I)

$\therefore$ f(8y, − 8x) = f(8 (−8x), −8 (8y)) = f(− 64, − 64y) ...(4)

From (3) and (4) we get, f(x, y) = f(−64x, −64y) ...(5)

or f(I, II) = f(−64I, −64II)

$\therefore$ f(− 64x, − 64y) = f(− 64 (− 63x), −64(− 64y))

$= f(2^{12}x, 2^{12}y)$ ...(6)

From (5) and (6) we get $f(x, y) = f(2^{12}x, 2^{12}y)$

$\Rightarrow f(x, 0) = f(2^{12}x, 0)$

Replace x by 2^x then $f(2^x, 0) = f(2^{x+12}, 0)$

$\Rightarrow g(x) = g(x + 12)$ $\{ g(x) = f(2^x, 0)\}$

Hence g(x) is periodic with period 12.

PROBLEM 4.23 Let → f : A → B be any function where A is a set containing the positive integral solution of the inequality $\operatorname{cosec}^{-1}(\operatorname{cosec} 2) > x^2 - 3x$ and B is the set of all divisors of the natural number 2010. If f(i) ≤ f(j) ∀ i < j, then find the total number of mappings from A to B.

SOLUTION We have $\operatorname{cosec}^{-1}\operatorname{cosec} 2 > x^2 - 3x$

$\Rightarrow x^2 - 3x - (\pi - 2) < 0$

$$\Rightarrow \frac{3-\sqrt{1+4\pi}}{2} < x < \frac{3+\sqrt{1+4\pi}}{2}.$$

$\therefore$ Positive integral solutions of above inequality are 1, 2, 3.

Now 2010 = 2 × 3 × 5 × 67, total number of divisors of 2010 = 16.

So A contains 3 elements and B contains 16 elements. Number of mapping are as follows :

$f(1) < f(2) < f(3) \Rightarrow 16C_3 = 560$

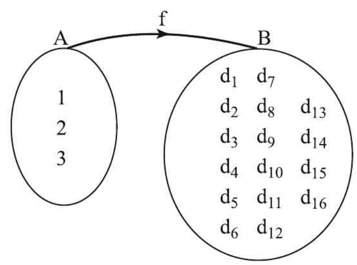
$f(1) = f(2) < f(3) \Rightarrow 16C_2 = 120$

$f(1) < f(2) = f(3) \Rightarrow 16C_2 = 120$

$f(1) = f(2) = f(3) \Rightarrow 16C_1 = 16$

Hence, the total number of mappings = 816.

Alternative :



a_i's are in increasing order from i = 2 to 15.

Number of mappings from A to B such that f (i) ≤ f (j) ∀ i < j is equivalent to number of linear arrangement of

$$\underbrace{d_2, d_3, \ldots\ldots, d_{16}}_{\text{alike}}, \underbrace{1, 2, 3}_{\text{alike}}$$

$d_2, d_3, \ldots. d_{16}$ are divisors in increasing order.

$$= 18C_3 = \frac{(18)(17)(16)}{6} = 51 \times 16 = 816.$$

PROBLEM 4.24 Consider a function $f(x) = \frac{x+a}{x-1}$, $x \in R - \{1\}$ where a is a real constant. If f is not a constant function, find the following

(i) f^{-1}, is it exist (ii) the range of f

(iii) $f\left(\frac{1}{f(f(x))}\right) - f\left(f\left(f\left(\frac{1}{x}\right)\right)\right)$.

SOLUTION

(i) $y = f(x) = \frac{x+a}{x-1}, (x \neq 1)$

$$f'(x) = \frac{(x-1)-(x+a)}{(x-1)^2} = \frac{-(a+1)}{(x+1)^2}$$

since f is not constant hence $a \neq -1$.
Hence $f'(x)$ is either always increasing or decreasing.
Therefore f^{-1} is defined

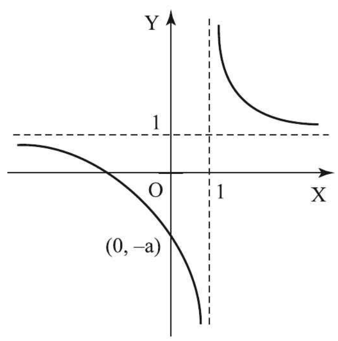
Now, $y = \dfrac{x+a}{x-1} \Rightarrow yx - y = x + a$

$\Rightarrow x(y-1) = y + a$

$$x = f^{-1}(y) = \frac{y+a}{y-1}$$

$$\Rightarrow f^{-1}(x) = \frac{x+a}{x-1}$$

(ii) The graph of f (for $a > 1$) is as shown.



Hence, the range is $R - \{1\}$

(iii) Again $f(x) = \dfrac{x+a}{x-1}$

$$f[f(x)] = \frac{f(x)+a}{f(x)-1} = \frac{\frac{x+a}{x-1}+a}{\frac{x+a}{x-1}-1}$$

$$= \frac{x+a+ax-a}{x+a-x+1} = \frac{x(a+1)}{a+1} = x \text{ (as } a \neq -1)$$

Now, $\dfrac{1}{(fof)(x)} = \dfrac{1}{x} = g(x)$, say

$$\therefore\ f(g(x)) = \frac{\frac{1}{x}+a}{\frac{1}{x}-1} = \frac{1+ax}{1-x} \qquad \ldots(1)$$

Similarly, $f\left(\dfrac{1}{x}\right) = \dfrac{\frac{1}{x}+a}{\frac{1}{x}-1} = \dfrac{1+ax}{1-x}$

$$f\left(f\left(\frac{1}{x}\right)\right) = \frac{\frac{1+ax}{1-x}+a}{\frac{1+ax}{1-x}-1} = \frac{1+ax+a-ax}{1+ax-1+x}$$

$$h(x) = f\left(f\left(\frac{1}{x}\right)\right) = \frac{1+a}{x(1+a)} = \frac{1}{x}$$

$$f(h(x)) = \frac{\frac{1}{x}+a}{\frac{1}{x}-1} = \frac{1+ax}{1-x} \qquad \ldots(2)$$

Hence from (1) and (2)

$$f\left(\frac{1}{f(f(x))}\right) - f\left(f\left(f\left(\frac{1}{x}\right)\right)\right) = 0.$$

PROBLEM 4.25 Let $f(x) = (x+1)(x+2)(x+3)(x+4) + 5$ where $x \in [-6, 6]$. If the range of the function is $[a, b]$ where $a, b \in N$ then find the value of $(a + b)$.

SOLUTION $f(x) = (x^2+5x+4)(x^2+5x+6)+5$

$= [(x^2+5x+5)-1]\,[(x^2+5x+5)+1]+5$

$= (x^2+5x+5)^2 - 1 + 5$

$f(x) = (x^2+5x+5)^2 + 4$

Hence $f(x)$ has a minimum value 4 when $x^2 + 5x + 5 = 0$.

i.e. when $x = \dfrac{-5 \pm \sqrt{5}}{2}$

Here $x = \dfrac{-(5+\sqrt{5})}{2} \in [-6, 6]$

Also maximum occurs at $x = 6$

$f(x)|_{max} = (36 + 30 + 5)^2 + 4 = (71)^2 + 4 = 5041 + 4 = 5045$

Thus, the range is $[4, 5045]$.

$\therefore\ a = 4;\ b = 5045 \Rightarrow a + b = 5049.$

PROBLEM 4.26 Find the period of the following functions :

(i) $f(x) = \text{sgn}(e^{-x}) + \sin x + |\sin x| + \min(\sin x, |x|)$

(ii) $\left[x+\frac{1}{2}\right]+\left[x-\frac{1}{2}\right] + 2[-x]$, where $[x]$ denotes the greatest integer function .

SOLUTION (i) $\text{sgn}(x) = \begin{cases} 1 & , x > 0 \\ 0 & , x = 0, \\ -1 & , x < 0 \end{cases}$

$\text{sgn}(e^{-x}) = 1 \qquad [\because e^{-x} > 0]$

$\text{sgn}(e^{-x})$ is constant function which is periodic with any length.

Period of $\sin x$ is 2π and period of $|\sin x|$ is π

$\therefore$ Period of $\sin x + |\sin x|$ is 2π.

$\min(\sin x, |x|) = \sin x \qquad (\because \sin x < |x|)$

Its period is 2π.

Hence $f(x) = 1 + 2\sin x + |\sin x|$ is periodic with period 2π.

(ii) Let $f(x) = \left[x+\frac{1}{2}\right]+\left[x-\frac{1}{2}\right] + 2[-x]$

If $x \in I$ then $f(x) = [x] + \left[\frac{1}{2}\right] + [x] + \left[-\frac{1}{2}\right]$

$+ 2(-[x]) = 0 - 1 = -1$

Similarly if $\notin$ x I then $f(x) = -3$.

With the help of graph, the function is periodic with period 1.

PROBLEM 4.27 Let l_1 be the line $4x + 3y = 3$ and l_2 be the line $y = 8x$. L_1 is the line formed by reflecting l_1 across the line $y = x$ and L_2 is the line formed by reflecting l_2 across the x-axis. If θ is the acute angle between L_1 and L_2 such that $\tan\theta = a/b$, where a and b are coprime then find $(a + b)$.

SOLUTION $l_1 : 4x + 3y = 3$

$$f(x) = y = \frac{3-4x}{3} \qquad ....(1)$$

Since $f(x)$ and $f^{-1}(x)$ are the mirror images of each other in the line $y = x$, hence we find $f^{-1}(x)$.

Now $y = f(x) \quad \Rightarrow \quad f^{-1}(y) = x$

from (1) $x = \frac{3(1-y)}{4}$; $\qquad f^{-1}(y) = \frac{3(1-y)}{4}$

$\therefore \; f^{-1}(x) = \frac{3(1-x)}{4} \Rightarrow 4y = 3 - 3x$

$L_1 = 3x + 4y - 3 = 0 \Rightarrow m_1 = -3/4$

Similarly $L_2 = y = -8x$ with $m_2 = -8$

If θ is the acute angle between the lines

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{-8+\frac{3}{4}}{1+(-8)\left(-\frac{3}{4}\right)}\right| = \left|\frac{-29}{28}\right|$$

$\Rightarrow \frac{29}{28}$

$\Rightarrow$ a = 29 and b = 28

$\therefore$ a + b = 29 + 28 = 57.

PROBLEM 4.28 Let A denote the set of all real numbers other than 0 and 1. A function $f : A \to R$ satisfies $f(x) + f\left(1-\frac{1}{x}\right) = \ln|x|$ for all $x \in A$, then find the value of $f(2009)$.

SOLUTION $f(x) + f\left(1-\frac{1}{x}\right) = \ell n|x| \qquad ....(1)$

Replacing x by $\frac{x-1}{x}$, we get

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = \ell n\left|\frac{x-1}{x}\right| \qquad ...(2)$$

Again replacing x by in (1), we get

$$f\left(\frac{1}{1-x}\right) + f(x) = \ell n\left|\frac{1}{1-x}\right| \qquad ...(3)$$

Adding (1) and (3) and substracting (2), we get

$$f(x) = \ell n\left|\frac{x}{1-x}\right|$$

$\therefore \; f(2009) = \ell n\left(\frac{2009}{2008}\right)$.

PROBLEM 4.29 Let $f : R - \{2\} \to R$ be a function satisfying the following functional equation,

$2f(x) + 3f\left(\frac{2x+29}{x-2}\right) = 100x + 80, \ \forall \ x \in R - \{2\}$. Determine f(x).

SOLUTION We have,

$$f(x) = \frac{3}{2} - f\left(\frac{2x+29}{x-2}\right) + 50x + 40 \qquad ...(1)$$

Replacing x by in the given functional equation we get,

$$f\left(\frac{2x+29}{x-2}\right) = -\frac{3}{2}f\left(\frac{2\left(\frac{2x+29}{x-2}\right)+29}{\left(\frac{2x+29}{x-2}\right)-2}\right)$$

$$+50\left(\frac{2x+29}{x-2}\right) + 40.$$

$$\Rightarrow \left(\frac{2x+29}{x-2}\right) = -\frac{3}{2}f(x) + 50\left(\frac{2x+29}{x-2}\right) + 40\, f\ldots \qquad (2)$$

Putting (2) in (1), we get,

$$f(x) = \frac{9}{4}\ f(x) - 75\left(\frac{2x+29}{x-2}\right) - 60 + 50x + 40$$

$$\Rightarrow \frac{9}{4}\ f(x) - f(x) = 20 - 50\,x + 75\left(\frac{2x+29}{x-2}\right)$$

$$\Rightarrow \frac{5}{4}\ f(x) = 20 - 50x + 75\left(\frac{2x+29}{x-2}\right)$$

$$\Rightarrow f(x) = 16 - 40x + 60\ \frac{(2x+29)}{(x-2)}.$$

PROBLEM 4.30 Prove that $f(n) = 1 - n$ is the only integer valued function defined on integers such that

(i) $f(f(n)) = n$ for all $n \in I$ and

(ii) $f(f(n + 2) + 2) = n$ for all $n \in I$ and

(iii) $f(0) = 1$.

SOLUTION The function $f(n) = 1 - n$ clearly satisfies conditions (i), (ii) and (iii). Conversely, suppose a function $f : I \to I$ satisfies (i), (ii) and (iii). Applying f to (ii) we get, $f(f(f(n + 2) + 2)\,)\,) = f(n)$ and this gives because of (i),

$$f(n + 2) + 2 = f(n), \qquad ...(1)$$

for all $n \in I$. Now using (1) it is easy to prove by induction on n that for all $n \in I$,

$$f(n) = \begin{cases} f(0) - n \text{ if } n \text{ is even} \\ f(1) + 1 - n \text{ if } n \text{ is odd} \end{cases}$$

Also by (iii), $f(0) = 1$. Hence by (i), $f(1) = 0$. Hence $f(n) = 1 - n$ for all $n \in I$.

PROBLEM 4.31 How many polynomials p(x) of degree atleast one an integer coefficient satisfy

$$16p(x^2) = (p(2x))^2, \text{ for all real numbers x ?}$$

SOLUTION Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a^1 x + a_0$ with $a_n \neq 0$, $n \geq 1$.

$$16p(x)^2 = (p(2x))^2$$

$$\Rightarrow 16(a_n x^{2n} + a_{n-1} x 2^{n-2} + ... + a_1 x^2 + a_0)$$

$$= (2^n a_n x^n + 2^{n-1a}{}_{n-1} x^{n-1} + ... + 2a_1 x + a_0)^2$$

Since the coefficients on both sides of the equality must agree, we must have

$$16an = 2^{2n} a^2{}_n \ \Rightarrow 2^4 = 2^{2n} a_n$$

Since $a_n \neq 0$. As a an is integer, we must have the followins $n = 1$, $a_n = 4$, $n = 2$, $a_n = 1$. Clearly we may not have $n \geq 3$. Thus, such polynomials are either linear or quadratic.

Also for $x = 0$, $16p(0) = (p(0)^2$ and therefore either $p(0) = 0$ or $p(0) = 16$.

For n = 1 we seek $p(x) = 4x + a$.

Solving $16(4x^2 + a) = (8x + a)^2$, we get $a = 0$

Therefore, $p(x) = 4x$.

For n = 2, let $p(x) = x^2 + ax + b$.

Solving $16(x^4 + ax^2 + b) = (4x^2 + 2ax + b)^2$, we get $a = 0$.

Since, $p(0) = 0$ or $p(0) = 16$, we must test $p(x) = x^2$ and $p(x) = x^2 + 16$. It is easy to see that only $p(x) = x^2$ satisfies the desired properties.

Therefore, $4x$ and x^2 are the only two polynomials.

PROBLEM 4.32 Suppose that f(x) is a function satisfying $|f(m + n) - f(m)| \le \frac{n}{m}$ for all positive rational number m and n. Prove that, for all natural numbers k,

$$\sum_{i=1}^{k} \left|f(2^k) - f(2^i)\right| \le \frac{k(k-1)}{2}.$$

SOLUTION Let $m = n = 2i$,

$$|f(2^{i+1}) - f(2^i)| \le 1$$

$$|f(2^k) - f(2^i)| \le |f(2^k) - f(2^{k-1})| + |f(2^{k-1} - f(2^{k-2})| \;..... \;| |f(2^{i+1}) - f(2^i)| \le k - i$$

Hence $\sum_{i=1}^{k} \left|f(2^k) - f(2^i)\right| \le \sum_{i=1}^{k} (k-i) = \frac{k(k-1)}{2}$.

PROBLEM 4.33 In each of the following cases, find all polynomials with real coefficients which satisfy the equation.

(a) $f(x^2 + x) = f(x)\, f(x + 1)$

(b) $f(g(x)) = f(x)\, g(x)$.

SOLUTION (a) First note that if x is a root of f(x) then so is $x^2 + x$. Similarly, if $x + 1$ is a root of f(x) then so is x2 + x. The latter statement is equivalent to saying that if x is a root of f(x) then

$$(x - 1)^2 + (x - 1) = x^2 - x$$

is also a root. However, if $x \ne 0$ is any root of f(x) in the complex plane, then

$$\max(|x^2 + x|, |x^2 - x|) > |x|.$$

Thus, if there were a nonzero root, we would be able to generate an infinite sequence of distinct roots. This could only happen if f(x) were a constant polynomial with f(x) = 0 or f(x) = 1.

The only other possibility is that f(x) is not constant, and 0 is the only root. In this case, $f(x) = x^k$ for some k.

(b) Examining the degree of each polynomial, we observe that

$$\deg f \cdot \deg g = \deg f + \deg g.$$

This is satisfied in nonnegative integers if both degrees are zero or both degrees are two. In the first case, we must have

$$f(x) = c,\ g(x) = 1.$$

where the constant c is arbitary.

When the degrees of both polynomials are two, we must have some point x in the complex plane such that $g(x) = 0$. Then

$$f(0) = f(g(x)) = f(x)\, g(x) = 0.$$

This implies that f(x) can be written in the form $f(x) = cx\,(x + a)$ for some constant a. It is easily checked that $f(x) = c\,(x^2 + ax)$

$$g(x) = x^2 + ax - a$$

is the general solution where a and c are arbitrary.

Things to REMEMBER

1. Two functions f(x) and g(x) are equal functions, if :
   (i) Domain of f (x) = Domain of g(x) = X
   (ii) $f(x) = g(x)$ for all $x \in X$

2.
   (i) $(f + g)(x) = f(x) + g(x)$ for all $x \in D_1 \cap D_2$
   (ii) $(f - g)(x) = f(x) - g(x)$ for all $x \in D_1 \cap D_2$
   (iii) $(fg)(x) = f(x)\, g(x)$ for all $x \in D_1 \cap D_2$

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for all $x \in D_1 \cap D_2 - \{x|g(x) \neq 0\}$

3. Methods of finding range :
   (i) Using concepts of (a) quadratic functions, (b) A.M.–G.M. inequality, and (c) trigonometric functions.
   (ii) The set of y–coordinates of the graph of a function is the range.
   (iii) Let $y = f(x) \Rightarrow x = g(y)$. Write conditions , if any, on transformation from $y = f(x)$ to $x = g(y)$ and al so write condition on 'y' so that g(y) produces real values of x lying in given domain.
   (iv) Find out all the critical points of f(x) in (a, b) ie. interior points where f'(x) = 0 or it does not exist.
   Let $M = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$
   and $m = \min\{f(a), f(c), f(c_2), \ldots, f(c_n), f(b)\}$
   Thus, the range is [m, M].
   (v) Range of a composite function is evaluated from inside to outside. This means that we need to evaluate innermost function and then the one outside it.
4. Tests for one-one and many-one functions
   (i) Solving $f(x_1) = f(x_2)$
   (ii) Horizontal Line Test
   (iii) If the function is either strictly increasing or strictly decreasing then it is one-one.
   (iv) If possible, find an element in the co-domain which is image of more than one element of the domain, then the function is many-one.
5. Tests for onto and into functions
   (i) A function f is onto if range of f = co-domain of f. It is into if range of f $\subset$ co-domain of f
   (ii) If possible, find an element in the co-domain which is not an image of any element of the domain, then function is into.
6. Let us consider mappings from set A to set B where they contain m and n elements respectively. No of ordered pairs in $A \times B = mn$.
   (i) No. of relations from A to B = 2^{mn}
   (ii) No. of functions = n^m
   (iii) No. of one-one functions $= {}^nP_m, n \geq m$
   (iv) No. of bijective function = n! if m = n
   (v) No. of onto functions
   $= \sum_{r=1}^{n} (-1)^{n-r} \cdot {}^nC_r \cdot r^m, \ 1 \leq n \leq m.$
7. Algebra of even/ odd functions

f(x) =	odd,	odd,	even
g(x) =	odd,	even,	even
f(x) ± g(x) =	odd,	neither,	even
f(x) g(x) =	even,	odd,	even
f(x)/g(x) =	even,	odd,	even
fog(x) =	odd,	even,	even

8. Properties of even / odd functions
   (i) A function which is both even and odd is the zero function. $f(x) = 0 \ \forall \ x \in R$.
   (ii) A non-zero constant function defined on R is an even function.
   (iii) A function whose domain is symmetric about origin can be written uniquely as sum of an even and odd function.
   $$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$
   (iv) An even function is many-one.
   (v) If x = 0 lies in the domain of an odd function

then f(0) = 0 i.e. graph must pass through origin.

(vi) Even extension : let f(x) be defined in [0, a], then, even extension is defined as

$$g(x) = \begin{cases} f(x); & 0 \le x \le a \\ f(-x); & -a \le x < 0 \end{cases}$$

(vii) Odd extension $g(x) = \begin{cases} f(x); & 0 \le x \le a \\ -f(-x); & -a \le x < 0 \end{cases}$

9.

(i) If T is the period of f(x), then period of function y = pf (ax + b) + q; a, b, p, q ∈ R is T/|a|

(ii) |sin x|, |cos x|, |cosec x|, |sec x|, |tan x| and |cot x| are periodic functions with period π.

(iii) Functions $\sin^n x$; $\cos^n x$; $\text{cosec}^n x$ and $\sec^n x$ are periodic with period π when n is even and 2π when n is odd.

On the other hand, tannx and $\cot^n x$ are periodic with period π whether n is even or odd.

(iv) If f is periodic with period T, then $\frac{1}{f(x)}$ is also periodic with period T.

(v) If f is periodic with period T then $(f(x))^{1/n}$ is also periodic with period T.

(vi) If f(x) is a periodic function with period T_1 and g(x) is a periodic function with period T_2, then the function f(x) + g(x) is also periodic with period T which is the L. C. M. of T_1 & T_2, provided f(x) and g(x) can not be interchanged by adding a positive number in x less than L.C.M. of T_1 & T_2. In the case when f(x) and g(x) get interchanged by adding a positive number in x less than L.C.M. of T_1 & T_2, then this number is the period of f(x) + g(x).

(vii) Consider the function h(x)

$$= \frac{f_1(x)............f_n(x)}{g_1(x)............g_m(x)},$$

where f_i and g_j are periodic for all i and j. Let their periods be T_i and T_j.

If the LCM of all T_i and T_j exists, then the function h(x) is periodic. However, the LCM need not be the fundamental period. To get the fundamental period, we need to check the LCM using

(a) graph of h(x), or,

(b) transformation of h(x) into sum of functions, or,

(c) trial of (LCM)/n using definition

(viii) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods. Further if g is a strictly increasing or decreasing function in the range of f, then T is the fundamental period of gof.

10. Some points of intersection of f and f^{-1} lie on the line y = x. However, it must be noted that f(x) and $f^{-1}(x)$ may intersect elsewhere also.

11. If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x,\ k \in R$.

(ii) $f(xy) = f(x) . f(y) \Rightarrow f(x) = 0$ or x^n, $n \in R$

(iii) $f(x + y) = f(x) . f(y) \Rightarrow f(x) = 0$ or a^x, $a > 0$

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx,\ k \in R$.

(v) Let a polynomial function f satisfy

$$f(x) + f\left(\frac{1}{x}\right) = f(x) . \left(\frac{1}{x}\right) f \ \forall x \in R - \{0\}$$

then $f(x) = \pm x^n + 1$, where $n \in W$.

OBJECTIVE EXERCISES

SINGLE CORRECT ANSWER TYPE

1. A value of 'a' for which the range of the function f(x) = [x [x]]. ([.] represent the G. I. F.), $x \in [0, a]$ has exactly 7 integers is
(A) 5.5 (B) 4.75 (C) 3.75 (D) None

2. The range of the function $f(x) = \sqrt{\sin(\cos x)} + \sqrt{\cos(\sin x)}$ is
(A) $\left[1, 1+\sqrt{\cos 1}\right]$ (B) $\left[\sqrt{\cos 1}, 1+\sqrt{\cos 1}\right]$ (C) $\left[\sqrt{\cos 1}, 1+\sqrt{\sin 1}\right]$ (D) $\left[1, 1+\sqrt{\sin 1}\right]$

3. The domain of the fucntion $f(x) = \sqrt{\log_{1/2}(\log_5([x^2]-3))}$ ([.] denotes greatest integer function):
(A) $[-3, -2] \cup [2, 3]$ (B) $(-3, -2] \cup [2, 3)$
(C) $(-3, -\sqrt{5}\,] \cup [\sqrt{5}, 3]$ (D) $(-2\sqrt{2}\,.\,-2) \cup (2, 2\sqrt{2}\,)$

4. Let $f(x) = \sqrt{\cot(5+3x)\ \ (\cot(5)+\cot(3x)) - \sqrt{\cot 3x} + 1}$, then domain is
(A) $R - \left\{\frac{n\pi}{3}\right\}, n \in I$ (B) $(2n+1)\frac{\pi}{6}, n \in I$
(C) $R - \left\{\frac{n\pi}{3}, \frac{n\pi-5}{3}\right\}, n \in I$ (D) $R - \left\{\frac{n\pi-5}{3}\right\}, n \in I$

5. The range of function $f : [0,1] \to R$, $f(x) = x^3 - x^2 + 4x + 2\sin^{-1} x$ is
(A) $[-\pi - 2, 0]$ (B) $[2, 3]$ (C) $[0, 4 + \pi]$ (D) $(0, 2 + \pi]$

6. The domain of definition of the function $f(x) = \log\left(\sqrt{10.3^{x-2} - 9^{x-1} - 1}\right) + \sqrt{\cos^{-1}(1-x)}$ is
(A) [0, 1] (B) [1, 2] (C) (0, 2) (D) (0, 1)

7. The domain of $f(x) = P(x)C_{Q(x)}$, where $P(x) = 19x - 9 - 2x^2$ and $Q(x) = \sqrt{x-4}$ is
(A) [4, 9] (B) [0, 9] (C) {1, 2, ..., 8,} (D) {4, 5, 8}

8. Let $f(x) = \sin^{23}x - \cos^{22}x$ and $g(x) = 1 + \frac{1}{2}\tan^{-1}|x|$, then the number of values of x in interval $[-10\pi, 20\pi]$ satisfying the equation f(x) = sgn (g(x)), is
(A) 6 (B) 10 (C) 15 (D) 20

9. If f (x) is defined on domain [0, 1] then f (2 sin x) is defined on
(A) $\bigcup_{n \in I} \left\{\left[2n\pi, 2n\pi+\frac{\pi}{6}\right] \cup \left[2n\pi+\frac{5\pi}{6}, (2n+1)\pi\right]\right\}$
(B) $\bigcup_{n \in I} \left[2n\pi,\ 2n\pi+\frac{\pi}{6}\right]$

(C) $\bigcup\limits_{n \in I} \left[2n\pi + \frac{5\pi}{6}, (2n+1)\pi\right]$

(D) None of these

10. The range of the function, $f(x) = \sqrt{-x^2+4x-3} + \sqrt{\frac{\sin \pi}{2}\left(\frac{\sin \pi}{2}(x-1)\right)}$ is

(A) [0, 2] (B) [0, 1] (C) [1, 4] (D) None of these

11. If $f(x) = \log_{e^2 x}\left(\frac{2\ln x + 2}{-x}\right)$ and g(x) = {x} then range of g(x) for the existence of f(g(x)) is

(A) $\left(0, \frac{1}{e}\right) - \left\{\frac{1}{e^2}\right\}$ (B) $\left(0, \frac{2}{e}\right) - \left\{\frac{1}{e^2}\right\}$ (C) $\left(0, \frac{3}{e}\right) - \left\{\frac{1}{e^2}\right\}$ (D) none of these

12. The number of one-one function f : {1, 2, 3, ..., 10} → {1, 2, 3,......,10} such that $f(i) \neq i$ but f(f(i)) for each i = 1, 2, 3,......,10 is

(A) 945 (B) 315 (C) 675 (D) None of these

13. A linear function f whose composition $f_0 f_0$ ....$_0$f (with f applied six times) is equal to 2x – 1 for all x is

(A) $2^{\frac{1}{6}}x - 2^{\frac{1}{6}}$ (B) $2^{\frac{1}{5}}x - 2^{\frac{1}{6}}$ (C) $2^{\frac{1}{6}}x - 2^{\frac{1}{6}} + 1$ (D) None of these

14. If $f(x) = \frac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x}$, then the fundamental period of f(x) is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) π (D) none of these

15. If the two roots of the equation $(c-1)(x^2+x+1)^2 - (c+1)(x^4+x^2+1) = 0$ are real and distinct and $f(x) = \frac{1-x}{1+x}$, then $f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right) =$

(A) – c (B) c (C) 2c (D) none of these

16. Let f : R → R be a given function and $A \subset R$ and $B \subset R$ then

(A) $f(A \cup B) = f(A) \cup f(B)$ (B) $f(A \cap B) = f(A) \cap f(B)$

(C) $f(A^c) = [f(A)]^c$ (D) None of these

17. Let $f_1(x) = \begin{cases} x & \text{for} \quad 0 \le x \le 1 \\ 1 & \text{for} \quad x > 1 \\ 0 & \text{otherwise} \end{cases}$

and $f_2(x) = f_1(-x)$ for all x

$f_3(x) = -f_2(x)$ for all x

$f_4(x) = f_3(-x)$ for all x.

Which of the following is necessarily true ?

(A) $f_4(x) = f_1(x)$ for all x (B) $f_1(x) = -f_3(-x)$ for all

(C) $f_2(-x) = f_4(x)$ for all x (D) $f_1(x) + f_3(x) = 0$ for all

18. If f(x) is even, periodic function defined for all $x \in R$ and has period 1then

(A) $f\left(x+\frac{1}{2}\right) = f(x)$ (B) $f\left(\frac{1}{3}+x\right) = f\left(\frac{2}{3}-x\right)$

(C) $f(x+1) = f(2x+1)$ (D) f(0) can not be zero

19. If $f : [a, \infty) \to R$ is defined as $f(x) = x + \frac{1}{x}$ then it is injective when

(A) $a \le -1$ (B) $0 < a \le 1$

(C) $a \ge 1$ (D) $a \in [-1, 0) \cup (0, 1)$

20. If $f(x) = \tan^{-1}x + \cot^{-1}x + \sin(3x) + \cos\left(\frac{2x}{3}\right)$, then

(A) f(x) is periodic with period 2π (B) f(x) is periodic with period $\frac{2\pi}{3}$

(C) f(x) is periodic with period 6π (D) f(x) is non-periodic

21. If $f : (-\infty, a] \to [b, \infty)$ defined as $f(x) = x^2 - 6x + 10$ is a bijective function and let the maximum value of a be α and the minimum value of b be β, then the ordered pair (α, β) is

(A) (–3, 1) (B) (–3, 37) (C) (3, 1) (D) (1, 3)

22. The number of functions *f* from {1, 2,.... 20} onto {1, 2,......,20} such that *f*(k) is a multiple of 3 whenever k is a multiple of 4 is

(A) 5! . 6! . 9! (B) 5^9 . 15! (C) 6^5 . 14! (D) 15! . 6!

23. The period of $\cos(x + 4x + 9x + \ldots\ldots\ldots + n^2 x)$ is $\pi/7$, then $n \in N$ is equal to

(A) 2 (B) 3 (C) 4 (D) 5

24. The number of bijective functions $f : A \to A$, where A = {1, 2, 3, 4} such that $f(1) \ne 3$, $f(2) \ne 1$, $f(3) \ne 4$, $f(4) \ne 2$ is

(A) 11 (B) 23 (C) 12 (D) 9

25. Let $f(x) = \begin{cases} 4 , & x < -1 \\ -4x , & -1 \le x \le 0 \end{cases}$ If f(x) is an even function on R then the definition of f(x) in $(0, \infty)$ is

(A) $f(x) = \begin{cases} 4x , & 0 < x \le 1 \\ 4 , & x > 1 \end{cases}$ (B) $f(x) = \begin{cases} 4x , & 0 < x \le 1 \\ -4 , & x > 1 \end{cases}$

(C) $f(x) = \begin{cases} 4 , & 0 < x \le 1 \\ 4x , & x > 1 \end{cases}$ (D) none of these

26. If $f(x) = -\frac{x|x|}{1+x^2}$ then $f^{-1}(x)$ equals

(A) $\sqrt{\frac{|x|}{1-|x|}}$ (B) $(\text{sgn } x)\sqrt{\frac{|x|}{1-|x|}}$ (C) $-\sqrt{\frac{x}{1-x}}$ (D) none of these

27. Let $f : R \to R$, $g : R \to R$ be two given function such that f is injective and g is surjective then which of the following is injective

(A) gof (B) fog (C) gog (D) fof

28. $f(x) = \left\{\frac{x^{2n}}{(x^{2n}\,\text{sgn}\,x)^{2n+1}}\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right)\right\}$, $x \ne 0$ and $n \ne N$ then f(x) is a/an

(A) even function (B) odd function

(C) one-one function (D) invertible function.

29. Let M be the set of all 2 × 2 matrices with entries from the set of real numbers, R. Then the function $f : M \to R$ defined $f(A) = |A|$ for every $A \in M$, is
(A) one-one and onto
(B) neither one-one nor onto
(C) one-one but not onto
(D) onto but not one-one

30. If $f(x) = 2 \tan 3x + 5\sqrt{1-\cos 6x}$ and g(x) is a function having the same period as that of f(x), then which of the following can be g(x) ?
(A) $(\sec^2 3x + \text{cosec}^2 3x)$
(B) $2 \sin 3x + 3 \cos 3x$
(C) $2\sqrt{1-\cos^2 3x} + \text{cosec}\, 3x$
(D) none of these

31. The function f is not defined for x = 0, but for all non zero real numbers x, $f(x) + 2f\left(\frac{1}{x}\right) = 3x$. The equation $f(x) = f(-x)$ is satisfied by
(A) exactly one real number
(B) exactly two real numbers
(C) no real numbers
(D) all non zero real numbers.

32. Given $f(x) = \sqrt{\frac{8}{1-x} + \frac{8}{1+x}}$ and $g(x) = \frac{4}{f(\sin x)} + \frac{4}{f(\cos x)}$ then g(x) is
(A) periodic with period $\pi/2$
(B) periodic with period π
(C) periodic with period 2π
(D) aperiodic

33. Let $g : R \to \left(0, \frac{\pi}{3}\right]$ is defined by $g(x) = \cos^{-1}\left(\frac{x^2-k}{1+x^2}\right)$. Then the possible values of 'k' for which g is surjective function, is
(A) $\left\{\frac{1}{2}\right\}$
(B) $\left(-1, -\frac{1}{2}\right]$
(C) $\left\{-\frac{1}{2}\right\}$
(D) $\left[-\frac{1}{2}, 1\right)$

34. If $f(x) = x^2 + bx + c$ and $f(2+t) = f(2-t)$ for all real numbers t, then which of the following is true?
(A) $f(1) < f(2) < f(4)$
(B) $f(2) < f(1) < f(4)$
(C) $f(2) < f(4) < f(1)$
(D) $f(4) < f(2) < f(1)$

35. The period of the function, $f(x) = [x] + [2x] + [3x] + \ldots\ldots + [nx] - \frac{n(n+1)}{2}x$, where $n \in N$ and [ ] denotes the greatest integer function, is
(A) 1
(B) n
(C) $\frac{1}{n}$
(D) non periodic

36. The minimum value of the function

$$f(x) = \frac{\sin x}{\sqrt{1-\cos^2 x}} + \frac{\cos x}{\sqrt{1-\sin^2 x}} + \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\text{cosec}^2 x - 1}}$$ is

(A) 4
(B) – 2
(C) 0
(D) 2

37. Let set A consists of 5 elements and set B consists of 3 elements. Number of functions that can be defined from A to B which are neither injective nor surjective, is
(A) 99
(B) 93
(C) 123
(D) none

38. Let $f(x) = x$ and $g(x) = |x|$ for all $x \in R$. Then the function $\varphi(x)$ satisfying $[\varphi(x) - f(x)]^2 + [\varphi(x) - g(x)]^2 = 0$ is
(A) $\varphi(x) = x, x \in [0, \infty)$
(B) $\varphi(x) = x, x \in R$
(C) $\varphi(x) = x, x \in (-\infty, 0]$
(D) $\varphi(x) = x + |x|, x \in R$

39. Define the function f (n) where n is a non negative integer satisfying f (0) = 1 and f (n) is defined for $n > 0$ as $f(n) = n\cdot\sum_{i=0}^{n-1} f(i)$. Let 2^m be the highest power of 2 that divides f (20). The value of m is
(A) 18 (B) 19 (C) 20 (D) 21

40. If for all x different from both 1 and 0 we have $f1(x) = \frac{x}{x-1}$, $f_2(x) = \frac{1}{1-x}$, and for all integers $n \geq 1$, we have $f_{n+2}(x) = \begin{cases} f_{n+1}(f_1(x)) & \text{if n is odd} \\ f_{n+1}(f_2(x)) & \text{if n is even} \end{cases}$ then f4(x) equals
(A) x (B) x – 1 (C) $f_1(x)$ (D) $f_2(x)$

41. If f(x) satisfies x + |f(x)| = 2 f(x) then f^{-1} (x) satisfies
(A) $3x + |f^{-1}(x)| = 2f^{-1}(x)$
(B) $x + |f^{-1}(x)| = 2f^{-1}(x)$
(C) $f^{-1}(x) - |x| = 2x$
(D) $3x - |f^{-1}(x)| = 2f^{-1}(x)$

42. The period of $f(x) = \frac{|\tan x| + |\cot x|}{|\sin x - \cos x| + |\sin x - \cos x|}$ is
(A) 2π (B) π (C) $\pi/2$ (D) none of these

43. If f(x) + f(x + a) + f(x + 2a) + ... + f(x + na) = constant $\forall$ x $\in$ R, and a $\neq$ 0 and f(x) is periodic, then period of f(x) is
(A) (n + 1) a (B) $e^{(x+1)a}$ (C) na (D) e^{na}

44. If the period of $\frac{\cos(\sin(nx))}{\tan(x/n)}$, n $\in$ N is 6π, then
(A) 3 (B) 2 (C) 6 (D) 1

45. The range of $f(x) = [1 + \sin x] + \left[2 + \sin\frac{x}{2}\right] + \left[3 + \sin\frac{x}{3}\right] + ... + \left[x + \sin\frac{x}{n}\right]$ $\forall$ x $\in$ [0, π], where [.] denotes the greatest integer function, is
(A) $\left\{\frac{n^2+n-2}{2}, \frac{n(n+1)}{2}\right\}$
(B) $\left\{\frac{n(n+1)}{2}\right\}$
(C) $\left\{\frac{n^2+n-2}{2}, \frac{n(n+1)}{2}, \frac{n^2+n+2}{2}\right\}$
(D) $\left\{\frac{n(n+1)}{2}, \frac{n^2+n+2}{2}\right\}$

46. Let f : R+ $\to$ R be a function defined by $f(x) = (l nx)^3 + (l nx)^2 + 3(l nx) + \sin^2 x$. Then f is
(A) one - one and onto
(B) one - one and into
(C) many one and onto
(D) many one and into

47. The domain of $f(x) = \ln(ax^3 + (a + b)x^2 + (b + c)x + c)$, where a > 0, $b^2 - 4ac = 0$, is
(A) $(-1, \infty) - \left\{-\frac{b}{2a}\right\}$
(B) $(1, \infty) - \left\{-\frac{b}{2a}\right\}$
(C) $(-1, 1) - \left\{-\frac{b}{2a}\right\}$
(D) none of these

48. The range of the function f: R $\to$ R, $f(x) = \frac{\left(\sqrt{x^2+1} - 3x\right)}{\sqrt{x^2+1} + x}$ is
(A) $(0, \infty)$ (B) $(-1, \infty)$ (C) $(-\infty, -1)$ (D) None of these

49. If P(x) be a polynomial satisfying the identity $P(x^2) + 2x^2 + 10x = 2x\,P(x + 1) + 3$, then P(x) is
(A) $2x + 3$ (B) $3x - 4$ (C) $3x + 2$ (D) $2x - 3$

50. $f(x - 1) + f(x + 1) = \sqrt{3}\,f(x)$, $\forall\, x \in R$ if $f(2) = 9$, then the value of $\sum_{r=0}^{9} f(2 + 12r)$ is
(A) 80 (B) 90 (C) 100 (D) 110

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

51. Suppose that the functions f (x) and g (x) satisfy the system of equations $f(x) + 3g(x) = x^2 + x + 6$
$2f(x) + 4g(x) = 2x^2 + 4$ for every x.
Find the value of 'x' for which f (x) = g (x) can be equal to
(A) – 2 (B) 2 (C) – 5 (D) 5

52. Let $f : \left[-\frac{\pi}{3}, \frac{2\pi}{3}\right] \to [0, 4]$ be a function defined as $f(x) = \sqrt{3}\sin x - \cos x + 2$. then $f^{-1}(x)$ is given by
(A) $\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}$ (B) $\sin^{-1}\left(\frac{x-2}{2}\right) + \frac{\pi}{6}$
(C) $\frac{2\pi}{3} - \cos^{-1}\left(\frac{x-2}{2}\right)$ (D) None of these

53. Let $f(x) = \max\{1 + \sin x, 1, -\cos x\}$, $x \in [0, 2\pi]$ and $g(x) = \max\{1, |x - 1|\}$ $x \in R$, then
(A) $g(f(0)) = 1$ (B) $g(f(1)) = 1$ (C) $f(g(1)) = 1$ (D) $f(g(0)) = \sin 1$

54. Let a function f defined on the set of all integers satisfy $f(0) \neq 0$, $f(1) = 3$ and $f(x)\,.\,f(y) = f(x + y) + f(x - y)$ for all integers x and y. Then
(A) $f(2) = 7$ (B) $f(3) = 21$ (C) $f(4) = 47$ (D) $f(7) = 843$

55. The real function f has the property than whenever $a, b, n \in N$ such that $a + b = 2^n$, the equation $f(a) + f(b) = x^2$ holds. Then
(A) $f(2) = 2$ (B) $f(14) = 14$ (C) $f(18) = 11$ (D) $f(46) = 25$

56. P(x) is a polynomial function satisfying $xP(x - n) = (x - 1)P(x)$ for some $n \in N$ and for all $x \in R$, then P(x) may be
(A) $x^2 + 2x$ (B) $2x$ (C) x^3 (D) $-2x$

57. If $f(x) = \sin(\{x\} + \sin ax)$ is periodic with period '1' then 'a' may be equal to (where {x} denotes fractional part of x):
(A) 0 (B) 2π (C) 4π (D) π

58. The domain of definition of the function, $f(x) = [2\tan\pi x]^{\log_{[2\tan\pi x]}\left(\frac{x^2+2x-3}{4x^2-4x-3}\right)}$ where [ ] denotes the greatest integer function is $\left[n + \frac{1}{4}, n + \frac{1}{2}\right)$; $n \in I$ then
(A) $n = 0$ (B) $n \leq 4$ (C) $n \geq 4$ (D) none of these

59. Which of the following is an odd function
(A) $(g(x) - g(-x))$ (B) $(g(x) - g(-x))^3$
(C) $\log\left(\frac{x^4 + x^2 + 1}{x^2 + x + 1}\right)$ (D) $xg(x).g(-x) + \tan(\sin x)$

60. A bijective function from R to R is given by
(A) e^x sgn x (B) $\cot^{-1}$ (cot 2x) (C) x |x| (D) ln $e^{x + \sin x}$

61. For the function $f(x) = \frac{\sin^2 x}{\sin x - \cos x} - \frac{\sin x + \cos x}{\tan^2 x - 1}$, $x \neq n\pi \pm \frac{\pi}{4}$, $n \in I$ which of the following does not hold good?
(A) f is even
(B) f is odd
(C) f is neither even nor odd but periodic with period 2π
(D) f is neither even nor odd and is aperiodic.

62. Which of the following functions have the same period?
(A) $f(x) = \sin^2 x + \cos^4 x + 2$
(B) $g(x) = \frac{1}{f(\sin x)} + \frac{1}{f(\cos x)}$ where $f(x) = \frac{1}{\sqrt{1-x^2}}$
(C) $h(x) = \frac{|\sin x| + |\cos x|}{|\sin x - \cos x|}$
(D) k (x) = cos(cos x) + cos(sin x)

63. Let $f(x) = x^3 - 2x$, $g(x) = x^3 - x|x|$, $h(x) = 3x^2 - |x| + 6$, $k(x) = x^4 - 2x + |x + 1|$. Which of the following functions is either odd or even?
(A) kog (B) koh (C) fog (D) fohog

64. Which of the following functions are aperiodic (where [x] denotes greatest integer function) ?
(A) $f(x) = 2 + (-1)^{[x]}$ (B) $f(x) = \frac{1}{x}\cos x$
(C) $f(x) = \tan\left(\frac{\pi}{4}(x - [x])\right)$ (D) f (x) = x + sin x

65. A period of the function, $f(x) = x + a - [x + b] + \sin \pi x + \cos 2\pi x + \sin 3\pi x + \cos 4\pi x + \ldots\ldots + \sin (2n - 1)\pi x + \cos 2n\pi x$ for every $a, b \in R$ is
(A) 2 (B) 4 (C) 1 (D) 0

66. If f: R → R is defined as $f(x) = e^{|x|} - e^{-x}$, then the correct statement(s) is / are
(A) f is one - one onto function (B) f is many one into function
(C) range of f is $[0, \infty)$ (D) range of f is $(-\infty, \infty)$

67. The function $f : (2, \infty) \to [8, \infty)$ defined by $f(x) = \frac{x^2}{x-2}$ is
(A) one to one function (B) many-one function
(C) into function (D) onto function

68. Which of the following statements are incorrect ?
(A) If f(x) and g(x) are one-one then f(x) + g(x) is also one-one.
(B) If f(x) and g(x) are one-one then f(x). g(x) is also one-one.
(C) If f(x) is odd then it is necessarily one-one.
(D) none of these

69. Let f : [– 1, 1] onto [3, 5] be a linear polynomial. Which of the following can be true?
(A) $f\left(\frac{-1}{2}\right) = \frac{7}{2}$ (B) $f^{-1}\left(\frac{15}{4}\right) = \frac{1}{4}$ (C) $f(0) \neq 4$ (D) $f\left(\frac{1}{2}\right) + f\left(\frac{-1}{2}\right) = 8$

70. Let $f : R \to R$ defined by $f(x) = \min.(|x|, 1-|x|)$

Then which of the following hold(s) good?

(A) Range of f is $(-\infty, 1]$
(B) f is aperiodic.
(C) f is neither even nor odd.
(D) f is neither injective nor surjective.

Comprehension - 1

Consider the function $f(x) = \sqrt{\dfrac{x+1}{x-1}}$.

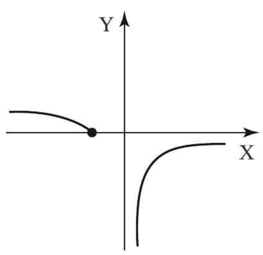
71. What is the domain of definition of f(x)?

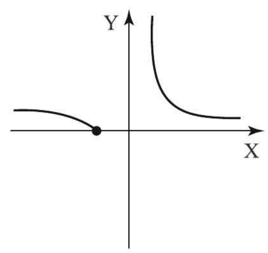
(A) $[-1, 1)$ (B) $[-1, 1]$ (C) $(-\infty, -1] \cup [1, \infty)$ (D) $(-\infty, -1] \cup (1, \infty)$

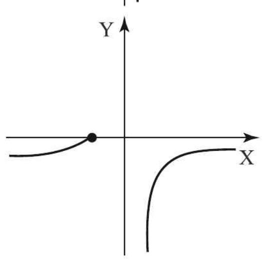
72. $f-1(x) =$

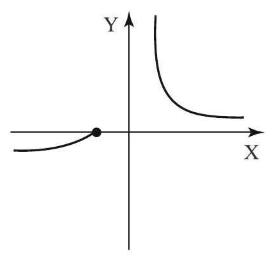
(A) $\dfrac{1-x^2}{1+x^2}$ (B) $\left(\dfrac{1+x}{1-x}\right)^2$ (C) $\dfrac{1+x^2}{1-x^2}$ (D) $\dfrac{1+x^2}{x^2-1}$

73. The graph of f most resembles :

(a) 

(b) 

(c) 

(d) 

(A) a (B) b (C) c (D) d

Comprehension - 2

For $x \neq 0, 1$, define $f_1(x) = x$, $f_2(x) = \dfrac{1}{x}$, $f_3(x) = 1 - x$, $f_4(x) = \dfrac{1}{1-x}$, $f_5(x) = \dfrac{x-1}{x}$, $f_6(x) = \dfrac{x}{x-1}$

This family of functions is closed under composition, that is the composition of any two of these functions is again one of these.

74. Let F be a function such that $f_1 oF = f_4$, then F is equal to

(A) f_1 (B) f_2 (C) f_3 (D) f_4

75. Let G be a function such that $Gof_3 = f_6$. Then G is equal to

(A) f_5 (B) f_4 (C) f_3 (D) f_2

76. Let J be a function such that $f_3oJof_2 = f_4$. Then J is equal to
(A) f_6 (B) f_5 (C) f_4 (D) f_3

Comprehension - 3

Let f(n) be the number of ones that occur in the decimal representation of all the number from 1 to n. There is a pattern in the function $f(10^n)$: $f(10) = 2$, $f(10^2) = 21$, $f(10^3) = 301$,...... $f(10^n) = n.10^n - 1 + 1$.

77. The value of f(110) is
(A) 2 + 1 (B) 30 (C) 33 (D) None of these

78. The value of $\sum_{r=1}^{n} f(10^r)$ is
(A) $\frac{1}{9}\left(n.10^n + n - \frac{10^n - 1}{9}\right)$ (B) $\frac{1}{9}\left(n.10^n - n + \frac{10^n - 1}{9}\right)$
(C) $\frac{1}{9}\left(n.10^n - n - \frac{10^n - 1}{9}\right)$ (D) None of these

79. The value of $\sum_{i=1}^{100} i\,.\,{}^{100}C_i\, 9^{100-i}$ is equal to
(A) $f(10^{100} - 1)$ (B) $f(10^{100}) - 1$ (C) $(f(10))^{100} - 1$ (D) None of these

Comprehension - 4

Let $f(x) = x^3 + x + 1$. Suppose g is a cubic polynomial such that $g(0) = -1$ and the roots of g are square of the roots of f.

80. The equation f(x) = 0 has
(A) atleast one positive root (B) atleast two negative roots
(C) exactly one negative root (D) None of these

81. The polynomial $g(x^2)$ is identical with
(A) $f(x^2)$ (B) $(f(x))^2$ (C) $2f(x)\,f(-x)$ (D) $-f(x)\,f(-x)$

82. The value of g(9) is
(A) 889 (B) 899 (C) 961 (D) None of these

83. If $f : A \to B$ defined by $f(x) = \sqrt{3|x| - 2x - 1}$ is bijective function A & B respectively cannot be
(A) $A = \{x : x < -1\}, B = [0, \infty)$ (B) $A = \{x : x \le \frac{-1}{5}\}, B = [0, \infty)$
(C) $A = \{x : x < -5\}, B = [0, \infty)$ (D) $A = \{x : x \ge 1\}, B = [0, \infty)$

84. If f(x) is a polynomial of degree 2 such that $f(0) = 1$ & $f(x + 2) - f(x) = 4x + 2$ then the polynomial is
(A) $x^2 + x + 1$ (B) $(2x - 1)^2 + 3$
(C) $x^2 - 2x + 1$ (D) $\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$

Comprehension - 5

Let a function f(x) be defined as follows for $n \in W$, $f(0) = 0$ and for $n > 0$, $k \in W$,

$$f(x) = \begin{cases} f(n-1)+3 & \text{if} \quad n = 6k, 6k+1 \\ f(n-1)+1 & \text{if} \quad n = 6k+2, 6k+5 \\ f(n-1)+2 & \text{if} \quad n = 6k+3, 6k+4 \end{cases}$$

85. The value of f(30) is
(A) 54 (B) 60 (C) 66 (D) None of these

86. The function satisfies the rule for all $n \in W$
(A) $f(n+3) = f(n) + 6$ (B) $f(n+4) = f(n) + 8$ (C) $f(n+6) = f(n) + 6$ (D) $f(n+6) = f(n) + 12$

87. The value of $\sum_{n=1}^{30} f(n)$
(A) 900 (B) 930 (C) 960 (D) None of these

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

Consider the function

$f(x) = ({}^{x+1}C_{2x-8})\,({}^{2x-8}C_{x+1})$

87. **Assertion (A) :** Domain of f(x) is singleton.

Reason (R) : Range of f(x) is singleton.

89. **Assertion (A) :** Let $S_n = 1 - 2 + 3 - 4 + ... + (-1)^{n-1} n$. The value of $\sum_{n=1}^{2009} S_n = S_{2009}$

Reason (R) : $Sn = \begin{cases} \dfrac{n+1}{2} & \text{if } n \text{ is odd} \\ -\dfrac{n}{2} & \text{if } n \text{ is even} \end{cases}$

90. **Assertion (A) :** Range of the function $f(x) = \cot^{-1}\{-x\}$ where {x} denotes the fractional part function, is $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$

Reason (R) : Since $0 \le \{-x\} < 1$, cot–1{–x} would be vary from $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$

91. **Assertion (A) :** The function $f(x) = x^4 + 2x + 3$ defined from R to R is not injective.

Reason (R) : Every polynomial function of even degree defined from R to R is always not injective.

92. Let $g : R \to R$ defined by $g(x) = \{\cos \pi[x] + e^x\}$, where [x] denotes the greatest integer function and {x} denotes the fractional part function.

Assertion (A) : g(x) is a periodic function.

Reason (R) : {x} is a periodic function.

93. **Assertion (A) :** f is an even function, g and h are odd functions, all 3 are polynomials. Given f (1) = 0, f (2) = 1, f (3) = – 5, g (1) = 1, g (–3) = 2, g (5) = 3, h (1) = 3, h (3) = 5 and h (5) = 1. The value of f(g(h(1))) + g(h(f(3))) + h(f(g(–1))) is equal to zero.

Reason (R) : If a polynomial function P(x) is odd then P(0) = 0.

94. Consider the function $f(x) = \ln \ln\left(\frac{x}{4e} + \frac{e^3}{x}\right)$.

Assertion (A) : The range of the function f (x) is R+.

Reason (R) : For two positive reals a and b, $\frac{a+b}{2} \geq \sqrt{ab}$.

95. Let $f(x) = \frac{x-1}{x(x-2)}$

Assertion (A) : The range of $f(x) = \frac{x-1}{x(x-2)}$ is the set of all real number.

Reason (R) : The range of $g(x) = \frac{(x-a)}{(x-c)(x-d)}$ is R if a ∈ (c, d), a, c, d ∈ R.

96. **Assertion (A) :** There are exactly 5 integer in the range of the function $f(x) = \frac{6}{1+7e^x}$.

Reason (R) : If the range of a function is [m, M] then the number of integers in the range is [M – m] where [.] is the G.I.F.

97. **Assertion (A) :** If a function f satisfies $f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$ for all x, y ∈ R, then the value of f(10) is – 48.

Reason (R) : By setting $y = \frac{x}{2}$, we see that the function is $f(x) = -\frac{x^2}{2} + 1$.

MATCH THE COLUMNS FOR JEE ADVANCED

98. Let p and q be positive integers and let f be a function defined for positive real numbers and attains only positive values such that f(xf(y)) = xpyq.

	Column-I		Column-II
(A)	f(t) is equal to	(P)	t^q
(B)	$(f(1))^{-p}$ f(t) is equal to	(Q)	t^{p2}
(C)	$2P\,(f(t))P\,f\left(\frac{1}{2}\right)$ is equal to	(R)	$t^{q/p}$
(D)	$\sqrt{f(f(t))^2}$ is equal to	(S)	t^p

99.

	Column-I		Column-II
(A)	Let f : [– 1, ∞) → (0, ∞) defined by $f(x) = e^{x^2+\|x\|}$, then f (x) is	(P)	one-one

(B)	Let $f : (1, \infty) \to [3, \infty)$ defined by $f(x) = \sqrt{10 - 2x + x^2}$, then f (x) is	(Q)	into
(C)	Let $f : R \to I$ defined by $f(x) = \tan^5\pi[x^2 + 2x + 3]$ where [ ] denotes greatest integer function, then f (x) is	(R)	many one
(D)	Let $f : [3, 4] \to [4, 6]$ defined by $f(x) = \lvert x - 1\rvert + \lvert x - 2\rvert + \lvert x + 3\rvert + \lvert x - 4\rvert$ then f (x)	(S)	onto
		(T)	periodic

100.

	Column-I		**Column-II**
(A)	Let f (x) be a function on $(-\infty, \infty)$ and $f(x + 2) = f(x - 2)$. If f (x) = 0 has only three real roots in [0, 4] and one of them is 4, then the number of real roots of f (x) = 0 in (– 8, 10], is	(P)	4
(B)	Let $r_1, r_2, r_3, \ldots, r_n$ be n positive integers, not necessarily distinct, such that $(x + r_1)(x + r_2) \ldots (x + r_n) = x^n + 56x^{n-1} + \ldots + 2009$ The possible value of n is	(Q)	5
(C)	If x and y are positive integers and 2xy = 2009 – 3y, then the number of ordered pairs of (x,y), is	(R)	8
(D)	If $x, y \in R$, satisfying the equation $\frac{(x-4)^2}{4} + \frac{y^2}{9} = 1$ then the difference between the largest and smallest value of the expression $\frac{x^2}{4} + \frac{y^2}{9}$, is	(S)	9

101. Let $f : R \to [\alpha, \infty)$, $f(x) = x^2 + 3ax + b$, $g(x) = \sin^{-1}\frac{x}{4}$ $(\alpha \in R)$.

	Column-I		**Column-II**
(A)	The possible integral values of 'a' for which f(x) is many one in interval [– 3, 5] is/are	(P)	– 2
(B)	Let a = – 1 and gof(x) is defined for $x \in [-1, 1]$ then possible integral values of b can be	(Q)	– 1
(C)	Let a = 2, $\alpha = -8$ the value(s) of b for which f(x) is surjective is/are	(R)	0
(D)	If a = 1, b = 2, then integers in the range of fog(x) is/are	(S)	1

102.

	Column-I		**Column-II**
(A)	The period of the function $f(x) = \sin\left(\cos\frac{x}{2}\right) + \cos(\sin x)$ equals $k\pi$ then k is equal to	(P)	2

(B)	The integral value(s) in the domain of definition of the function, f (x) = arc cos $\left[\frac{3x^2 - 7x + 8}{1 + x^2}\right]$ where [*] denotes the greatest integer function, is	(Q)	3
(C)	Let f(x) = $\sin\sqrt{[a]}\,x$. If f is periodic with fundamental period π, then the possible integral value(s) of 'a' is/are (where [] denotes the greatest integer function)	(R)	4
(D)	If the values of x satisfying the equation $[x]^2 - 5[x] + 6 = 0$, then integral value of x, is/are (where [ ] denotes the greatest integer function)	(S)	5

REVIEW EXERCISES for JEE ADVANCED

1. Find the domain of the following functions :

   (i) $f(x) = \sqrt{\frac{(|x|-3)(2-|x|)}{x(x-2)}}$.

   (ii) $f(x) = \log_2|4-x^2| + \log_2\left(-1+\left|x-\frac{1}{2}\right|\right)$

   (iii) $f(x) = \sqrt{-\log_{\left(|x|+\frac{5}{2}\right)}\left(\frac{x-5}{2x-3}\right)}$.

2. Find the domain of the following functions :

   (i) $\sqrt[6]{4^x + 8^{\frac{2}{3}(x-2)} - 52 - 2^{2(x-1)}}$

   (ii) $\log_2\left(\sqrt{x-1} - \sqrt{5-x}\right)$

   (iii) $\sqrt{1-\log_x \log_2(4^x - 12)}$.

3. Find the range of the following functions, where [.] denotes greatest integer function :

   (i) $y = \frac{e^x}{[x]+1}$, $x \ge 0$,

   (ii) $y = |\sin x| + |\cos x|$, $0 \le x \le \pi$

   (iii) $y = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$,

4. Find the range of the following functions :

   (i) $f(x) = \tan^{-1}\left(\log_{\frac{4}{5}}\left(5x^2 - 8x + 4\right)\right)$.

(ii) $f(x) = \cos^{-1}\sqrt{\log_{[x]}\frac{|x|}{x}}$, where [.] denotes greatest integer function.

(iii) $f(x) = \sqrt{\tan x} + \sqrt{\cot x}$.

5. Find the domain and range of the following functions :

(i) $y = \log_{\sqrt{5}}\left(\sqrt{2}(\sin x - \cos x) + 3\right)$.

(ii) $f(x) = \frac{x^2 - 5x + 4}{x^2 - 3x + 2}$.

(iii) $f(x) = \sqrt{\cos(\sin x)}$.

(iv) $y = \sec^{-1}\left(\log_3 \tan x + \log_{\tan x} 3\right)$.

6. Find the range of the following functions :

(i) $f(x) = \sin^{-1}\left(\frac{|\sin x|}{\sin x}\right)^{[x]}$

(ii) $f(x) = \log_a(\cos a^{3x+2})$, $a > 1$

(iii) $y = f(x)$ where $2^x + 4^y = 4$

7. Find fog(x) if $f(x) = [x] + \{x\}^2$ and $g(x) = [x] + \sqrt{\{x\}}$ and also find the range of fog(x).

8. Find the set of values of a for which the function $f : R \to R$ given by $f(x) = x^3 + (a + 2)x^2 + 3ax + 5$ is one-one.

9. Classify the following functions as injective, surjective both or none :

(a) $f : R \to R$, $f(x) = x^3 - 6x^2 + 13x - 6$.

(b) $f : \left[\frac{-1}{2}, +\infty\right) \to R$, $f(x) = (x^2 + x + 5)(x^2 + x - 3)$.

10. Find all functions f defined on the set of all real numbers with real values, such that $f(x^2 + f(y)) = y + f(x)^2$ for all x, y.

11. Prove that $f : (-1, 1) \to R$ defined by $f(x) = \begin{cases} \frac{x}{1+x}, & -1 < x \le 0 \\ \frac{x}{1-x}, & 0 < x < 1 \end{cases}$ is a bijective function.

12. Let $f : X \to Y$ be a function defined by $f(x) = a\sin\left(x + \frac{\pi}{4}\right) + b\cos x + c$. If f is both one-one and onto, find sets X and Y.

13. Check whether $f : (-\infty, 2) \cup (3, \infty) \to R$ $f(x) = ln(x^2 - 5x + 6)$ is a bijective function or not. If not then choose a suitable longest domain and co-domain for which the above function becomes bijective.

14. If $f(x) = \frac{a^x}{a^x + \sqrt{a}}$ $(a > 0)$, show that $f(x) + f(1 - x) = 1$. Hence or otherwise, evaluate $\sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right)$.

15. If $f(2 - x) = f(2 + x)$ and $f(7 - x) = f(7 + x)$ and $f(0) = 0$ then find the minimum number of roots of $f(x) = 0$, where $|x| \le 100$

16. Let $g(x) = f(x) - 1$. If $f(x) + f(1 - x) = 2\ \forall\ x \in R$, then find the line about which g(x) is symmetrical.

17. Let $f : R \to \left(0, \frac{2\pi}{3}\right]$ defined as $f(x) = \cot^{-1}(x^2 - 4x + \alpha)$. Find the smallest integral value of α such that f (x) is into function.

18. Let $f : R \to [-2, 2]$ be defined by $f(x) = \cos\left[\frac{1}{2}\pi^2\right]x + \sin\left[-\frac{1}{2}\pi^2\right]x$, $[x]$ denoting the greatest integer function. Prove that f is an into function.

19. Find period of : $f(x) = (\cos 2x + \tan 2x) + (\cos 2^2x + \tan 2^2x) + \ldots\ldots\ldots\ldots\ldots + (\cos 2^nx + \tan 2^nx)$

20. Let $f : R \to R$ and $g : R \to R$ be functions defined by $f(x) = \begin{cases} 2x & , x < 1 \\ 2x^2 - 1, & x \ge 1 \end{cases}$ $g(x) = \begin{cases} x + 2, & x < 0 \\ 2x & , x \ge 0 \end{cases}$

find (i) f + g (ii) fg (iii) fog

21. If for all real values of u and v, $2f(u) \cos v = f(u + v) + f(u - v)$, prove that, for all real values of x :
(a) $f(x) + f(-x) = 2a \cos x$.
(b) $f(\pi - x) + f(-x) = 0$.
(c) $f(\pi - x) + f(x) = -2b\sin x$. Deduce that $f(x) = a \cos x - b \sin x$, where a, b are arbitrary constants.

22. Let f and g be real valued functions such that $f(x + y) + f(x - y) = 2 f(x). g(y)$ $\forall x, y \in R$ and $f : R \to [-1,1]$ onto function then prove that $| g(y) |$ $\forall y \in R$.

23. Prove that if the graph of the function, $y = f(x)$, defined throughout the number scale, is symmetrical about two lines $x = a$ and $x = b, (a < b)$, then the function is a periodic one.

24. Find the inverse of the following function :
(a) $f : R \to (-1, 1)$, $f(x) = \dfrac{-x\,|\,x\,|}{1 + x^2}$.
(b) $f : R \to R$, $f(x) = \begin{cases} x, & -\infty < x < 1 \\ x^2, & 1 \le x \le 4 \\ 2^x. & 4 < x < \infty \end{cases}$.

25. Let $f : W \to W$ where $f(x) = f(x - 1) + f(x - 2)$ for $x \ge 2$. Also, $f(0) = 0$ and $f(1) = 1$. Prove that
(a) $f(x + 1) > f(x)$ $\forall x \ge 2$
(b) $f(5x)$ is divisible by 5 $\forall$ x in the domain of f.
(c) there exists precisely three integers for which $f\{f(x)\} = f(x)$.

26. Find the range of $f(x) = \begin{cases} 2 - x^2 & , x < -1 \\ \sin^{-1} x, & -1 \le x \le 1 \\ \{x^2\} & x > 1 \end{cases}$.

27. If $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$, $g(x) = |\sin x| - |\cos x|$ and $\varphi(x) = f(x)\, g(x)$ (where [.] denotes the greatest integer function) then find the respective fundamental periods of f(x), g(x) and $\varphi(x)$.

28. Find all real values of x for which the expression $\sqrt{\log_{x^2-4}\frac{3}{4} - \log_{x^2-4x+3}\frac{3}{4}}$ is a real number.

29. If $f(x) + f(x + a) + f(x + 2a) + \ldots\ldots\ldots\ldots + f(x + na) = \text{constant}$ $\forall x \in R$ and $a > 0$ and f(x) is periodic, then find the period of f(x).

30. Find the domain of the function $y = f(x)$ where $3y + 2^{x^4} = 2^{4x^2-1}$.

31. Check whether the given functions are onto or into.
   (a) $f : R \to R$, $f(x) = x^3 - 4x + [\sin x]$
   (b) $f : R \to [-1, 1]$, $f(x) = \dfrac{\sin \pi x}{x^2+1}$
32. Let f be defined for all $x \in R$, so that $(f(x))n = f(nx)$, n is an odd integer. Find f(x)
33. If f(x) is an odd function, then find the parity of the function $g(x) = f(f(x)) + xf(x^2) + \text{sgn}(f(x))$.
34. Find whether f(x) is into or onto : $f : R \to (-\pi, \pi)$ $f(x) = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} + \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$.
35. Find the fundamental period of the function $f(x) = \cos(\tan x + \cot x).\cos(\tan x - \cot x)$.

TARGET EXERCISES for JEE ADVANCED

1. Find the domain and range of
   $f(x) = [\log_5 \cos^{-1}\sqrt{x^2+5x+6}\,] + \sec^{-1}[\{x\}+1]$, where [.] denotes the greatest integer function and {.} denotes the fractional part function.
2. If a function is defined as $f(x) = \sqrt{\log_{\varphi(x)} g(x)}$, where $g(x) = |\sin x| + \sin.x$, $\varphi(x) = \sin x + \cos x$, $0 \le x \le \pi$. Then find the domain of f(x).
3. Find the domain of the following functions :
   (i) $f(x) = \sin^{-1}\left(\log_{|x|}\sqrt{|x|-x}\right)$
   (ii) $f(x) = e^{\sin^{-1}\left(\frac{x}{2}\right)} + \tan^{-1}\left[\dfrac{x}{2}-1\right] + \ell n\left(\sqrt{x-[x]}\right)$
   where [.] denotes greatest integer function
   (iii) $f(x) = \sqrt{\sqrt{4\sin^2 x - 1}\ \log_{\sin x}\dfrac{x-5}{2x-1}}$.
4. Find the range of the following functions :
   (i) $f(x) = \sqrt{\ell n(\cos(\sin x))}$.
   (ii) $f(x) = (\tan^{-1} x)^2 + \dfrac{2}{\sqrt{1+x^2}}$
   (iii) $f(x) = \begin{cases} \dfrac{e^x}{1-e^x}, & x < 0 \\ \sqrt{1-x}, & 0 \le x \le 1 \\ x-1, & 1 < x \le 2 \end{cases}$.
5. Find the domain and range of the following functions :
   (i) $f(x) = \log_2\left(\sqrt{x-4} + \sqrt{6-x}\right)$.

(ii) $y = \sec^{-1}(2x - x^2)$.

(iii) $y = \sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{\ldots\ldots\ldots\infty}}}$.

(iv) $y = \log_{\left[\frac{x}{\pi}\right]} \sin x$, where [.] denotes the greatest integer function.

6. Let $f(x) = x^2 - 2x, x \in R$ and $g(x) = f(f(x) - 1) + f(5 - f(x))$. Show that, $g(x) \geq 0\ \forall\ x \in R$.

7. If $f(x) = \log_{100x}\left(\dfrac{2\log_{10} x + 2}{-x}\right)$; $g(x) = \{x\}$, where $\{x\}$ denotes the fractional part of x. If the function fog(x) exists, then find the maximum possible range of g(x).

8. Draw the graph of $f(x) = \begin{cases} x+1 & ; \quad x \leq 1 \\ (x-1)^2; & \quad x > 1 \end{cases}$ and $g(x) = \begin{cases} \tan x & ; \quad -\frac{\pi}{2} < x \leq \frac{\pi}{4} \\ \tan x - 2; & \quad \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$ and hence find fog(x) and gof(x).

9. If $f(x) = \begin{cases} 1+x^2, & x \geq 0 \\ 3-x^2, & x < 0 \end{cases}$ and

$g(x) = \begin{cases} \tan x, & 0 \leq x < \pi/2 \\ \operatorname{cosec} x, & -\pi/2 \leq x < 0 \end{cases}$ then find fog(x).

10. Compute the inverse of the function

$f : R - \{1\} \to R^+ - \{2\},\ f(x) = 2^{\frac{x}{x-1}}$. Also find domain and range of f^{-1}.

11. Find the minimum value of ‘a’ and ‘b’ for which $f(x) = x^x : [a, \infty) \to [b, \infty)$ be an invertible function.

12. Check whether $f : R \to R, f(x) = [x] + \sqrt{\{x\}}$ (where [.] and {.} represents greatest integral and fractional part function respectively) is an invertible or not, if yes then find its inverse. Also solve the equation $f(x) = f^{-1}(x)$.

13. Let $f(x) = x^2 + 3x - 3, x \geq 0$. n points $x_1, x_2, \ldots\ldots\ldots x_n$ are so chosen on the x-axis such that :

(a) $\dfrac{1}{n}\sum_{i=1}^{n} f^{-1}(x_i) = f\left(\dfrac{1}{n}\sum_{i=1}^{n} x_i\right)$.

(b) $\sum_{i=1}^{n} f^{-1}(x_i) = \sum_{i=1}^{n} x_i$, where f^{-1} denotes the inverse of f. Find the A.M. of x_i ’s.

14. $f(x) = \begin{cases} x^4 \tan \dfrac{\pi x}{2}, & |x| < 1 \\ x|x|, & |x| \geq 1 \end{cases}$. Prove that f(x) is an odd function.

15. If $g : D \to R$ be a function such that $g(x) = \ln \ln \ln \ln \ldots\ldots\ldots \ln (4(x^2 + x + 1) + \sin(\pi x))$, (log operated n times, $n \in N$), then find the least value of n for which g becomes onto.

16. Let $f(x) = \max \{\sin t : 0 \leq t \leq x\}$ $h(x) = [f(x) - g(x)]$ where [.] is G.I.F. $g(x) = \min \{\sin t : 0 \leq t \leq x\}$ Find the range of h(x).

17. Find whether f(x) is one-one or many-one :

(i) $f(x) = \ln (x + \sqrt{x^2+1})^{(x-1)}$ (ii) $f(x) = \cos\left(\frac{1}{2}\left(\sin^{-1}x + \frac{\pi}{2}\right)\right)$.

18. Lef $f : R \to [-1, 1]$. Find whether

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$ is into or onto.

19. For what value of 'a' is the function from $R \to (0, \pi)$, $f(x) = \cos^{-1}\left(\frac{a-1-e^x}{e^x+1}\right)$ is onto.

20. If f and g are two distinct linear functions defined on R such that they map [–1, 1] onto [0, 2] and $h : R - \{-1, 0, 1\} \to R$ defined by $h(x) = \frac{f(x)}{g(x)}$, then show that $|h(h(x)) + h(h(1/x))| > 2$.

21. Let $f : [3, \infty) \to [1, \infty)$ be defined by $f(x) = \pi^{x(x-3)}$. Show that f is a bijective function and find its inverse. How many solutions exist for the equation $f(x) = f^{-1}(x)$.

22. Let $f(x) = \max \{1 + \sin x, 1 - \cos x, 1\}$, $x \in [0, 2\pi]$ and $g(x) = \max \{1, |x - 1|\}$ $x \in R$. Show that gof(x) is a constant function.

23. Check whether the following functions are even, odd or neither even nor odd :

(i) $f(x) = \dfrac{2x(\sin 2x + \tan x)}{2\left[\dfrac{x+2\pi}{\pi}\right]-3}$ [.] denotes greatest integer function.

(ii) $f(x) = \int_0^x \ell n\left(\frac{1-t}{1+t}\right)dt$

24. If $a, b \in R^+$, $f(a + x) = b + [b^3 + 1 - 3b^2f(x) + 3b \{f(x)\}^2 - \{f(x)\}^3]^{1/3}$ for all $x \in R$, prove that f(x) is periodic.

25. Find the values of 'p' so that gof is defined whenever f(x) is defined, where

$f(x) = 2^{\cos^{-1}}\left(\frac{2\sin x+1}{2\sqrt{2}\sin x}\right)$ and $g(x) = \sin^{-1}\left(\frac{1+px^2}{px}\right)$

26. $f : R \to R$, $f(x) = \dfrac{3x^2+mx+n}{x^2+1}$. If the range of this function is [– 4, 3), then find the value of $m^2 + n^2$.

27. If f(x) satisfies the equation

$$\begin{vmatrix} f(x-3) & f(x+4) & f\left[(x+1)(x-2)-(x-1)^2\right] \\ 5 & 4 & -5 \\ 5 & 6 & 15 \end{vmatrix} = 0$$

for all real x, then show that f(x) is periodic with period 7.

28. Find out the integral values of n if 3π is a period of the function $f(x) = \cos nx \cdot \sin (5/n)x$.

29. A function f maps every sequence of integers to another sequence of integers as follows :

$$(f(a))_n = \begin{cases} 1, \text{ if } n = 0 \\ n a_{n-1}, \text{ if } n \neq 0 \end{cases}$$

If $(f(b))_n = b_n$, find b_{2010}.

30. Let f be a function from the set of positive integers to the set of real numbers i.e., $f : N \to R$, such that
(i) $f(1) = 1$
(ii) $f(1) + 2 f(2) + 3 f(3) + ... + n f(n) = n (n + 1) f(n)$ for $n \geq 2$, then find the value of $(f(1994))^{-1}$.

31. Let $f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$. Prove that $f(4x^3 - 3x) + 3f(x) = 0, \ \forall\ x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ also prove that $f(x) = 0, \ \forall\ x \in \left[-1, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right]$.

32. Find all functions f defined on the set of positive reals which take positive real values and satisfy:
$f(x(f(y)) = yf(x)$ for all x, y; and $f(x) \to 0$ as $x \to \infty$.

33. Find all functions f defined on the non-negative reals and taking non-negative real values such that:
$f(2) = 0$, $f(x) \neq 0$ for $0 \leq x < 2$, and
$f(xf(y))\, f(y) = f(x + y)$ for all x, y.

34. Let S be the set of all real numbers greater than –1. Find all functions $f : S \to S$ such that $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y, and $f(x)/x$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

35. Determine all functions $f : R \to R$ such that $f(x - f(y)) = f(f(y)) + x\, f(y) + f(x) - 1$ for all x, y in R.

Previous Years' Questions (JEE ADVANCED)

A. Fill in the Blanks

1. The values of $f(x) = 3 \sin\left(\sqrt{\frac{\pi^2}{16} - x^2}\right)$ lie in the interval.......................... **[IIT - 1983]**

2. The domain of the function $f(x) = \sin^{-1}(\log_2 \frac{x^2}{2})$ is given by.............. **[IIT - 1984]**

3. Let A be a set of n distinct elements. Then the total number of distinct functions from A to A is .... and out of these............are onto function **[IIT - 1985]**

4. If $f(x) = \sin \ln\left(\frac{\sqrt{4-x^2}}{1-x}\right)$, then domain of f(x) is..........and its range is.............. **[IIT - 1985]**

5. There are exactly two distinct linear functions..............and ....which map [–1, 1] onto [0, 2]. **[IIT - 1989]**

6. If f is an even function defined on the interval (–5, 5), then four real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ are ...... **[IIT - 1996]**

B. True/False

7. If $f(x) = (a - xn)^{1/n}$ where $\alpha < 0$ and n is positive integer, then $f[f(x)] = x$. **[IIT - 1983]**

8. The function $f(x) = \frac{x^2+4x+30}{x^2-8x+18}$ is not one to - one. **[IIT - 1983]**

9. If $f_1(x)$ and $f_2(x)$ are defined on domains D_1 and D_2 respectively, then $f_1(x) + f_2(x)$ is defined on $D_1 \cup D_2$. **[IIT - 1988]**

C. Multiple Choice Questions with One Correct Answer

10. Let R be the set of real numbers. If f R $\to$ R is a function defined by $f(x) = x^2$, then f is : **[IIT - 1979]**

(A) Injective but not surjective
(B) Surjective but not injective
(C) Bijective
(D) None of these

11. The domain of definition of the equation of the function $y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$ is **[IIT - 1983]**

(A) (–3, –2) excluding – 2.5
(B) [0, 1] excluding 0.5
(C) [–2, 1) excluding 0
(D) None of these

12. Which of the following functions is periodic ? **[IIT - 1983]**

(A) $f(x) = x - [x]$ where [x] denotes the largest integer less than or equal to the real number x
(B) $f(x) = \sin 1/x$ for $x \neq 0$, $f(0) = 0$
(C) $f(x) = x \cos x$
(D) None of these

13. If function $f(x) = \frac{1}{2} - \tan\left(\frac{\pi x}{2}\right)$; $(-1 < x < 1)$ and $g(x) = \sqrt{3+4x-4x^2}$, then the domain of gof is - **[IIT - 1990]**

(A) (–1, 1)
(B) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(C) $\left[-1, \frac{1}{2}\right]$
(D) $\left[\frac{1}{2}, -1\right]$

14. The value of b and c for which the identity $f(x + 1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$, are **[IIT - 1992]**

(A) b = 2, c = 1
(B) b = 4, c = – 1
(C) b = –1, c = 4
(D) None

15. The value of the parameter α, for which the function $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itself, if- **[IIT - 1992]**

(A) –2
(B) –1
(C) 1
(D) 2

16. Let $f(x) = \sin x$ and $g(x) = \ln |x|$. If the ranges of the composite function fog and gof are R_1 and R_2 respectively, then **[IIT - 1994]**

(A) $R_1 = \{u : -1 < u < 1\}$, $R_2 = \{v : -\infty < v < 0\}$

(B) $R_1 = \{u : -\infty < u \le 1\}, R_2 = \{v : -1 \le v \le 0\}$
(C) $R_1 = \{u : -1 < u < 1\}, R_2 = \{v : -\infty < v < 0\}$
(D) $R_1 = \{u : -1 \le u \le 1\}, R_2 = \{v : -\infty < v \le 0\}$

17. Let $f(x) = (x+1)^2 - 1, (x \ge -1)$. Then these set $\{x : f(x) = f^{-1}(x)\}$ is - **[IIT - 1995]**

(A) $\left\{0, -1\frac{-3+i\sqrt{3}}{2}, \frac{-3-i\sqrt{3}}{2}\right\}$ (B) $\{0, 1, -1\}$
(C) $\{0, -1\}$ (D) Empty

18. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then (gof) (x) = **[IIT - 1996]**

(A) –2 (B) –1 (C) 2 (D) 1

19. If the function f : [1, ∞) is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is - **[IIT - 1999]**

(A) $\left(\frac{1}{2}\right)^{x(x-1)}$ (B) $\frac{1}{2}\left(1+\sqrt{1+4\log_2 x}\right)$
(C) $\frac{1}{2}\left(1-\sqrt{1+4\log_2 x}\right)$ (D) Not defined

20. The domain of definition of the function, y (x) given by the equation, $2^x + 2^y = 2$ is : **[IIT - 2000]**
(A) $0 < x \le 1$ (B) $0 \le x \le 1$ (C) $-\infty < x \le 0$ (D) $-\infty < x < 1$

21. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then $f(\theta)$ - **[IIT - 2000]**
(A) ≥ 0 only where $\theta \ge 0$ (B) ≤ 0 for all real θ
(C) ≥ 0 for all real θ (D) ≤ 0 only where $\theta \le 0$

22. Let $g(x) = 1 + x - [x]$ & $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$. Then for all x, f (g (x)) is equal to **[IIT - 2001]**

(A) x (B) 1 (C) f(x) (D) g(x)

23. If $f : [1, \infty) \to [2, \infty)$ is given by, $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals : **[IIT - 2001]**

(A) $\frac{x + \sqrt{x^2 - 4}}{2}$ (B) $\frac{x}{1 + x^2}$ (C) $\frac{x - \sqrt{x^2 - 4}}{2}$ (D) $1 - \sqrt{x^2 - 4}$

24. The domain of definition of $f(x) = \frac{\log_2(x+3)}{x^2 + 3x + 2}$ is : **[IIT - 2001]**

(A) $R - \{-1, -2\}$ (B) $(-2, \infty)$ (C) $R - \{-1, -2, -3\}$ (D) $(-3, \infty) - \{-1, -2\}$

25. Let E = {1, 2, 3, 4 } & F = {1, 2}. Then the number of onto functions from E to F is **[IIT - 2001]**
(A) 14 (B) 16 (C) 12 (D) 8

26. Let $f(x) = \frac{\alpha x}{x + 1}, x \ne -1$. Then for what value of α is $f(f(x)) = x$? **[IIT - 2001]**

(A) $\sqrt{2}$ (B) $-\sqrt{2}$ (C) 1 (D) -1.

27. Suppose $f(x) = (x+1)^2$ for $x \ge -1$. If g(x) is the function whose graph is the reflection of the graph of f(x) with respect to the line y = x, then g(x) equals - **[IIT - 2002]**

(A) $-\sqrt{x}-1, x \geq 0$ (B) $\frac{1}{(x+1)^2}, x > -1$

(C) $\sqrt{x+1}, x \geq -1$ (D) $\sqrt{x}-1, x \geq 0$

28. Let function f: R → R be defined by f(x) = 2x + sin x for x → R, then f is **[IIT - 2002]**
(A) one-to-one and onto (B) one-to-one and onto
(C) onto but NOT one-one (D) neither one-to-one nor onto

29. If f :[0, ∞) → [0, ∞), and f(x) = $\frac{x}{1+x}$ then f is : **[IIT - 2003]**
(A) one-one and onto (B) one-one but not onto
(C) onto but not one-one (D) neither one-one nor onto

30. Domain of definition of the function

$f(x) = \sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ for real valued x, is - **[IIT - 2003]**

(A) $\left[-\frac{1}{4},\frac{1}{2}\right]$ (B) $\left[-\frac{1}{2},\frac{1}{2}\right]$ (C) $\left(-\frac{1}{2},\frac{1}{9}\right)$ (D) $\left[-\frac{1}{4},\frac{1}{4}\right]$

31. Range of the function $f(x) = \frac{x^2+x+2}{x^2+x+1}$; x ∈ R is **[IIT - 2003]**
(A) (1, ∞) (B) (1, 11/7] (C) (1, 7/3] (D) (1, 7/5]

32. If f(x) = sin x + cos x, g(x) = $x^2 - 1$, then g(f(x)) is invertible in the domain - **[IIT - 2004]**
(A) [0, π/2] (B) [–π/4, π/4] (C) [–√3, √3] (D) (√3, 2]

33. If the functions f(x) and g(x) are defined on R → R such that $f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}$;
$g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$ then (f – g) (x) is **[IIT - 2005]**
(A) one-one & onto (B) neither one-one nor onto
(C) one-one but not onto (D) onto but not one-one

34. X and Y are two sets and f: X → Y. If {f(c) = y; c ⊂ X, y ⊂ Y} and {f^{-1}(d) = x; d ⊂ Y, x ⊂ X}, then the true statement is **[IIT - 2005]**
(A) $f(f^{-1}(b)) = b$ (B) $f^{-1}(f(a)) = a$
(C) $f(f^{-1}(b)) = b$, b ⊂ y (D) $f^{-1}(f(a)) = a$, a ⊂ x

35. Let f(x) = x^2 and g(x) = sin x for all x ∈ R. Then the set of all x satisfying (f o g o g o f) (x) = (g o g o f) (x), where (f o g) (x) = f(g(x)), is :-
(A) $\pm\sqrt{n\pi}$, n ∈ {0, 1, 2, ...} (B) $\pm\sqrt{n\pi}$, n ∈ {1, 2, ...}
(C) $\frac{\pi}{2} + 2n\pi$, n ∈ {.....–2, –1, 0, 1, 2, ...} (D) 2nπ, n ∈ {.....–2, –1, 0, 1, 2, ...}

36. Let f(x) = x^2 and g(x) = sinx for all x ∈ ℝ. Then the set of all x satisfying (f ∘ g ∘ g ∘ f) (x) = (g ∘ g ∘ f) (x), where (f ∘ g)(x) = f(g)x)), is **[IIT - 2011]**
(A) $\pm\sqrt{n\neq}, n \in \{0, 1, 2, ...\}$ (B) $\pm\sqrt{n\neq}, n \in \{1, 2, ...\}$
(C) $\frac{\pi}{2} + 2n\pi, n \in \{..., -2, -1, 0, 1, 2, ...\}$ (D) 2nπ, n ∈ {..., –2, –1, 0, 1, 2, ...}

37. The function f: [0, 3] $\rightarrow$ [1, 29], defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is **[IIT - 2012]**
(A) one-one and onto.
(B) onto but not one-one.
(C) one-one but not onto.
(D) neither one-one nor onto.

D. Multiple Choice Questions with One or More Than One Correct Answer

38. If $y = f(x) = \frac{x+2}{x-1}$ then **[IIT - 1984]**
(A) $x = f(y)$
(B) $f(1) = 3$
(C) y increases with x for $x < 1$
(D) f is a rational function x

39. If $f(x) = 3x - 5$, then $f^{-1}(x)$: **[IIT - 1998]**
(A) is $\frac{1}{3x-5}$
(B) $\frac{x+5}{3}$
(C) Does not exist as f is not one-one
(D) Does not exist as f is not onto

40. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin \sqrt{x})^2$, then **[IIT - 1998]**
(A) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$
(B) $f(x) = \sin x$, $g(x) = |x|$
(C) $f(x) = x^2$, $g(x) = \sin \sqrt{x}$
(D) f and g cannot be determined

41. Let f: (–1, 1) $\rightarrow$ R be such $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\neq}{4}\right) \cup \left(\frac{\neq}{4}, \frac{\neq}{2}\right)$. Then the value(s) of $f\frac{1}{3}$ is (are) **[IIT - 2012]**
(A) $1 - \sqrt{\frac{3}{2}}$
(B) $1 + \sqrt{\frac{3}{2}}$
(C) $1 - \sqrt{\frac{2}{3}}$
(D) $1 + \sqrt{\frac{2}{3}}$

E. Subjective Problems

42. Find the domain and range of the function $f(x) = \frac{x^2}{1+x^2}$. Is the function one-to-one ? **[IIT - 1978]**

43. Let f be a one-one functions with domain {x, y, z} and range {1, 2, 3}. It is given that exactly one of the following statements is true and the remaining two are false $f(x) = 1, f(y) \neq 1, f(z) \neq 2$ determine $f^{-1}(1)$. **[IIT - 1982]**

44. Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$ where the function f satisfies the relation $f(x + y) = f(x).\, f(y)$ for all natural numbers x, y and further $f(1) = 2$. **[IIT - 1992]**

45. A function f : R $\rightarrow$ R, where $f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the interval of values of α for which f is onto. Is the function one-to-one for $\alpha = 3$? Justify your answers. **[IIT - 1996]**

46. Let $f(x) = Ax^2 + Bx + C$ where A, B, C are real numbers. Prove that if f(x) is an integer when ever x is an integer, then the numbers 2A, A + B and C are all integers. Conversely, prove that if the number 2A, A + B and C are all integers then f(x) is an integer whenever x is an integer. **[IIT - 1998]**

F. Match the Column

47. **Column-I** **Column-II**

(A) The set $\left\{RE\left(\frac{2iz}{1-z^2}\right): z \text{ is a comples number } |Z| = 1, z \neq \pm 1\right\}$ (P) $(-\infty,-1)\cup(1,\infty)$

(B) The domain of the function $f(x)=\sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ (Q) $(-\infty,0)\cup(0,\infty)$

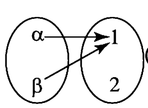
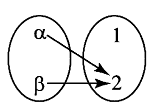
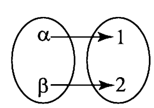
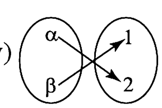
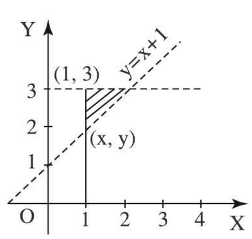
(C) If $f(\theta)=\begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$ than the set $\left[f(\theta): 0 \le \theta < \frac{\pi}{2}\right]$ (R) $[2,\infty)$

(D) If $f(x) = x^{3/2}(3x-10)$, $x \ge 0$, then f(x) is increasing in (S) $(-\infty,-1]\cup[1,\infty)$

(t) $(-\infty,-1]\cup[1,\infty)$

ANSWERS

Concept Problems [A]

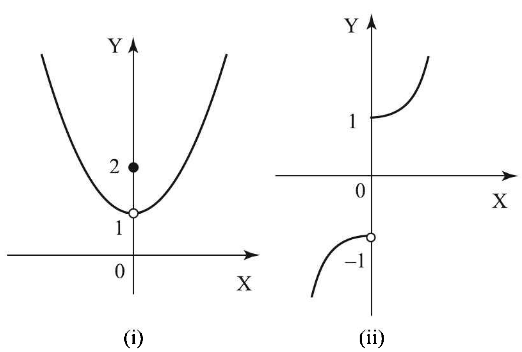
1. (i) No; (ii) No
2. (b) and (c) are ; (a) is not
3. (a) is ; (b) and (c) are not
4. (i)  (ii)  (iii)  (iv) 
5. g(x) and h(x).
7. (i) True (ii) False (iii) True
8. (i) and (iii) are functions
9. 

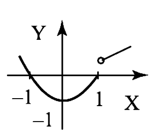
Not a function.

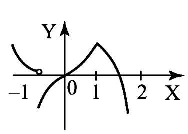
Practice Problems [A]

10. 64
11. 5050
12. 2/3.
13. (b) and (c)
14. (b), (c) and (d)
15. (c) Equals –3. (d) $f(x) = 3x$ for all x.
16. (i) $f(x) = 1$, $f(x) = 2x$, $f(x) = -1$.
    (ii) $f(x) = 3.2^x$, $f(x) = 0$, $f(x) = 2[x]$, where [x] is the greatest integer less than or equal to x.
    (iii) $f(x) = 0$, $f(x) = 2x$, $f(x) = x^3$
    (iv) $f(x) = 0$, $f(x) = \log^3 x$, $f(x) = -4 \log^6 x$

Practice Problems [B]

1. (i), (iii), (iv)
2. $A(x) = \dfrac{bx(h-x)}{h}$, $p(x) = 2\left\{x + \dfrac{b(h-x)}{h}\right\}$
3. $\sqrt{x^2+16} + \sqrt{x^2-8x+25}$
4. 

5. (i) 

(ii) 

6. (a) $\begin{cases} x+2 , & -2 \le x < 0 \\ 2 , & 0 \le x \le 2 \\ 4-x , & 2 < x \le 4 \end{cases}$ (b) $\begin{cases} x+1 , & -2 \le x < 0 \\ x , & 0 \le x \le 2 \\ 6-2x , & 2 < x \le 3 \end{cases}$

(c) $\begin{cases} 1+1/2 & , -2 \le x < 0 \\ 1+x/2 & , 0 \le x < 2 \\ 2x-4 & , 2 < x \le 3 \end{cases}$

8. (a) $y = 10^4/x - 1, x > 0$

(b) $y = \log_2\left(\dfrac{x^3-8}{x^2-2}\right) - x, x \in (-\sqrt{2}, \sqrt{2}) \cup (2, \infty)$

(c) $y = -4 \pm 4\sqrt{x+1}\ \ x \ge -1.$

9. (a) $y = 1 + \sqrt{x^2+1}, x \in R,$

(b) $y = 1 - \sqrt{1-x^2}, x \in [-1, 1].$

10. $y^5 - y - x = 0.$

Practice Problems [C]

1. (i) Yes (ii) No
2. (a) $(2, \infty)$ (b) $(0, 1) \cup (1, \infty)$

   (c) $[-1, 1]$ (d) $R - \dfrac{n\pi}{2}$, where n is an integer.
5. All have different graphs.
6. (i) $\dfrac{1}{x^3} - \dfrac{4}{x^2} + \dfrac{5}{x} - 2$ (ii) $x^3 - 4x^2 + 5x - 2 + \dfrac{1}{x}$

   (iii) $\dfrac{1}{x^3 - 4x^2 + 5x - 2}$ (iv) $x^3 - 4x + 5 - \dfrac{2}{x}$

   (v) $5x - 20 + \dfrac{25}{x} - \dfrac{10}{x^2}$.
7. (i) $[-4, 2]$ (ii) $[-4, 2]$ (iii) $[-4, 2]$

   (iv) $[-4, 2) \cup (3, \infty)$ (v) -2 (vi) 0

   (vii) 0 (viii) undefined

   (ix) 5 (x) 1/5
8. (i) 13 (ii) 5981 (iii) 10
9. $f/g = \begin{cases} \dfrac{x^2-4x+3}{x-3} & , 0 \le x < 3 \\ \dfrac{x-4}{x-3} & , 3 \le x < 4 \\ \dfrac{x-4}{x^2-7x+10} & , 4 \le x \le 6, x \ne 5 \end{cases}$

   Domain: $x \in [0, 6] - \{3, 5\}$

PRACTICE PROBLEMS [D]

1. (i) $(-\infty, -\frac{1}{2}] \cup (0, 1) \cup (2, \infty)$ (ii) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

   (iii) $(1-\sqrt{2}, 0) \cup (1+\sqrt{2}, 3)$
2. (i) $R - \{0, 1\}$ (ii) $x \in [2, 4)$ (iii) $[-4, -\pi] \cup [0, \pi]$ (iv) $[0, 1]$
3. (i) $(3, 4)$ (ii) $\{n\pi : n \in I\}$ (iii) $[10^3, 10^4)$ (iv) $[2, \infty)$
4. (i) $(-\infty, -3) \cup (-2, 0) \cup (0, 2) \cup (3, \infty)$

   (ii) $(-3, -2) \cup (0, 2) \cup (3, \infty)$

   (iii) $R - \{-3, -2, 2, 3\}$
5. (i) $(1, 2) \cup (3, \infty)$ (ii) $x \in [4, 6]$

   (iii) $\{1\}$ (iv) $\{-1/2\}$
6. (i) $[-1, -1/2] \cup [0, 1/2] \cup \{1\}$

   (ii) $(-\sqrt{8}, -1] \cup [1, \sqrt{8})$ (iii) $[2, \infty)$ (iv) $R - \{0\}$
7. (i) $[2n\pi - \sin^{-1}(\sqrt{2} - 1), 2n\pi) \cup (2n\pi + \pi, 2n\pi + \pi + \sin^{-1}(\sqrt{2} - 1)], n \in I$

   (ii) $(2n\pi - \cos^{-1}(-1/4), 2n\pi] \cup \{2n\pi + \pi/2\}$

   (iii) $[2n\pi - \pi/2, 2n\pi + \pi/2]$
9. $(-\sqrt{3}, -1) \cup (1, \sqrt{3})$
10. $\left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, \infty\right)$
11. $(8, 10)$
12. 6

PRACTICE PROBLEMS [E]

1. $D_f : R - \{(4+2\sqrt{5}), (4-2\sqrt{5})\}$

   $R_f : \left(-\infty, -\frac{1}{4}\right] \cup \left[-\frac{1}{20}, \infty\right)$
2. (i) $[-1, 1]$ (ii) $[6, 10]$
3. (i) $[0, \sqrt{3}]$ (ii) $\left[0, \frac{1}{2}\right]$ (iii) $\left[1, \frac{3}{2}\right)$ (iv) $[0, \infty)$
4. (i) $\{1, 2\sqrt{3}\}$ (ii) $[9, \infty)$
5. (i) $[3/4, 2]$ (ii) $\{0\}$

6. $[-1, 1/2]$
7. 1
8. 28
9. (i) $\left[\frac{-106-\pi}{2}, \frac{46+\pi}{2}\right]$ (ii) $[-1, 3]$
10. $\left(-\infty, \frac{\pi}{2}\right]$
11. $\left[2, \frac{\pi^2}{4}\right)$
12. (i) $[1, \infty)$ (ii) $[6, \infty)$
13. (i) $\left[\sqrt{2}, \sqrt{10}\right]$ (ii) $(-\infty, \log_2 \frac{3}{8}]$ (iii) $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \frac{\pi}{3}\right]$ (iv) R

CONCEPT PROBLEMS [B]

1. (i) $(f \circ g)(x) = 4x^2 - 6x + 1$ (ii) $(g \circ f)(x) = 4x^2 - 6x + 1$
   (iii) $(g \circ g)(x) = 4x - 9$ (iv) $(f \circ f)(x) = x^4 + 6x^3 + 14x^2 + 15x + 5$
2. (i) range of f = {1, 2, 4}, range of g = {1, 2, 3, 4}, range of h = {1, 3}
   (ii)

x	(f o g) (x)	(h o f) (x)	(g o g) (x)
1	1	3	4
2	4	1	3
3	2	1	2
4	1	3	1

3. (i) 6 (ii) –60 (iii) – 80
4. (i) 5 (ii) $2x - 1$ (iii) $4x - 3$
5. $-1 \pm \sqrt{2}$
6. (i) $[0, \sqrt{3}]$ (ii) $(-\infty, 0]$ (iii) $[-2, 1]$
   (iv) $\sqrt{1-x}$ (v) $[-\sqrt{3}, \sqrt{3}]$ (vi) $-\sqrt{\sqrt{3-x^2}+2}$
8. Given $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x)=\frac{3x+x^3}{1+3x^2}$ then show that fog (x) equals 3f(x).
10. Infinitely many
11. None
12. No.

13. $|x|$
14. x
16. gof and gog
17. $fog = x^2$; $gof = \begin{cases} -x^2 & x \le 0 \\ -x & x > 0 \end{cases}$

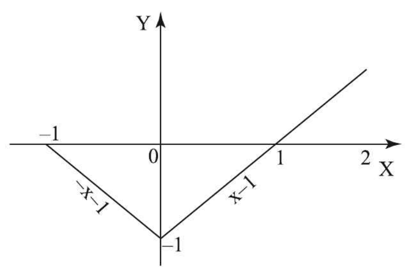
Practice Problems [F]

19. 4
20. 4 21. (b) Infinitely many
22. $y = 8x^4 - 8x^2 + 1.$
23. $f\{f(x)\} = 2 + x,\ x \in [0, 1]$
$= 4 - x,\ x \in (1, 2]$
$= 2 - x,\ x \in [2, 3]$
24. $f(g(x)) = \begin{cases} \{x\} + 1, & -1 \le x \le 2 \\ 5, & 2 \le x < 3 \end{cases}$
25. $fog\ (x) = \begin{cases} x^2 + 1\ , & -1 \le x \le 1 \\ 2x^2 + 1\ , & 1 < x \le \sqrt{2} \end{cases}$;

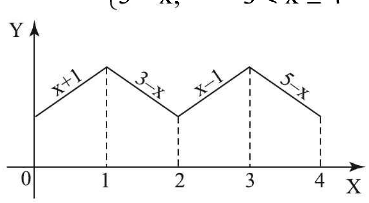
$gof\ (x) = \begin{cases} (x+1)^2\ , & -2 \le x < 1 \\ x + 3\ , & x = 1 \end{cases}$

or simply $(x + 1)^2, -2 \le x \le 1.$

26. $fog\ (x) = \begin{cases} -(1+x); & -1 \le x \le 0; \\ x - 1; & 0 < x \le 2; \end{cases}$



$gof(x) = \begin{cases} x + 1; & 0 \le x < 1 \\ 3 - x; & 1 \le x \le 2 \\ x - 1; & 2 < x \le 3 \\ 5 - x; & 3 < x \le 4 \end{cases}$



27. (i) $|x|^{1/2^n}$

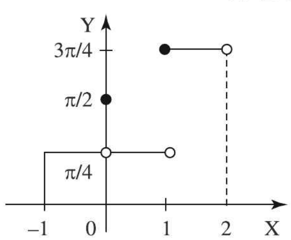
(ii) $a_n = \begin{cases} x & \text{if} \quad n = 3m, m \in I \\ \dfrac{1}{1-x} & \text{if} \quad n = 3m+1, m \in I \\ \dfrac{x-1}{x} & \text{if} \quad n = 3m+2, m \in I \end{cases}$,

Domain = R – {0, 1}

28. $f(g(x)) = \begin{cases} 6+4x \ , & -2 \le x \le 1 \\ -2x \ , & -1 \le x \le 0 \\ x \ , & 0 \le x \le 2 \\ 6-2x \ , & 2 \le x \le 4 \end{cases}$

Practice Problems [G]

1. All except g(x)
2. (a)Yes; (b)Yes (c)Yes
3. (a) No, (b) yes
4. No
5. f : {(1, 3), (2, 2), (3, 1), (4, 5), (5, 4)}
6. No
7. Domain : [– 1 , 2), Range : $\left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ f is many one



Concept Problems [C]

1. not surjective.
2. [–1, 3]
3. (i) injective but not surjective
   (ii) neither injective nor surjective
   (iii) bijective (iv) neither injective nor surjective
   (v) surjective but not injective
4. f(n) = n + 1, g(n) = n–1
5. (iii)
6. both one-one and onto.

7. neither one-one nor onto
10. Let A = {1, 2} and B = {a, b, c}.
11. Yes
12. many one into
13. (a) 9 (b) 6
14. $(-\infty, -1]$
15. (i) one-one onto (ii) one-one into (iii) one-one onto

PRACTICE PROBLEMS [H]

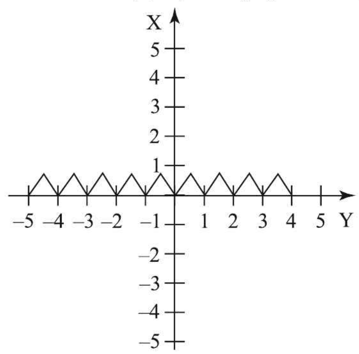
17. (i) many-one, into
18. $(-\infty, -2] \cup [2, \infty)$
20. only (iv)
23. 6
24. consider f(x) = x + 1 and $g(x) = \begin{cases} x-1 & \text{if } x>1 \\ 1 & \text{if } x=1 \end{cases}$.
25. {(1.1), (2, 3), (3, 4), (4, 2)}; {(1, 1), (2, 4), (3, 2), (4, 3)} & {(1, 1), (2, 4), (3, 3), (4, 2)}

PRACTICE PROBLEMS [I]

1. (i) Even (ii) Neither
   (iii) Odd (iv) Neither
3. $y = f_e + f_o$ where $f_e = (\sin(x+1)\sin^3(2x-3) + \sin(x-1)\sin^3(2x+3))/2$, $fo = (\sin(x+1)\sin^3(2x-3) - \sin(x-1)\sin^3(2x+3))/2$
4. $y = f_e + f_o$ where $f_e = \cos\frac{\pi}{8}\cos x - \sin\frac{\pi}{12}\cos 2x$, $f_o = -\sin\frac{\pi}{8}\sin x + \cos\frac{\pi}{12}\sin 2x$.
5. Odd extension is not posssible since f(0) is 0.
7. $-x - \sin(\cos x) + \tan(\sin x)$
10. $x = \frac{1}{2}$
11. 3
15. even.
16. even
17. even
18. All are even
19. –1
20. –3

Concept Problems [D]

1. (i) 1 (ii) $\frac{2\pi}{3}$ (iii) $\pi/4$ (iv) $\pi/2$
4. No
5. 4
6. The function is periodic of period 1. If x ∈ |0, 1| then ||x|| = min (x, 1 – 1).



8. Yes, 2π

Practice Problems [J]

9. (i) $2n\pi$; (ii) non-periodic; (iii) π ; (iv) π
10. (i) $\pi/2$; (ii) 2; (iii) 2π ; (iv) π
11. (i) Not periodic (ii) π (iii) 2π (iv) 2π
12. (i) 10π ; (ii) 12 ; (iii) 2π ; (iv) π
14. 2
15. (i) 2π (ii) 4π
16. 36
18. 4, – 2 , 2

Concept Problems [E]

1. $f^{-1}(x) = \frac{x^3+2}{x^3-1}$
2. 2
3. $a + d = 0$
4. $f^{-1}(x) = 2 + \sqrt{x-5}$

5. $\dfrac{1+\sqrt{2}}{2}$
6. 3

Practice Problems [K]

8. $-\infty, -3] \cup [0, \infty)$

10. (i) $-1+\sqrt{-x-2}$ (ii) $\dfrac{2\pi}{3}-\sin^{-1}x$

11. $R-\{\pm 2\}$, $f^{-1}(x)=\dfrac{x-|m|}{m^2-4}$

13. If $f(x)=\begin{cases} x^3 & , \quad x \le 0 \\ x^2 & , \quad 0 \le x < 1 \\ 2x-1 & , \quad 1 \le x \le 2 \\ \dfrac{x^3}{9} & , \quad x > 3 \end{cases}$ then show that

(i) $f^{-1}(-1)=-1$ (ii) $f^{-1}(-8)=-2$ (iii) $f^{-1}(1)=1$

15. 1

16. $y=\log_e\left(\dfrac{x+1}{3-x}\right)^{1/2}$

17. $(fog)(x)=e^{3x-2;}$ $(gof)(x)=3e^x-2$; Domain $(0, \infty)$, $(-2, \infty)$.

18. $g(x)=\left(\dfrac{x^{1/3}-b}{a}\right)^{1/2}$

19. $b=1$

20. $f^{-1}(x)=3+\dfrac{\sqrt{4x-7}}{2}$; $x=2$.

Concept Problems [F]

1. $\dfrac{23}{3}$
2. $-\dfrac{2x^4+x^2-3}{5x^2}$
3. $x-\dfrac{\pi}{6}$
4. 2000
6. $P(x)=x^2+c$
7. -15

Practice Problems [L]

8. 20
9. 15

10. 13
12. 16
13. $f(x) = kx, k \in R$
14. $f(x) = 4.\sqrt[3]{\dfrac{x^2(x+1)}{1-x}}$
16. (i) $f(x) = x + 1$ (ii) $f(x) = x$
(iii) $f(x) = \pm e^{x-2}$ (iv) $f(x) = x^2$
19. $f(x) = x + a$, $g(x) = x + b$ where a and b are arbitrary.
20. 1

Objective Exercises

1. C	2. C	3. C	4. B
5. C	6. C	7. D	8. C
9. A	10. A	11. A	12. A
13. D	14. C	15. B	16. D
17. B	18. B	19. C	20. C
21. C	22. D	23. B	24. D
25. A	26. B	27. D	28. A
29. D	30. A	31. B	32. A
33. C	34. B	35. A	36. B
37. B	38. A	39. C	40. C
41. D	42. C	43. A	44. C
45. D	46. A	47. A	48. B
49. A	50. B	51. AD	52. BC
53. AB	54. ACD	55. ABC	56. BD
57. ABC	58. ABC	59. ABD	60. CD
61. ABD	62. ABD	63. BCD	64. BD
65. AB	66. BC	67. BD	68. ABC
69. ABD	70. BD	71. D	72. D
73. B	74. D	75. A	76. C
77. C	78. D	79. B	80. C
81. D	82. B	83. D	84. D
85. B	86. D	87. B	87. B
89. B	90. A	91. A	92. D
93. A	94. D	95. A	96. C
97. D	98. (A)–(R), (S); (B) –(R); (C) – (R); (D)–(P)		

99. (A) Q, R; (B) P, Q; (C) Q, R, T; (D) P, S

100. (A)–(S); (B) –(P); (C) – (Q); (D)–(R)
101 (A) P,Q,R,S (B) P, Q,R (C) S (D) R,S]
102. (A) R; (B) P, Q, R, S; (C) R; (D) P, Q]

Review Exercises for JEE Advanced

1. (i) $[-3, -2] \cup (0, 2) \cup (2, 3)$
   (ii) $(-\infty, -1/2) \cup (3/2, \infty) - \{\pm 2\}$ (iii) $(-\infty, -2] \cup (5, \infty)$
2. (i) $[3, \infty)$ (ii) $(3, 5]$ (iii) $(\log_4 13, 2]$
3. (i) $[1, \infty)$ (ii) $[1, \sqrt{2}]$ (iii) $\{p\}$
4. (i) $R_f : \left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$ (ii) $\left\{\frac{\pi}{2}\right\}$ (iii) $[2, \infty)$
5. (i) $D_f : x \in R, R_f : [0, 2]$ (ii) Df : $R - \{1, 2\}, R_f : R - \{1, 3\}$
   (iii) $D_f : R, R_f : \left[\sqrt{\cos 1}, 1\right]$
   (iv) $D_f : x \in \left(2n\pi, 2n\pi + \frac{\pi}{2}\right) \cup \left((2n\pi + \pi), \left(2n\pi + \frac{3\pi}{2}\right)\right)$
   $-\left\{x \mid x = 2n\pi + \frac{\pi}{4} \text{ or } 2n\pi + \frac{5\pi}{4}\right\}, n \in I$
   $R_f : \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] - \left\{\frac{\pi}{2}\right\}$
6. (i) $\left\{\pm\frac{\pi}{2}\right\}$ (ii) $(-\infty, 0]$ (iii) $(-\infty, 1)$
7. (iv) fog(x) = x; Range is $(-\infty\ \infty)$
8. $a \in [1, 4]$
9. (a) bijective(b) Injective but not surjective
10. f(x) = x
12. $X = \left[-\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha\right]$, $Y = [c - \gamma, c + \gamma]$, where $\alpha = \tan^{-1}\left(\frac{a + b\sqrt{2}}{a}\right)$, $\gamma = \sqrt{a^2 + b^2 + \sqrt{2}\,ab}$
13. Domain = $(-\infty, 2)$, Co-domain = $(-\infty\ \infty)$
14. 2n – 1
15. 21
16. x = 1/2
18. 4
19. π

20. (i) $y = \begin{cases} 3x+2, & x<0 \\ 4x, & 0 \le x < 1 \\ 2x^2+2x-1, & x \ge 1 \end{cases}$ (ii) $y = \begin{cases} 2x^2+4x, & x<0 \\ 4x^2, & 0 \le x < 1 \\ 4x^3-2x, & x \ge 1 \end{cases}$

(iii) $y = \begin{cases} 2x+4, & x<-1 \\ 2x^2+8x+7, & -1 \le x < 0 \\ 4x, & 0 \le x < \frac{1}{2} \\ 8x^2-1, & x \ge \frac{1}{2} \end{cases}$

24. (i) (a) $\begin{cases} +\sqrt{\dfrac{-x}{1+x}}, & x \in (-1, 0) \\ -\sqrt{\dfrac{x}{1-x}}, & x \in (0, 1) \end{cases}$ (b) $\begin{cases} x, & -\infty < x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ \log_2 x, & 16 < x < \infty \end{cases}$

26. $\left(-\infty, \dfrac{\pi}{2}\right]$

27. $\pi, \pi, \pi/2$

28. $(-\infty, -2) - \{-\sqrt{5}\}$

29. $(n + 1)a$

30. $|x| < \sqrt{2+\sqrt{3}}$

31. (a) Into (b) Into

32. $f(x) = 0, 1, -1, ax, a > 0.33.$ odd

34. into

35. $\pi/2$

Target Exercises for JEE Advanced

1. $Df = \left(\dfrac{-5-\sqrt{5}}{2}, -3\right] \cup \left[-2, \dfrac{-5+\sqrt{5}}{2}\right)$; Range is I – N.

2. $x \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right)$

3. (i) $(-\infty, -2] \cup \left[-\left(\dfrac{1}{2}\right)^{1/3}, 0\right)$ (ii) $(-2, 2) - \{-1, 0, 1\}$

(iii) $x \in \left\{-\frac{7\pi}{6}\right\} \cup \left[-\frac{11\pi}{6}, -4\right] \cup \left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right]$

and $x \neq 2n\pi + \frac{\pi}{2}$, $n \in I$, $n \neq 0, -1$

4. (i) $\{0\}$ (ii) $\left[2, \frac{\pi^2}{4}\right)$ (iii) $[0, \infty)$

5. (i) Df : [4, 6], Rf : $\left[\frac{1}{2}, 1\right]$

(ii) $D_f : \left(-\infty, 1-\sqrt{2}\right] \cup \{1\} \cup \left[1+\sqrt{2}, \infty\right)$, $R_f : \{0\} \cup \left(\frac{\pi}{2}, \pi\right]$

(iii) $D_f : x \in R$, $R_f : [0, 2]$ (iv) $D_f : \bigcup_{n \in N} (2n\pi, (2n+1)\pi)$, $R_f : (-\infty, 0]$

7. $(0, 10^{-2}) \cup (10^{-2}, 10^{-1})$

8. $fog(x) = \begin{cases} \tan x + 1 & ; -\frac{\pi}{2} < x \le \frac{\pi}{4} \\ \tan x - 1 & ; \frac{\pi}{4} < x \le \tan^{-1}(3) \\ (\tan x - 3)^2 & ; \tan^{-1}(3) < x < \frac{\pi}{2} \end{cases}$ and

$$gof(x) = \begin{cases} \tan(x+1) & ; -\frac{\pi}{2} - 1 < x \le \frac{\pi}{4} - 1 \\ \tan(x+1) - 2 & ; \frac{\pi}{4} - 1 < x \le 1 \\ \tan\left((x-1)^2\right) & ; 1 < x \le 1 + \sqrt{\frac{\pi}{4}} \\ \tan\left((x-1)^2\right) - 2 & ; 1 + \sqrt{\frac{\pi}{4}} < x < 1 + \sqrt{\frac{\pi}{2}} \end{cases}$$

9. $fog(x) = \begin{cases} 2 - \cot^2 x; & -\pi/2 \le x < 0 \\ \sec^2 x; & 0 \le x < \pi/2 \end{cases}$

10. $f^{-1}(x) = \frac{\log_2 x}{\log_2 x - 1}$, Domain : R+ – {0}, Range : R – {1}

11. $a = \frac{1}{e}$, $b = e^{-1/e}$

12. $f^{-1}(x) = [x] + \{x\}^2$, solution is the set of integers

13. $\frac{1}{n}\sum_{i=1}^{n} x_i = 1$

15. n = 3

16. 0, 1, 2 }

17. (i) many-one (ii) one-one18. Into

19. $a = 2$	21. 1	23. (i) odd (ii) even
25. $p \le -\frac{1}{2}$	26. 16	28. ± 3, ± 15
29. $b_{2010} = 2010\,!$	30. 3988	

32. f(x) = 1/x.   33. f(x) = 0 for x ≥ 2, and 2/(2 – x) for 0 ≤ x < 2.

34. f(x) = – x/(x+1).   35. f(x) = 1 – x2/2.

Previous Years' Questions (JEE Advanced)

1. $\left[0, \frac{3}{\sqrt{2}}\right]$

2. [–2, –1] ∪ [1, 2]

3. nn, n!

4. (–2, 1), [–1, 1]

5. x + 1 and –x + 1

6. $\frac{-3\pm\sqrt{5}}{2}, \frac{-1\pm\sqrt{5}}{2}$	7. T	8. T	9. F
10. D	11. C	12. A	13. B
14. B	15. B	16. D	17. C
18. D	19. B	20. D	21. C
22. B	23. A	24. D	25. A
26. D	27. D	28. A	29. B
30. A	31. C	32. B	33. A
34. D	35. A	36. A	37. B
38. AD	39. B	40. A	41. A,B

42. R, [0, 1); f is not one to one

44. a = 3

45. 2 < a < 14, No

47.

	p	q	r	s	t
a	○	○	○	●	○
b	○	○	○	○	●
c	○	○	●	○	○
d	○	○	●	○	○